# 4 $K$-theory as the universal localising invariant

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In the last lecture we generalised $K_0$ to the pair $K_0, K_1$ by moving from the set $\mathrm{Proj}(R)_{/\cong}$ to the groupoid $\mathrm{Proj}(R)^{\cong}$. In this lecture we generalise this process to get higher $K_0, K_1, K_2, \ldots$ by considering $\infty$-groupoids. These form an $\infty$-category.

## 4.1 Simplicial sets

References for simplicial sets and $\infty$-groupoids (which used to be called Kan complexes):

1. May, Simplicial objects in algebraic topology.
2. Bousfield–Kan, Homotopy limits, completions and localizations.
3. Goerss–Jardine, Simplicial homotopy theory.

| | | | | | | |
|---|---|---|---|---|---|---|
| Groups (groupoids with one object) | $\subseteq$ | Groupoids (categories for which every morphism is invertible) | $\subseteq$ | Categories | $\to$ | Directed graphs |
| $\infty$-groups | $\subseteq$ | $\infty$-groupoids | $\subseteq$ | $\infty$-categories | $\subseteq$ | Simplicial sets |

Recall that a directed graph consists of a set $G_0$ of vertices, a set $G_1$ of edges and two morphisms

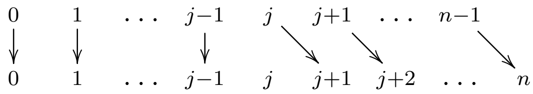
$$d_0, d_1 : G_1 \rightrightarrows G_0$$

which associate to each edge $e \in G_1$ a source $d_1 e \in G_0$ and a target $d_0 e \in G_0$.

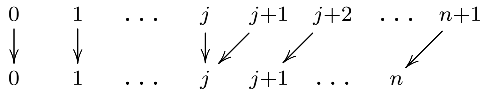
We can generalise this in higher dimensions by allowing "$n$-dimensional edges" for all $n \in \mathbb{N}$. The information of all these higher edges and how they are related to each other is organised in the concept of a *simplicial set*.

**Definition 4.1.1.** We write $\Delta \subseteq \text{LinOrdSet}$ for the full subcategory of the category LinOrdSet of linearly ordered sets whose objects are finite and non-empty. In other words, those linearly ordered sets which are isomorphic to the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$ for some $n \geq 0$. Morphisms are those morphisms of sets $p : [n] \to [m]$ such that $i \leq j \implies p(i) \leq p(j)$.

**Example 4.1.2.** For each $0 \leq j \leq n$ with $n \neq 0$, the *face* morphism $\delta_j : [n-1] \to [n]$ are defined as the unique injection which does not have $j$ in its image.




**Example 4.1.3.** For each $0 \leq j \leq n$ the *degeneracy* morphism $\sigma_j : [n+1] \to [n]$ is defined as the unique surjection which sends both $j$ and $j+1$ to $j$.




**Exercise.** Show that every morphism $[n] \to [m]$ can be written as a composition of face and degeneracy morphisms.

**Definition 4.1.4.** The category of simplicial sets $\mathcal{S}\text{et}_\Delta$ is the category of functors $\Delta^{op} \to \mathcal{S}\text{et}$, so

$$\mathcal{S}\text{et}_\Delta := \text{PSh}(\Delta)$$

Given such a functor $X : \Delta^{op} \to \mathcal{S}\text{et}$ we write $X_n := X([n])$. Elements of $X_n$ are called *$n$-simplices* of $X$..

**Example 4.1.5.** For any simplicial set $X : \Delta^{op} \to \mathcal{S}\text{et}$ the morphisms

$$d_j : X_n \to X_{n-1}.$$

corresponding to the $\delta_j$ are called *face* morphisms. For $x \in X_n$ we call $d_j x$ the *$j$th face of $x$*. The morphisms

$$s_j : X_n \to X_{n+1}.$$

corresponding to the $\sigma_j$ are called *degeneracy* morphisms.

**Example 4.1.6** ($\Delta^n$)**.** For each $n$, the functor $\Delta^n := \hom_\Delta(-, [n]) : \Delta^{op} \to \mathcal{S}\text{et}$ defines a simplicial set. By Yoneda's Lemma, for any $X \in \mathcal{S}\text{et}_\Delta$,

$$\hom_{\mathcal{S}\text{et}_\Delta}(\Delta^n, X) \cong X_n.$$

**Example 4.1.7** ($\partial\Delta^n$)**.** Consider the morphisms of simplicial sets $\delta_j : \Delta^{n-1} \to \Delta^n$. We define

$$\partial\Delta^n = \bigcup_{j=0}^{n} \delta_j(\Delta^{n-1})$$

as the union of these faces. Explicitly, $(\partial\Delta^n)_j \subseteq (\Delta^n)_j = \hom_\Delta([j],[n])$ is the set of morphisms $[j] \to [n]$ of linearly ordered sets which are not surjective. This can also be described as the colimit

$$\partial\Delta^n = \varinjlim_{[i]\subsetneq[n]} \Delta^i$$

In particular, for any other simplicial set $X$ we have

$$\hom(\partial\Delta^n, X) = \varprojlim_{\substack{[i]\subsetneq[n]\\ n-2\le i\le n-1}} X_i.$$

That is, a morphism $\partial\Delta^n \to X$ is the same thing as a set of $(n-1)$-simplicies $x_0, \ldots, x_n \in X_{n-1}$ satisfying $d_i x_j = d_j x_i$.

