5 Recent advances; open problems

Contents

5.1 Weibel's Conjecture

Theorem 5.1.1 (Weibel's Conjecture [Wei80], Kerz–Strunk–Tamme [KST18]). *For a Noetherian scheme X of finite Krull dimension d, we have $K_i(X) = 0$ for all $i < -d$.*

There were many previous partial cases of Theorem 5.1.1. A non-exhaustive list is: [Bas68], [Wei89], [Hae04], [Kri10a], [Cis13], [Kel14], [KS17], [KS22]. Many of them relied on versions of Theorem 5.1.2.

Theorem 5.1.2 (Descent for blow-ups in regular centres [SGA71, Exposé VII], [TT90]). *Suppose $Z \to X$ is a regular immersion of classical schemes. Then there is a cartesian square*

$$\begin{array}{ccc} K(X) & \longrightarrow & K(Z) \\ \downarrow & & \downarrow \\ K(Bl_X Z) & \longrightarrow & K(E) \end{array}$$

where $Bl_X Z$ is the blow-up of X along Z, and E is the exceptional divisor.

If $Z \to X$ is not regular, then the above theorem fails! This can be corrected by using formal completions. The idea to use formal completions can be considered as a version of Grothendieck's *theorem on formal functions.* Such formal completions in K-theory were considered by many authors such as [KS02], [Cor06], [Kri10b], [GH06], [GH11], [Mor16], [Mor18].

Theorem 5.1.3 ([KST18, Thm. A]). *Suppose X is a Noetherian scheme, $Y \to X$ is proper, $Z \to X$ is a closed immersion, and the induced morphism $Y \setminus E \to X \setminus Z$*

is an isomorphism, then

$$\begin{array}{ccc} K(X) & \longrightarrow & \varprojlim_n K(Z_n) \\ \downarrow & & \downarrow \\ K(Y) & \longrightarrow & \varprojlim_n K(E_n) \end{array}$$

is a cartesian square of spectra, where Y_n and E_n denote the nth infinitesimal thickenings of Z in X and E in Y respectively.

Theorem 5.1.3 was proven using *derived schemes.* These are schemes where instead of rings, we use *derived rings* such as simplicial rings or animated rings.

5.2 Prismatic cohomology and K-theory of finite rings

Theorem 5.2.1 (Antieau–Krause–Nikolaus, [AKN22])**.** *We can now calculate the K-theory of rings of the form $\mathbb{Z}/p^k$ to quite high degrees using computers.*

As $K(\mathbb{Z}/p^k;\mathbb{Z}_p) \simeq \tau_{\geq 0}TC(\mathbb{Z}/p^k;\mathbb{Z}_p)$ by [DGM13, HM03], it is enough to determine TC of these rings. To do so, we use the filtration on TC constructed by Bhatt–Morrow–Scholze in [BMS19]. If R is a quasisyntomic ring, there is a complete decreasing filtration $F^{\geq\star}_{\mathrm{syn}}TC(R;\mathbb{Z}_p)$ with associated graded pieces

$$F^{=i}_{\mathrm{syn}}TC(R;\mathbb{Z}_p) \simeq \mathbb{Z}_p(i)(R)[2i],$$

where $\mathbb{Z}_p(i)(R)$ is the weight i syntomic cohomology of R introduced in [BMS19]. The syntomic complexes provide a p-adic analogue of the motivic filtration on K-theory.

There is a spectral sequence which larger vanishes and we end up with a three term complex.

Theorem 5.2.2 (Antieau-Krause-Nikolaus [AKN22])**.** *For $i > 1$, there is an explicit cochain complex*

$$\mathbb{Z}_p^{in-1} \xrightarrow{syn_0} \mathbb{Z}_p^{2(in-1)} \xrightarrow{syn_1} \mathbb{Z}_p^{in-1}$$

quasi-isomorphic to $\mathbb{Z}_p(i)(\mathbb{Z}/p^n)$. The terms are free $\mathbb{Z}_p$-modules of the given ranks in cohomological degrees 0, 1, and 2.

This can be put into a computer. A new theorem coming out of this is:

Theorem 5.2.3 (Even vanishing theorem [AKN22])**.** *If $i > \frac{p}{2(p-1)^2}(p^n-1)$, then $H^2(\mathbb{Z}_p(i)(\mathbb{Z}/p^n)) = 0$ and hence $K_{2i-2}(\mathbb{Z}/p^n) = 0$ if additionally $i > 2$.*

This gives a quantitative bound on the vanishing of even K-groups, extending earlier results of Angeltveit [Ang11].

