3 $K_{\leq 1}$

Contents

Recall that if $X \subseteq \mathbb{A}^n$ is an affine variety, then all information (except the embedding into $\mathbb{A}^n$) is contained in the ring $\mathcal{O}_X(X)$. That is, up to isomorphism, we can reconstruct the variety X from the ring $\mathcal{O}_X(X)$. More precisely, we have an equivalence of categories:

$$\{\text{affine } k\text{-varieties}\} \simeq \{\text{finitely generated } k\text{-algebras}\}^{op}$$

In this lecture I want to work with the larger category of *affine schemes*. This is equivalent to, and sometimes defined as, the opposite of the category $\mathcal{R}\text{ing}$ of commutative rings with unit. That is, in this lecture we will work with rings. If I want to think of a ring as a geometric object I will write $\text{Spec}(R)$, but in this lecture you should just think of this as notation. I don't want to talk about locally ringed topological spaces.

$$\{\text{affine schemes}\} \cong \mathcal{R}\text{ing}^{op}$$
$$\text{Spec}(R) \leftrightarrow R$$

3.1 K_0

Last time we considered $G_0(X)$. For a ring R, this is defined as:

$$G_0(R) = \frac{\mathbb{Z}\left[\begin{array}{c}\text{finitely generated}\\ R\text{-modules}\end{array}\right]}{\left\langle [M]-[L]-[N] \;\middle|\; \begin{array}{c}0 \to L \to M \to N \to 0\\ \text{is exact}\end{array}\right\rangle}$$

The group G_0 is good for many things, but not everything.

Example 3.1.1 (Limitations of G_0)**.**

1. G_0 **doesn't detect nilpotent elements**: Consider $R = k[x]/(x^2)$ where k is a field. Every $M \in \mathcal{C}\text{oh}(R)$ is a finite direct sum of copies of

$$M = k[x]/(x^2) = R \quad \text{or} \quad M' = R/(x) \cong k.$$

We have an exact sequence $0 \to k \xrightarrow{x} R \to k \to 0$, so in $G_0(R)$ we get $[R] = [k] + [k] = 2[k]$. Thus $G_0(R) \cong \mathbb{Z} \cong G_0(k)$. More generally, for a Noetherian ring R with nilradical $\operatorname{Nil}(R)$, we have

$$G_0(R) \cong G_0(R/\operatorname{Nil}(R)).$$

We will see below that for local rings R we have $K_1(R) = R^*$. In particular, K_1 can see nilpotents.

2. G_0 **cannot see certain singularities.** For example, consider the cusp $X = V(y^2 - x^3) \subseteq \mathbb{A}^2$ and the affine line $Y = \mathbb{A}^1$. Both have $G_0(X) \cong G_0(Y) \cong \mathbb{Z}$, even though X has a cusp singularity while Y is smooth. However, for integral Noetherian rings of (Krull) dimension one, we have $K_0(R) \cong \mathbb{Z} \oplus \operatorname{Pic}(R)$, [Weibel II.2.6.3]. Since $\operatorname{Pic}(X) \neq 0$ while $\operatorname{Pic}(\mathbb{A}^1) = 0$, we get $K_0(X) \neq K_0(\mathbb{A}^1)$, so K_0 (defined below) can distinguish these cases where G_0 cannot.

3. **Functoriality**: The functor G_0 has functoriality of a cohomology *with compact support* rather than a "cohomology theory". More precisely, G_0 is covariant $G_0(X) \to G_0(Y)$ for projective morphisms $X \to Y$ of varieties, and contravariant $G_0(X) \to G_0(Y)$ for flat morphisms $Y \to X$, but it is not contravariant for all morphisms.

4. **No ring structure in general**: The semiring structure on $\mathcal{C}\mathrm{oh}(X)_{/\cong}$ coming from $\otimes$ does not descend in general to a ring structure on $G_0(X)$.

In this lecture instead of all coherent sheaves we will focus on vector bundles.