**Definition 4.1.8** ($\Lambda^n_j$)**.** For $0 \le j \le n$ we define the *jth horn* as the union

$$\Lambda^n_j = \bigcup_{i\neq j} \delta_i(\Delta^{n-1}).$$

Equivalently, $(\Lambda^n_j)_i \subseteq (\Delta^n)_i = \hom_\Delta([i],[n])$ is the set of those $[i] \to [n]$ whose image does *not contain* the subset $\{0, 1, \ldots, j-1, j+1, \ldots, n\}$.

**Example 4.1.9** ($\operatorname{Sing} X$)**.** Define

$$\Delta^n_{\text{top}} := \left\{ (x_0, \ldots, x_n) \mid 0 \le x_i \le 1; \sum_{i=0}^{n} x_i = 1 \right\} \subseteq \mathbb{R}^{n+1}$$

to be the convex hull of the standard basis vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. So $\Delta^0_{\text{top}}$ is a point, $\Delta^1_{\text{top}}$ is a line segment, $\Delta^2_{\text{top}}$ is a triangle, $\Delta^3_{\text{top}}$ is a tetrahedron, …

Any morphism $p : [n] \to [m]$ in $\Delta$ defines an $\mathbb{R}$-linear morphism $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$; $e_i \mapsto e_{p(i)}$, which restricts to a continuous morphism $\Delta^n_{\text{top}} \to \Delta^m_{\text{top}}$. In this way we get a functor

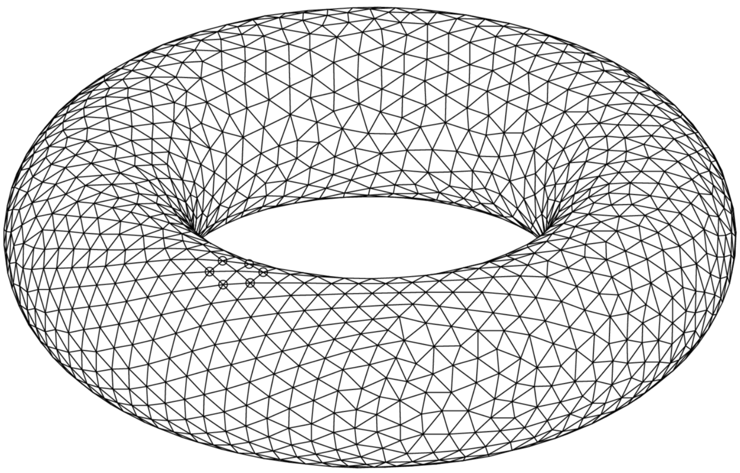
$$\Delta \to \mathcal{T}\text{op}; \qquad [n] \mapsto \Delta^n_{\text{top}}$$

from $\Delta$ to the category of topological spaces. For any other topological space $X$, the assignment

$$\operatorname{Sing} X : [n] \mapsto \hom_{\mathcal{T}\text{op}}(\Delta^n_{\text{top}}, X)$$

defines a simplicial set. Explicitly,

1. $\mathrm{Sing}_0\, X$ is the set of points of $X$,
2. $\mathrm{Sing}_1\, X$ is the set of paths in $X$,
3. $\mathrm{Sing}_2\, X$ is the set of triangles in $X$,
4. ...



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**Example 4.1.10** (Product of simplicial sets)**.** For simplicial sets $X$ and $Y$, their product $X \times Y$ is defined by

$$(X \times Y)_n = X_n \times Y_n$$

with structure morphisms acting componentwise.

**Example 4.1.11** (Mapping simplicial sets)**.** For simplicial sets $X$ and $Y$, the *mapping simplicial set* $\mathrm{Map}(X, Y)$ has $n$-simplices given by

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X, Y)_n = \hom_{\mathcal{S}\mathrm{et}_\Delta}(X \times \Delta^n, Y)$$

The maps associated to $[n] \to [m]$ are induced by the corresponding $\Delta^n \to \Delta^m$.

## 4.2 Infinity groupoids

**Definition 4.2.1** (Kan fibration)**.** A morphism $f : X \to Y$ of simplicial sets is a *Kan fibration* if for every $0 \leq j \leq n$ with $0 \neq n$ and commutative square

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

a dashed morphism exists making two triangles commutative. A simplicial set $X$ is an $\infty$-*groupoid* if the canonical morphism $X \to \Delta^0$ is a Kan fibration.