Corollary 5.2.4 (Order formula [AKN22]). *For any* $\mathbb{Z}/p^n$*,*

$$\frac{\#K_{2i-1}(\mathbb{Z}/p^n;\mathbb{Z}_p)}{\#K_{2i-2}(\mathbb{Z}/p^n;\mathbb{Z}_p)} = p^{i(n-1)}.$$

5.3 Failure of the telescope conjecture

The telescope conjecture was one of the seven Ravenel conjectures from 1984 concerning the stable homotopy groups of spheres and chromatic homotopy theory. While six of the seven conjectures were eventually proven, the telescope conjecture remained open until 2023.

One way to understand the objects appearing in the conjecture is in terms of *tensor triangulated geometry.* The category of spectra $\mathcal{S}\mathrm{pt}$ as a monoidal structure, and we can pretend it is $D^b(X)$ for some (hypothetical) topological space X. The points of the topological space $\mathrm{Spec}(\mathcal{S}\mathrm{pt})$ correspond to certain monoid objects in $\mathcal{S}\mathrm{pt}$. These are indexed by primes p and a natural number n. In this setting one implicitly fixes a prime p, and talks about the *n-th Morava K-theory* $K(n)$. Just as with $D^b(R)$ in scheme theory, we can perform certain "localisations" in $\mathcal{S}\mathrm{pt}$.

Conjecture 5.3.1 (Ravenel's telescope conjecture, [Rav84]). For each prime p and height $n \geq 0$, the telescopic localization $L_{T(n)}$ agrees with the $K(n)$-localization $L_{K(n)}$ on the category of spectra.

Here, $T(n)$ denotes the nth telescope, a spectrum built from the Bousfield-Kuhn functor, and $K(n)$ is the nth Morava K-theory spectrum.

Theorem 5.3.2 (Burklund-Hahn-Levy-Schlank, [BHLS23]). *The telescope conjecture is false. For each prime p and height $n+1 \geq 2$, there exist spectra X such that $L_{T(n+1)}X \not\simeq L_{K(n+1)}X$.*

Proof sketch. The authors construct explicit counterexamples using algebraic K-theory. They show that the $T(n+1)$-localized algebraic K-theory of $BP\langle n\rangle^{(h\mathbb{Z})}$ is not $K(n+1)$-local, where $BP\langle n\rangle^{(h\mathbb{Z})}$ denotes a certain truncated Brown-Peterson spectrum with additional structure. □

5.4 Atiyah-Hirzebruch spectral sequence

There is a generalisation of the Grothendieck–Riemann-Roch isomorphism that was conjectured by Lichtenbaum, Quillen, and Beilinson in various forms. It was constructed by multiple authors. See [Weibel, VI.4.4] for a historical account.

Theorem 5.4.1. *For X smooth over a field, there are functors $\mathbb{Z}(n) : Sm_k^{op} \to D(\mathbb{Z})$ and a spectral sequence*

$$H^{p-q}(X, \mathbb{Z}(-q)) \implies K_{-p-q}(X)$$

which degenerates rationally to give isomorphisms

$$K_n(X)_{\mathbb{Q}} \cong \oplus_{i=0}^{d} H^{2i-n}(X, \mathbb{Q}(i)) \cong \oplus_{i=0}^{d} CH_{d-i}(X, n)_{\mathbb{Q}}.$$

Here, Bloch's higher Chow groups $CH_*(X, n)$ are more general versions of the Chow groups $A_*(X)$ from the second lecture. The group $CH_j(X, n)$ is generated by cycles in $\mathcal{Z}_j(X \times \Delta_k^n)$ which intersect the boundary properly. Here $\Delta_k^n = V(\sum x_m - 1) \subseteq \mathbb{A}^{n+1}$ and the boundary is the intersection with the axes $V(x_0 x_1 \dots x_n)$ (Similar to Δ_{top}^n from the last lecture).

Example 5.4.2.

1. $\mathbb{Z}(n) = 0$; $n < 0$
2. $\mathbb{Z}(0) = \mathbb{Z}$;
3. $\mathbb{Z}(1) \cong \mathbb{G}_m[1]$.

There are a number of models for the complexes $\mathbb{Z}(n)$. As mentioned above, one of them is via Bloch's higher Chow groups, [Blo86]. Another is via the *slice filtration*, [Voe02]. This is a sequence of presheaves of spectra

$$\cdots \to f_{n+1}K \to f_n K \to \cdots \to f_0 K = K \in \mathrm{PSh}(Sm_k, \mathcal{S}\mathrm{pt})$$

such that

$$\mathbb{Z}(n)[2n] = \mathrm{cofib}(f_{n+1}K \to f_n K),$$

[Lev08]. The presheaf $f_n K$ is essentially, the colimit of all maps the form $(\mathbb{P}^1, \infty)^{\wedge n} \wedge (\Sigma^\infty X_+)[i] \to K$ in the Morel-Voevodsky stable homotopy category.