Definition 3.1.2. A *vector bundle* on a variety X is a coherent sheaf $\mathcal{E}$ that is locally of constant rank, meaning that for every point $x \in X$, there exists a basic open $U \ni x$ such that $\mathcal{E}|_U \cong \mathcal{O}_U^{\oplus r}$ for some integer $r \geq 0$.

Example 3.1.3 (Examples of vector bundles)**.**

1. The structure sheaf $\mathcal{O}_X$ is a vector bundle of rank 1.

2. The $\mathcal{O}(D)$ (for $D \in \mathcal{Z}_{d-1}(X)$ on a smooth irreducible X of dimension d) are vector bundles of rank 1.

3. The $\mathcal{O}(d)$ in $\mathcal{C}\mathrm{oh}(\mathbb{P}^n)$ are vector bundles of rank 1.

4. A variety X is smooth of dimension d if and only if Ω_X is a vector bundle of rank d.

5. If E and F are vector bundles, then $E \oplus F$ is a vector bundle.

6. If $E \oplus F \cong G$ where G is a vector bundle, then both E and F are vector bundles.

Recall that for affine varieties (and more generally, for affine schemes) the category of $\mathcal{C}\mathrm{oh}(X)$ is equivalent to the category of finitely generated $\mathcal{O}_X(X)$-modules. We can also identify the subcategory of vector bundles.

Proposition 3.1.4. *Let X be an affine variety and $R = \mathcal{O}_X(X)$. Then we have an equivalence of categories:*

$$\left\{ \begin{array}{c} \textit{vector bundles} \\ \textit{on } X \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{finitely generated} \\ \textit{projective } R\textit{-modules} \end{array} \right\}$$

Note that a module P is finitely generated and projective *if and only if there exists some Q and an isomorphism $P \oplus Q \cong R^{\oplus n}$.*

Algebraic K-theory of a ring R is then defined as follows.

Definition 3.1.5 (K_0)**.** For a ring R, we define $K_0(R)$ as:

$$K_0(R) = \frac{\mathbb{Z}\left[\begin{array}{c} \text{finitely generated} \\ \text{projective } R\text{-modules} \end{array} \right]}{\left\langle [P] - [N] - [Q] : \begin{array}{c} 0 \to N \to P \to Q \to 0 \text{ exact} \\ \text{with } N, P, Q \text{ projective} \end{array} \right\rangle}$$

Remark 3.1.6. Since surjections $P \twoheadrightarrow Q$ towards projective modules Q have sections $P \hookleftarrow Q$, for sequences as above we have $P \cong N \oplus Q$ and so $K_0(R)$ can also be defined as:

$$K_0(R) = \frac{\mathbb{Z}\left[\begin{array}{c} \text{finitely generated} \\ \text{projective } R\text{-modules} \end{array} \right]}{\langle [P \oplus Q] - [P] - [Q] \rangle}$$

This description shows that $K_0(R)$ is the group completion of the abelian monoid $(\mathrm{Proj}(R)_{/\cong}, \oplus)$ of isomorphism classes of projective R-modules. That is, the map $(\mathrm{Proj}(R)_{/\cong}, \oplus) \to K_0(R)$ is the unique homomorphism of abelian monoids such that for every abelian group A,

$$\hom_{\mathcal{A}\mathrm{b}}(K_0(R), A) \xrightarrow{\sim} \hom_{\mathrm{CommMon}}(\mathrm{Proj}(R)_{/\cong}, A)$$

Definition 3.1.7 (Regular ring)**.** A Noetherian ring R is called *regular* if every finitely generated R-module admits a finite resolution by finitely generated projective R-modules. That is, for every $M \in \mathcal{C}\mathrm{oh}(R)$ there exists an exact sequence

$$0 \to P_0 \to P_1 \to \cdots \to P_n \to M \to 0$$

with each $P_i \in \mathrm{Proj}(R)$.

Remark 3.1.8. Usually regularity is defined in terms of regular sequences. The equivalence to the above definition is an actual theorem requiring substantial commutative algebra, [Stacks Project, Tag 00O7].