**Example 4.2.2.** If $X$ is a topological space, then $\operatorname{Sing} X$ is an $\infty$-groupoid. In fact, by the homotopy hypothesis, there is an equivalence of $\infty$-categories between the $\infty$-category of topological spaces and the $\infty$-category of $\infty$-groupoids.

**Definition 4.2.3** (Homotopy groups)**.** For an $\infty$-groupoid $X$ and a basepoint $x \in X_0$, the $n$th homotopy group $\pi_n(X, x)$ is defined as follows. Consider

$$Z_{n,x} := \left\{ \Delta^n \xrightarrow{f} X \;\middle|\; \begin{array}{ccc} \partial\Delta^n & \longrightarrow & x \\ \downarrow & & \downarrow \\ \Delta^n & \xrightarrow{f} & X \end{array} \quad \text{commutes} \right\}$$

We define an equivalence relation on $Z_{n,x}$: two morphisms $f, g \in Z_{n,x}$ are equivalent if there exists a morphism $H : \Delta^{n+1} \to X$ such that

$$\begin{aligned} H|_{\delta_0(\Delta^n)} &= f \\ H|_{\delta_i(\Delta^n)} &= x \qquad\qquad i = 1, \dots, n-1 \\ H|_{\delta_n(\Delta^n)} &= g \end{aligned}$$

Then

$$\pi_n(X, x) = Z_{n,x} / \sim .$$

**Remark 4.2.4.** The lifting property defining $\infty$-groupoids ensures this actually is an equivalence relation. It is not an equivalence relation in a general simplicial set.

**Example 4.2.5.** If $n = 0$ then $\pi_0(X, x)$ is the set of connected components of the $\infty$-groupoid $X$.

**Example 4.2.6.** Suppose $p : E \to B$ is a Kan fibration between $\infty$-groupoids, $e \in E$, $b = p(e)$, and $F = \{b\} \times_B E$. Then there is a long exact sequence of groups

$$\dots \pi_{n+1}(B, b) \to \pi_n(F, e) \to \pi_n(E, e) \to \pi_n(B, b) \to \dots$$

for $n > 0$ ending with an exact sequence of pointed sets

$$\dots \to \pi_1(B, b) \to \pi_0(F, e) \to \pi_0(E, e) \to \pi_0(B, b).$$

Here, a sequence $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$ of pointed sets is exact if $f(A) = g^{-1}(c)$.

**Definition 4.2.7** (Weak equivalence)**.** A morphism $f : X \to Y$ of $\infty$-groupoids is a *weak equivalence* if it induces isomorphisms on all homotopy groups:

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is a bijection for all $n \geq 0$ and all basepoints $x \in X_0$, and $\pi_0 X \to \pi_0 Y$ is surjective.

**Example 4.2.8.** In the notation of Example 4.2.6, if all homotopy groups of $F$ are trivial and $\pi_0(E) \to \pi_0(B)$ is surjective, then $E \to B$ is a weak equivalence.

**Example 4.2.9.** The inclusion

$$X = \{z \in \mathbb{C} \mid |z| = 1\} \to \mathbb{C} \setminus \{0\} = Y$$

induces a weak equivalence $\operatorname{Sing} X \to \operatorname{Sing} Y$ of $\infty$-groupoids. Indeed, the existence of a deformation retract $Y \to X$; $z \mapsto z/|z|$ implies all homotopy groups are isomorphic.

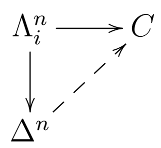
**Example 4.2.10** (Homotopy equivalence)**.** Two $\infty$-groupoids $X$ and $Y$ are *homotopy equivalent* if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identity morphisms. Here, two morphisms $a, b : A \to B$ are *homotopic* if there exists a morphism $H : A \times \Delta^1 \to B$ such that $H|_{A\times\{0\}} = a$ and $H|_{A\times\{1\}} = b$. By Whitehead's theorem for $\infty$-groupoids, a morphism is a homotopy equivalence if and only if it is a weak equivalence.

## 4.3 Infinity categories

The standard reference for $\infty$-categories is [Lurie, Higher Topos Theory].

There are also a number of other texts, e.g., [Haugseng, Yet another introduction to infinity categories].

**Definition 4.3.1** (Boardman, Vogt, 1973)**.** An $\infty$-*category* is a simplicial set $C$ such that for every $0 < i < n$ and each diagram




there exists a (not necessarily unique) dashed arrow making a commutative triangle.

**Example 4.3.2.** If $I \in \mathcal{S}\mathrm{et}_\Delta$ and $C \in \mathcal{C}\mathrm{at}_\infty$, then $\operatorname{Map}(I, C) \in \mathcal{C}\mathrm{at}_\infty$. A morphism $D \to C$ between two $\infty$-categories is called a *functor*.