We would like to extend this spectral sequence to non-smooth schemes. In fact, there is a version for quasi-projective varieties converging to G-theory, but we would like one that captures K-theory.

Definition 5.4.3. A presheaf of spectra F is a *procdh sheaf* if it sends formal abstract blowup squares to cartesian squares of spectra.[1]

[1] We also require that it be a Nisnevich sheaf, but I don't want to get distracted with that here.

The canonical forgetful functor admits a left adjoint

$$\mathrm{Shv}_{procdh}(\{\text{qcqs schemes}\}) \leftrightarrows \mathrm{PSh}(Sm) : L_{procdh}$$

The following theorem is a combination of an observation of Bhatt–Lurie, and a result of K.-Saito.

Theorem 5.4.4 ([KS24, Thm. 1.8]). *The left adjoint sends K-theory to K-theory.*

If we push the slice filtration through this left adjoint, then we obtain a spectral sequence with graded pieces $L_{procdh}\mathbb{Z}(n)[2n]$. Consequently, we obtain a spectral sequence.

Corollary 5.4.5 ([KS24]). *For any Noetherian scheme of finite dimension there is a convergent spectral sequence*

$$H^{2i-j}_{procdh}(X, \mathbb{Z}(i)_{procdh}) \implies K_j(X)$$

Remark 5.4.6. [EM23] (over a field), [Bou24] (mixed characteristic), show that $\mathbb{Z}(i)_{procdh}$ can be obtained from $\mathbb{Z}(i)_{cdh}$ and $\mathbb{Z}(i)^{TC}$, and also prove many properties about this (e.g., projective bundle formula).

5.5 Open problems

Here is a somewhat random selection of open conjectures.

Conjecture 5.5.1 (Parshin's conjecture). For any smooth projective variety X defined over a finite field, the higher algebraic K-groups vanish up to torsion:

$$K_i(X) \otimes \mathbb{Q} = 0, \quad i > 0$$

Conjecture 5.5.2 (Finite generation conjecture for K-theory). For any ring R that is finitely generated over $\mathbb{Z}$, the groups $K_n(R)$ should be finitely generated.

Conjecture 5.5.3 (Beilinson–Soulé vanishing). For X a smooth variety, for all $i < 0$ one has

$$H^i(X, \mathbb{Z}(n)) = 0.$$

Conjecture 5.5.4 (Vandiver's conjecture, see [Wei13, Conj.VI.10.8]). If ℓ is an irregular prime, then the group $\mathrm{Pic}(\mathbb{Z}[\zeta_\ell + \zeta_\ell^{-1}])$ has no ℓ-torsion.

Vandiver's conjecture has been verified for all primes up to 163 million.

Theorem 5.5.5 (Connection to K-theory, [Wei13, Thm.VI.10.10]). *If Vandiver's conjecture holds for ℓ, then the ℓ-primary torsion subgroup of $K_{4k-2}(\mathbb{Z})$ is cyclic for all k.*

If Vandiver's conjecture holds for all ℓ, then the groups $K_{4k-2}(\mathbb{Z})$ are cyclic for all k.

Conjecture 5.5.6 ((One of) Beilinson's conjecture(s), [Nek94, Conj.6.5(2)]). Let X be a smooth projective variety over $\mathbb{Q}$. Then

$$\mathrm{ord}_{s=n} L(h^{2n-1}(X), s) = \dim_{\mathbb{Q}} CH^n(X)_0 \otimes \mathbb{Q}$$

where $L(h^{2n-1}(X), s)$ is the L-function associated to the motive $h^{2n-1}(X)$, and $CH^n(X)_0$ denotes the Chow group of codimension n cycles on X that are homologically equivalent to zero.

References

[AKN22] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. On the K-theory of $\mathbb{Z}/p^n$ – announcement. *arXiv preprint arXiv:2204.03420*, 2022.

[Ang11] Vigleik Angeltveit. On the algebraic K-theory of Witt vectors of finite length. *arXiv preprint arXiv:1101.1866*, 2011.

[Bas68] Hyman Bass. *Algebraic K-theory*. Benjamin, New York, 1968.