Corollary 3.1.9. *For regular Noetherian rings R, e.g., $R = \mathcal{O}_X(X)$ when X is a smooth affine variety, we have*

$$G_0(R) = K_0(R).$$

Remark 3.1.10 (Sketch of proof)**.** By induction, we see that for resolutions as in the above definition, we have $[M] = \sum_{i=0}^{n}(-1)^i[P_i]$ in $G_0(R)$. In particular, $K_0(R) \to G_0(R)$ is surjective. Similarly, any relation in $G_0(R)$ can be replaced by a relation in $K_0(R)$ only involving projective modules.

3.2 K_1

Definition 3.2.1 (Milnor square, [[Weibel, Exam.I.2.6])**.** A *Milnor square* is a pullback square of surjections

$$\begin{array}{ccc} R & \xrightarrow{p} & R' \\ \downarrow & & \downarrow \\ S & \xrightarrow[q]{} & S' \end{array}$$

Remark 3.2.2.

1. Explicitly, we are asking that $R = \ker(S \oplus R' \to S')$.
2. Often in the definition of Milnor squares there is the condition that $\ker(p) \cong \ker(q)$, but this is automatic from the above formulation.

Remark 3.2.3. Surjections of rings correspond to closed immersions of schemes, and pullbacks of rings correspond to pushouts of schemes. That is $R = S \times_{S'} R'$ means $\mathrm{Spec}(R) = \mathrm{Spec}(S) \sqcup_{\mathrm{Spec}(S')} \mathrm{Spec}(R')$.

Theorem 3.2.4 ([Weibel, Thm.II.2.9])**.** *Suppose we have a Milnor square as above. Then there is a long exact sequence*

$$GL_\infty(S') \to K_0(R) \to K_0(S) \oplus K_0(R') \to K_0(S')$$

where $GL_\infty(S') = \varinjlim(GL_1(S') \to GL_2(S') \to GL_3(S') \to \dots)$.

Remark 3.2.5. In fact, one might expect that such a sequence exists because the category $\mathrm{Proj}(R)$ is equivalent to a category whose objects are triples (P, Q, ϕ) consisting of an S-module P, an R'-module Q and an isomorphism $\phi : P \otimes_S S' \cong Q \otimes_{R'} S'$, [Weibel, Theorem I.2.7].

The failure of injectivity suggests that we need to keep track of more information than just isomorphism classes. Automorphisms seem to be important.

Observation 3.2.6.

1. $K_0(R)$ is the group completion of the monoid $(\mathrm{Proj}(R)_{/\cong}, \oplus)$.

2. Automorphisms seem to be important (Theorem 3.2.4).

Instead of working with isomorphism classes $\mathrm{Proj}(R)_{/\cong}$, let's consider the *groupoid*

$$\mathrm{Proj}(R)^{\cong}.$$

This is the category whose objects are finitely generated projective modules and whose morphisms are isomorphisms.

The groupoid $\mathrm{Proj}(R)^{\cong}$ has a symmetric monoidal structure

$$\mathrm{Proj}(R)^{\cong} \times \mathrm{Proj}(R)^{\cong} \to \mathrm{Proj}(R)^{\cong}$$

given by direct sum $\oplus$ and the isomorphisms $P \oplus Q \cong Q \oplus P$. We want to form its group completion. That is, a universal functor

$$\mathrm{Proj}(R)^{\cong} \to \mathcal{G}$$

towards a *grouplike* symmetric monoidal groupoid. That is, a symmetric monoidal groupoid such that for every object X the functor $X \oplus -$ is an equivalence. *Universal* means that for any grouplike symmetric monoidal groupoid $\mathcal{G}'$ it should induce an equivalence of groupoids

$$\mathrm{Fun}(\mathcal{G}, \mathcal{G}') \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Proj}(R)^{\cong}, \mathcal{G}')$$

where Fun is the groupoid of monoidal functors.