(Exercise: Show that for any $X \in \mathcal{S}\mathrm{et}_\Delta$, if $Y$ is an $\infty$-groupoid, resp. $\infty$-category, then so is $\operatorname{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X, Y)$)

**Example 4.3.3.** Let $C$ be a small category. Considering the ordered sets $[n]$ as categories[1] $\{0 \to 1 \to \cdots \to n\}$ the assignment

$$N : [n] \mapsto \operatorname{Fun}([n], C)$$

[1]So, for $0 \leq i, j \leq n$ there is exactly one morphism $i \to j$ if $i \leq j$, and no morphisms otherwise.

sending $[n]$ to the set of functors $[n] \to C$ defines a simplicial set. This is called the *nerve* of $C$. Explicitly,

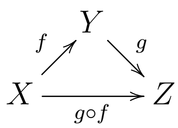
1. $N(C)_0$ is the set of objects of $C$,
2. $N(C)_1$ is the set of (all) morphisms in $C$,
3. The two morphisms $N(C)_1 \rightrightarrows N(C)_0$ induced by the two functors $[0] \rightrightarrows [1]$ send morphisms in $N(C)_1$ to their source and target.

$$(X \xrightarrow{f} Y) \qquad \mapsto \qquad X, Y$$

4. The morphism $N(C)_0 \to N(C)_1$ induced by $[1] \to [0]$ sends each object to its identity morphism.

$$X \qquad \mapsto \qquad (X \xrightarrow{\mathrm{id}_X} X)$$

5. $N(C)_2$ is the set of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.
6. The three maps $d_0, d_1, d_2 : N(C)_2 \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} N(C)_1$ induced by the three faithful functors $[1] \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} [2]$ send $\xrightarrow{f}\xrightarrow{g}$ to $g$, $g \circ f$, and $f$ respectively.

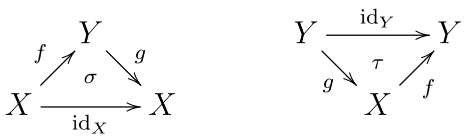



$$\mapsto \qquad (Y \xrightarrow{g} Z), \quad (X \xrightarrow{g \circ f} Y), \quad (X \xrightarrow{f} Y)$$

7. More generally, $N(C)_n$ is the set of sequences of $n$ composable morphisms $\xrightarrow{f_1} \dots \xrightarrow{f_n}$ and the various maps $N(C)_n \to N(C)_m$ come from various combinations of composition and inserting identities.

**Definition 4.3.4.** Let $C \in \mathcal{C}\mathrm{at}_\infty$. Elements of $C_0$ are called *objects* and elements of $C_1$ are called 1-morphisms, or often just *morphisms*. Given two morphisms $f, g \in C_1$ such that $d_0 f = d_1 g$ (equivalently, a morphism of simplicial sets $\Lambda^2_1 \to C$), for any factorisation $\Lambda^2_1 \to \Delta^2 \overset{\sigma}{\dashrightarrow} C$, the morphism $d_1\sigma \in C_1$ will be called a *composition* of $g$ and $f$. For any object $X \in C_0$, the morphism $s_0 X \in C_1$ is called the *identity morphism* of $X$, and written $\mathrm{id}_X$.

**Example 4.3.5.** A morphism $f : X \to Y$ in an $\infty$-category is called an *equivalence* if there exists a morphism $g : Y \to X$ and 2-cells $\sigma$ and $\tau$ of the form



**Definition 4.3.6** (Mapping space)**.** For an $\infty$-category $C$ and objects $x, y \in C_0$, the *mapping space* $\mathrm{Map}_C(x, y)$ is defined as the pullback

$$\mathrm{Map}_C(x, y) := \{x\} \times_C \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^1, C) \times_C \{y\}$$

in the 1-category $\mathcal{S}\mathrm{et}_\Delta$ where the fiber products are taken with respect to the source and target maps $d_1, d_0 : C_1 \to C_0$. The *morphism set* is

$$\hom_C(x, y) = \pi_0 \mathrm{Map}_C(x, y).$$

**Example 4.3.7.** Any $\infty$-groupoid is an $\infty$-category. In particular, for any topological space $X$, the simplicial set $\mathrm{Sing}\, X$ is an $\infty$-category.

**Example 4.3.8.** There exists an $\infty$-category $\mathcal{G}\mathrm{pd}_\infty$ whose objects are small $\infty$-groupoids and whose mapping spaces are equivalent to the mapping simplicial set defined above.

$$\mathrm{Map}_{\mathcal{G}\mathrm{pd}_\infty}(X, Y) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X, Y).$$

**Example 4.3.9.** For any $\infty$-category $C$, there is a maximal sub-$\infty$-groupoid

$$C^{\cong} \subseteq C$$

called the *core* of $C$. It has the same objects as $C$, but only the invertible morphisms. More precisely, $(C^{\cong})_n$ consists of those $n$-simplices $x \in C_n$ such that all images of $x$ in $C_1$ are invertible morphisms in $C$.