[BHLS23] Robert Burklund, Jeremy Hahn, Ishan Levy, and Tomer M. Schlank. K-theoretic counterexamples to Ravenel's telescope conjecture. *arXiv preprint arXiv:2310.17459*, 2023.

[Blo86] Spencer Bloch. Algebraic cycles and higher K-theory. *Advances in Mathematics*, 61(3):267–304, 1986.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publications mathématiques de l'IHÉS*, 129:199–310, 2019.

[Bou24] Tess Bouis. Motivic cohomology of mixed characteristic schemes. *arXiv preprint arXiv:2412.06635*, 2024.

[Cis13] Denis-Charles Cisinski. Descente par éclatements en K-théorie invariante par homotopie. *Annals of Mathematics*, 177(2):425–448, 2013.

[Cor06] Guillermo Cortiñas. The obstruction to excision in K-theory and in cyclic homology. *Inventiones mathematicae*, 164(1):143–173, 2006.

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[EM23] Elden Elmanto and Matthew Morrow. Motivic cohomology of equicharacteristic schemes. *arXiv preprint arXiv:2309.08463*, 2023.

[GH06] Thomas Geisser and Lars Hesselholt. Bi-relative algebraic K-theory and topological cyclic homology. *Inventiones mathematicae*, 166(2):359–395, 2006.

[GH11] Thomas Geisser and Lars Hesselholt. On relative and bi-relative algebraic K-theory of rings of finite characteristic. *Journal of the American Mathematical Society*, 24(1):29–49, 2011.

[Hae04] Christian Haesemeyer. Descent properties of homotopy K-theory. *Duke Mathematical Journal*, 125(3):589–620, 2004.

[HM03] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Annals of Mathematics*, 158(1):1–113, 2003.

[Kel14] Shane Kelly. Vanishing of negative K-theory in positive characteristic. *Compositio Mathematica*, 150(8):1425–1434, 2014.

[Kri10a] Amalendu Krishna. An Artin-Rees theorem in K-theory and applications to zero cycles. *Journal of Algebraic Geometry*, 19(3):555–598, 2010.

[Kri10b] Amalendu Krishna. An Artin-Rees theorem in K-theory and applications to zero cycles. *Journal of Algebraic Geometry*, 19(3):555–598, 2010.

[KS02] Amalendu Krishna and Vasudevan Srinivas. Zero-cycles and K-theory on normal surfaces. *Annals of Mathematics*, 156(1):155–195, 2002.

[KS17] Moritz Kerz and Florian Strunk. On the vanishing of negative homotopy K-theory. *Journal of Pure and Applied Algebra*, 221(7):1641–1644, 2017.

[KS22] Moritz Kerz and Florian Strunk. On negative K-groups in positive characteristic. *Mathematische Zeitschrift*, 301:2071–2085, 2022.

[KS24] Shane Kelly and Shuji Saito. A procdh topology. *arXiv preprint arXiv:2401.02699*, 2024.

[KST18] Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic K-theory and descent for blow-ups. *Inventiones mathematicae*, 211(2):523–577, 2018.

[Lev08] Marc Levine. The homotopy coniveau tower. *Journal of Topology*, 1(1):217–267, 2008.

[Mor16] Matthew Morrow. Pro *cdh*-descent for cyclic homology and K-theory. *Journal of the Institute of Mathematics of Jussieu*, 15(3):539–567, 2016.

[Mor18] Matthew Morrow. Pro unitality and pro excision in algebraic K-theory and cyclic homology. *Journal für die reine und angewandte Mathematik*, 736:95–139, 2018.

[Nek94] Jan Nekováø. Beilinson's conjectures. In *Motives*, volume 55 of *Proceedings of Symposia in Pure Mathematics*, pages 537–570. American Mathematical Society, Providence, RI, 1994.

[Rav84] Douglas C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106(2):351–414, 1984.

[SGA71] *Théorie des intersections et théorème de Riemann-Roch*, volume 225 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[TT90] R. W. Thomason and Thomas Trobaugh. Higher algebraic K-theory of schemes and of derived categories. *The Grothendieck Festschrift*, III:247–435, 1990.

[Voe02] Vladimir Voevodsky. Open problems in the motivic stable homotopy theory. I. *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, 3:3–34, 2002.

[Wei80] Charles A. Weibel. K-theory and analytic isomorphisms. *Inventiones mathematicae*, 61(2):177–197, 1980.

[Wei89] Charles A. Weibel. Homotopy algebraic k-theory. *Contemporary Mathematics*, 83:461–488, 1989.

[Wei13] Charles A. Weibel. *The K-book: An introduction to algebraic K-theory*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, 2013.