**Example 4.3.10.** There exists an $\infty$-category $\mathcal{C}\mathrm{at}_\infty$ whose objects are small $\infty$-categories and whose mapping spaces are equivalent to

$$\mathrm{Map}_{\mathcal{C}\mathrm{at}_\infty}(C, D) \simeq \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(C, D)^{\cong}.$$

**Example 4.3.11.** Let $R \in \mathcal{R}\mathrm{ing}$. A *bounded chain complex* of projectives is a sequence of morphisms

$$[\ldots \overset{d_{n+1}}{\to} P_n \overset{d_n}{\to} P_{n-1} \overset{d_{n-1}}{\to} \ldots\ldots]$$

in $\mathrm{Proj}(R)$ such that $d_{n-1} d_n = 0$ and only finitely many $P_n$ are non-zero. A *morphism* of chain complexes $f_\bullet : P_\bullet \to Q_\bullet$ is a sequence a morphisms $f_n : P_n \to Q_n$ making commutative squares. A *homotopy* of morphisms $h : f_\bullet \sim g_\bullet$ is a sequence of morphisms $h_n : P_n \to Q_{n+1}$ such that $f_n - g_n = d_{n+1} h_n + h_{n-1} d_n$.

There exists an $\infty$-category $D^b(R)$ whose objects are bounded chain complexes of projectives and morphisms are

$$\hom_{D^b(R)}(P_\bullet, Q_\bullet) \cong \frac{\{f_\bullet : P_\bullet \to Q_\bullet\}}{\text{homotopy}}$$

For more details on this example, see [Lurie, Higher Algebra, §1.3.1 and §1.3.2].

**Example 4.3.12.** The simplicial set $\Delta^1$ is an $\infty$-category but not an $\infty$-groupoid (the unique non-identity morphism is not invertible). The pushout $\Delta^0 \sqcup_{\partial\Delta^1} \Delta^1$ of simplicial sets is not an $\infty$-category. (The unique non-degenerate 1-simplex cannot be composed with itself).

## 4.4 Limits

**Definition 4.4.1** (Limits in $\infty$-categories)**.** Let $C \in \mathcal{C}\mathrm{at}_\infty$ and $I \in \mathcal{S}\mathrm{et}_\Delta$. Given an object $X \in C$, write $\gamma(X) \in \mathrm{Fun}(I, C)$ for the constant functor $I \to \Delta^0 \xrightarrow{X} C$.

For a morphism $F : I \to C$, a *limit* of $F$ is an object $\varprojlim F \in C$ together with a morphism $\gamma(\varprojlim F) \to F$ in $\mathrm{Fun}(I, C)$ such that for any object $X \in C$, the natural map

$$\mathrm{Map}_C(X, \varprojlim F) \to \mathrm{Map}_{\mathrm{Fun}(I,C)}(\gamma(X), F)$$

is an equivalence of $\infty$-groupoids. Dually, a *colimit* of $F$ is an object $\varinjlim F \in C$ together with a natural transformation $F \to \gamma(\varinjlim F)$ such that for any object $X \in C$, the natural map

$$\mathrm{Map}_C(\varinjlim F, X) \to \mathrm{Map}_{\mathrm{Fun}(I,C)}(F, \gamma(X))$$

is an equivalence of $\infty$-groupoids.

**Example 4.4.2** (Initial and terminal objects)**.** An *initial object* $\varnothing$, resp. *terminal object* $*$, in an $\infty$-category $C$ is a limit, resp. colimit, of the unique functor $\varnothing \to C$ from the empty $\infty$-category. Equivalently, it is an object such that for any $X \in C$, the mapping space $\mathrm{Map}_C(\varnothing, X)$, resp. $\mathrm{Map}_C(X, *)$, is *contractible*, i.e., all homotopy groups are trivial, or equivalently, $\mathrm{Map} \cong \Delta^0$.

1. In the $\infty$-category of $\infty$-groupoids $\mathcal{G}\mathrm{pd}_\infty$:
   - Initial object $\varnothing$, resp. terminal object $*$: the empty $\infty$-groupoid, resp. the point $\Delta^0$. Note that just as we can have very large categories which are equivalent to the punctual category, we can have quite large $\infty$-groupoids which are terminal objects. For example,
     (a) for any $n$, the $\infty$-groupoid $\mathrm{Sing}\,\mathbb{R}^n$ is a terminal object of $\mathcal{G}\mathrm{pd}_\infty$.
     (b) For any $X \in \mathcal{G}\mathrm{pd}_\infty$ and $x \in X$, the $\infty$-groupoid $\mathrm{Map}(\Delta^1, X) \times_X \{x\}$ of paths towards $x$ is a terminal object of $\mathcal{G}\mathrm{pd}_\infty$.
2. In the $\infty$-category of pointed $\infty$-groupoids:
   - Initial and terminal object: $\Delta^0$ (the point, which is both initial and terminal, making this a pointed category)

3. In the $\infty$-category $\mathcal{C}\mathrm{at}_\infty$ of $\infty$-categories:

   - Initial object $\varnothing$, resp. terminal object $*$: the empty $\infty$-category $\varnothing$, resp. the terminal $\infty$-category $\Delta^0$ with one object and only identity morphisms
   - Terminal object: $*$ (the terminal $\infty$-category with one object and only identity morphisms)

4. In the derived category $D^b(R)$:

   - Initial object: 0 (the zero chain complex)
   - Terminal object: 0 (the zero chain complex)

   As in $\mathcal{G}\mathrm{pd}_\infty$ we can have "large" objects which are also initial / terminal. For example $[\cdots \to 0 \to P \xrightarrow{=} P \to 0 \to \dots]$ is equivalent to 0 for any $P$. So it is also an intial / terminal object.

**Example 4.4.3** (Products and disjoint unions)**.** Products and Coproducts are limits and colimits over $\Delta^1 = \Delta^0 \sqcup \Delta^0$.

1. In $\mathcal{G}\mathrm{pd}_\infty$ and $\mathcal{C}\mathrm{at}_\infty$, products and coproducts are as in the 1-category $\mathcal{S}\mathrm{et}_\Delta$.
2. In $D^b(R)$ coproducts and products are isomorphic:

$$(P_\bullet \times Q_\bullet)_n = P_n \oplus Q_n = (P_\bullet \sqcup Q_\bullet)_n.$$

**Example 4.4.4** (Pullbacks)**.**

1. In $\mathcal{G}\mathrm{pd}_\infty$ the limit of a diagram $X \to Z \leftarrow Y$ is modelled by the simplicial set

$$X \times_Z \mathrm{Map}(\Delta^1, Z) \times_Z Y.$$

   If $X = \{z\} = Y$ is a vertex of $Z$, then the limit $\{z\} \times_Z \{z\}$ in $\mathcal{G}\mathrm{pd}_\infty$ is the space of loops from $z$ to $z$.

2. In $\mathcal{C}\mathrm{at}_\infty$: The limit of a diagram $C \to D \leftarrow E$ is modelled by the simplicial set

$$C \times_D \mathrm{Map}(\mathrm{Iso}(\Delta^1), D) \times_D E.$$

   where $\mathrm{Iso}(\Delta^1)$ is the free $\infty$-groupoid generated by $\Delta^1$. Explicitly, we have $\mathrm{Iso}(\Delta^1)_m = \prod_{i \in [m]} \{0, 1\}$

3. In $D^b(R)$: For chain complexes $P_\bullet \xrightarrow{f} R_\bullet \xleftarrow{g} Q_\bullet$, the limit is modelled by the complex
$$\mathrm{Cone}(P_\bullet \oplus Q_\bullet \xrightarrow{(f,-g)} R_\bullet)[-1].$$
Here, for a general morphism of complexes $A_\bullet \xrightarrow{f} B_\bullet$ one defines
$$\mathrm{Cone}(A_\bullet \xrightarrow{f} B_\bullet)_n = B_n \oplus A_{n-1}$$
with differential $\left[\begin{smallmatrix} d_B & f \\ 0 & d_A \end{smallmatrix}\right]$ and
$$A_\bullet[m]_n = A_{n-m}$$

**Example 4.4.5.** Suppose that a quasi-projective variety $X$ is the union of two basic opens $X = U \cup V$. Then the derived category $D^b(X)$ is equivalent to the pullback
$$D^b(X) = D^b(U) \times_{D^b(U\cap V)} D^b(V)$$
in $\mathcal{C}\mathrm{at}_\infty$ where we define $D^b(Y) := D^b(\mathcal{O}_Y(Y))$ for affine varieties. More generally, if $X$ is a union of finitely many basic opens $X = \cup_{i=1}^n U_i$ then the derived category is the limit
$$D^b(X) = \varprojlim_{i_0<\cdots<i_m} D^b(U_{i_0} \cap \cdots \cap U_{i_n})$$

Pushouts in $\mathcal{C}\mathrm{at}_\infty$ and $\mathcal{G}\mathrm{pd}_\infty$ are not so explicit in general.

**Example 4.4.6.** The pushout of a diagram $P_\bullet \leftarrow R_\bullet \to Q_\bullet$ in $D^b(R)$ is modelled by the chain complex
$$\mathrm{Cone}(R_\bullet \to P_\bullet \oplus Q_\bullet).$$

## 4.5 Stable infinity categories

**Definition 4.5.1.** An $\infty$-category $C$ is *stable* if it satisfies the following conditions:

(Sta0) It is pointed. That is, it admits both an initial object $\varnothing$ and a final object $*$ and an equivalence $\varnothing \cong *$. We write 0 for such an object.

(Sta1) It admits *fibres* and *cofibres*. That is, for every $f : X \to Y$, both
$$\mathrm{fib}(f) := X \times_Y 0 \quad \text{and} \quad \mathrm{cof}(f) = 0 \sqcup_X Y$$
exist.

(Sta2) A commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array} \tag{1}$$

is cartesian if and only if it is cocartesian.

**Example 4.5.2.** For $R \in \mathcal{R}\mathrm{ing}$ the category $D^b(R)$ is stable.

**Example 4.5.3.** Suppose that $C$ is a pointed $\infty$-category. The stabilisation $\mathrm{Sp}(C)$ is defined as the limit

$$\mathrm{Sp}(C) = \varprojlim(\dots \overset{\Omega}{\to} C \overset{\Omega}{\to} C \overset{\Omega}{\to} C)$$

in $\mathcal{C}\mathrm{at}_\infty$ where $\Omega X := * \times_X *$. Every object in $\mathrm{Sp}(C)$ gives rise to a sequence of objects $E_0, E_1, \dots$ in $C$ equipped with equivalences $E_{n-1} \tilde{\to} \Omega E_n$. We also have a canonical functor

$$\Omega^\infty : \mathrm{Sp}(C) \to C$$

namely, projection to the last component.

**Proposition 4.5.4** ([HA, Cor.1.4.2.23])**.** *Let $C$ be an $\infty$-category which admits finite limits, and $T$ a stable quasi-category. Then composition with the functor $\Omega^\infty$ induces an equivalence of $\infty$-categories*

$$\mathrm{Fun}^{Lex}(T, \mathrm{Sp}(C)) \to \mathrm{Fun}^{Lex}(T, C)$$

*where* $\mathrm{Fun}^{Lex}$ *means the full subcategory of functors sending finite limits to finite limits, i.e.,* left exact *functors.*

**Definition 4.5.5.** The stabilisation of the category of pointed $\infty$-groupoids is called the category of *spectra* and is denoted

$$\mathcal{S}\mathrm{pt} = \mathrm{Sp}(\mathcal{G}\mathrm{pd}_{\infty,*}).$$

**Remark 4.5.6.** The equivalences $E_n \tilde{\to} \Omega E_{n+1}$ induces isomorphisms

$$\pi_i E_n \tilde{\to} \pi_{i+1} E_{n+1}.$$

Given a spectrum $E$ we define

$$\pi_i E := \pi_{i+j} E_j; \qquad i \in \mathbb{Z}$$

for any $j$ such that the right hand side is defined.

**Example 4.5.7.** Suppose that $C \subseteq D$ in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ is a full sub-$\infty$-category closed under finite limits and finite colimits. The *Verdier localisation* $D/C$ is a stable $\infty$-category with the same objects as $D$ and mapping spaces

$$\mathrm{Map}_{D/C}(X,Y) = \varinjlim_{X'\to X} \mathrm{Map}_D(X',Y)$$

where the colimit is over the $\infty$-category of morphisms $s : X' \to X$ such that $\mathrm{fib}(s) \in C$.

**Example 4.5.8.** The canonical functor $D^b(\mathbb{Z}) \to D^b(\mathbb{Q})$ identifies $D^b(\mathbb{Q})$ with the Verdier quotient of $D^b(\mathbb{Z})$ by the full sub-$\infty$-category $D^b(\mathbb{Z})_{\mathrm{tor}} \subseteq D^b(\mathbb{Z})$ of those complexes $P_\bullet$ whose homology groups $H_n(P) = \frac{\ker(P_n \to P_{n-1})}{\mathrm{im}(P_{n+1}\to P_n)}$ are torsion.

$$D^b(\mathbb{Z})/D^b(\mathbb{Z})_{\mathrm{tor}} \cong D^b(\mathbb{Q}).$$

**Example 4.5.9.** Let $X \in \mathcal{Q}\mathrm{Proj}$. The coherent $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ induce a sequence of stable sub-$\infty$-categories

$$0 = C_{-1} \subset C_0 \subset \cdots \subset C_n = D^b(\mathbb{P}^n \times X).$$

Namely, $C_i$ is generated by $\mathcal{O}, \dots, \mathcal{O}(i)$. We have equivalences

$$C_i/C_{i-1} \cong D^b(X).$$

via the functors

$$\begin{aligned} D^b(X) &\to D^b(\mathbb{P}^n \times X) \\ F &\mapsto \mathcal{O}(i) \otimes p^*F \end{aligned}$$

## 4.6 Localising invariants

The original reference for the characterisation of algebraic $K$-theory as the universal localising invariant is [Antieau–Gepner–Tabuada]. For a course about this see [Hebestreit–Wagner, Algebraic and Hermitian K-Theory, Winter Term 2020/21, University of Bonn]

Write

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$$

for the $\infty$-category whose objects are stably $\infty$-categories and morphisms are exact functors. That is, functors which preserve finite limits and finite colimits.

**Definition 4.6.1** (Idempotent completion)**.** A stable $\infty$-category $C$ is *idempotent complete* if every idempotent endomorphism $p : X \to X$ (i.e., $p \circ p \simeq p$) admits a splitting. That is, there exists an isomorphism

$$X \cong Y \oplus Z$$

such that $p$ is identified with the composition $X \to Y \to X$. We write $\mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}} \subseteq \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ for the full subcategory of idempotent complete stable $\infty$-categories. The inclusion admits a left adjoint

$$(-)^\natural : \mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}} \to \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}}$$

called idempotent completion.

**Proposition 4.6.2.** *A square*

$$\begin{array}{ccc} C & \xrightarrow{i} & D \\ \downarrow & & \downarrow{\scriptstyle p} \\ 0 & \longrightarrow & E \end{array}$$

*of small stable $\infty$-categories is a bifibre square in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{ex}}$ if it is a cartesian square in $\mathcal{C}\mathrm{at}_\infty$ and $E$ is (equivalent to) the Verdier localisation $D/C$ of $D$ along $C$. Such a square is a fibre square in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}}$ if $E \cong (D/C)^\natural$.*

**Definition 4.6.3.** A functor $F : \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}} \to \mathcal{S}\mathrm{pt}$ is called a *localising invariant* if it sends bifibre squares to bifibre squares.

## 4.7 $K$-theory

**Theorem 4.7.1.** *The $\infty$-category of localising functors $F : \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}} \to \mathcal{S}\mathrm{pt}$ equipped with a natural transformation*

$$(-)^{\cong} \to \Omega^\infty F$$

*admits an initial object $K$.*

**Remark 4.7.2.** That is, $K$ is the localising invariant "as close as possible" to the core functor which sends $C \in \mathcal{C}\mathrm{at}_\infty^{\mathrm{perf}}$ to its associated $\infty$-groupoid $C^{\cong}$. Note that in a precise mathematical sense, $D^b(R)$ is the closest stable $\infty$-category to $\mathrm{Proj}(R)$, [Lurie, Higher Algebra, Thm.1.3.3.2, Thm.1.3.3.8].

**Theorem 4.7.3.** *For $n \leq 1$ and $R \in \mathcal{R}\mathrm{ing}$, the homotopy groups*

$$\pi_n K(D^b(R))$$

*are the groups $K_n(R)$ we saw last time.*

**Example 4.7.4.** Using the decomposition associated to $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$, we have

$$K(D^b(\mathbb{P}^n_R)) \cong \bigoplus_{i=0}^{n} K(R).$$

**Example 4.7.5.** For a finite field $\mathbb{F}_q$ with $q$ elements, we have:

$$K_n(\mathbb{F}_q) = 0 \quad \text{for } n \leq -1 \tag{2}$$
$$K_0(\mathbb{F}_q) \cong \mathbb{Z} \tag{3}$$
$$K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z} \tag{4}$$
$$K_{2i}(\mathbb{F}_q) = 0 \quad \text{for } i \geq 1 \tag{5}$$
$$K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1)\mathbb{Z} \quad \text{for } i \geq 1 \tag{6}$$

The pattern in positive degrees follows from Quillen's computation, while negative $K$-groups vanish since $\mathbb{F}_q$ has finite global dimension.

**Example 4.7.6.** For the field of rational numbers $\mathbb{Q}$:

$$K_n(\mathbb{Q}) = 0 \quad \text{for } n \leq -1 \tag{7}$$
$$K_0(\mathbb{Q}) \cong \mathbb{Z} \tag{8}$$
$$K_1(\mathbb{Q}) \cong \mathbb{Q}^\times \tag{9}$$
$$K_2(\mathbb{Q}) \cong (\mathbb{Z}/4)^* \times \prod_{p \text{ odd prime}} (\mathbb{Z}/p)^* \tag{10}$$

The computation of $K_2(\mathbb{Q})$ is due to Tate and follows from Gauss's first proof of quadratic reciprocity. For higher $K$-groups, Borel proved that (modulo torsion):

$$K_{4k+1}(\mathbb{Z})/\text{tors} = \mathbb{Z} \quad \text{for } k > 0$$
$$K_i(\mathbb{Z})/\text{tors} = 0 \quad \text{for } i > 2; i \neq 4k+1$$

**Example 4.7.7** (Matsumoto's theorem)**.** For any field $k$, Matsumoto's theorem states that the second $K$-group is given by

$$K_2(k) = \frac{k^\times \otimes_{\mathbb{Z}} k^\times}{\langle a \otimes (1-a) \mid a \neq 0, 1 \rangle}.$$

The relations $a \otimes (1-a) = 0$ are called the *Steinberg relations.*