Observation 3.2.7. Suppose that $\Phi : \mathrm{Proj}(R)^{\cong} \to \mathcal{G}$ is a functor towards a grouplike symmetric monoidal groupoid. Use $\oplus$ for operations on both groupoids and $\mathbb{O}$ for the unit object. So $- \oplus \mathbb{O}$ is isomorphic to the identity functor.

1. Since $X \oplus - : \mathcal{G} \to \mathcal{G}$ is an equivalence for any X, we have

$$\mathrm{Aut}_{\mathcal{G}}(\mathbb{O}) \cong \mathrm{Aut}_{\mathcal{G}}(X)$$

for all objects X.

2. For each n, we have a group homomorphism $\mathrm{GL}_n(R) = \mathrm{Aut}(R^{\oplus n}) \to \mathrm{Aut}_{\mathcal{G}}(\Phi(R^{\oplus n})) \cong \mathrm{Aut}_{\mathcal{G}}(\mathbb{O})$ compatible with inclusions $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$. This gives a map

$$\mathrm{GL}_\infty(R) \to \mathrm{Aut}_{\mathcal{G}}(\mathbb{O}).$$

3. By the Eckmann-Hilton argument, since composition $\circ$ and direct sum $\oplus$ both give operations on the set $\mathrm{Aut}(\mathbb{O})$, and they distribute over each other:

$$(\alpha \oplus \beta) \circ (\gamma \oplus \delta) = (\alpha \circ \gamma) \oplus (\beta \circ \delta)$$

the group $\mathrm{Aut}(\mathbb{O})$ is abelian. Therefore we get a group homomorphism

$$\frac{\mathrm{GL}_\infty(R)}{[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]} \to \mathrm{Aut}_{\mathcal{G}}(\mathbb{O})$$

Here we write $[G, G] = \langle ghg^{-1}h^{-1} \rangle$ for the commutator subgroup of a group G. So $G \to G/[G, G]$ is the largest abelien quotient of G.

Definition 3.2.8 (K_1)**.** For a ring R, we define:

$$K_1(R) = \frac{\mathrm{GL}_\infty(R)}{[\mathrm{GL}_\infty(R), \mathrm{GL}_\infty(R)]}.$$

Here is some evidence that this is a good definition.

Theorem 3.2.9 ([Weibel, Thm.III.2.6])**.** *Suppose we have a Milnor square as above. Then there is a long exact sequence*

$$K_1(R) \longrightarrow K_1(S) \oplus K_1(R') \longrightarrow K_1(S') \longrightarrow K_0(R) \longrightarrow K_0(S) \oplus K_0(R') \longrightarrow K_0(S')$$

3.3 $K_{<0}$

Now we move on to negative K-theory. One motivation for negative K-theory comes from trying to extend exact sequence to the right.

Here is our (admittedly weak) hint as to what a good definition might be.

Theorem 3.3.1 (Fundamental Theorem for K_1, [Weibel, III.3.6])**.** *For every ring R, there is an exact sequence*

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t, t^{-1}]) \to K_0(R) \to 0.$$

Definition 3.3.2 (Negative K-theory)**.** For a ring R and $n > 0$, we inductively define $K_{-n}(R)$ to be the cokernel

$$K_{-n}(R) := \mathrm{coker}\Big(K_{-n+1}(R[t]) \oplus K_{-n+1}(R[t^{-1}]) \to K_{-n+1}(R[t, t^{-1}]) \Big).$$

Theorem 3.3.3 ([Weibel, III.4.3])**.** *Suppose we are given a Milnor square as above. Then the sequence of Theorem 3.2.9 continues as :*

$$\dots \longrightarrow K_0(R) \longrightarrow K_0(S) \oplus K_0(R') \longrightarrow K_0(S') \longrightarrow K_{-1}(R) \longrightarrow K_{-1}(S) \oplus K_{-1}(R') \longrightarrow K_{-1}(S') \longrightarrow \dots \longrightarrow K_{-n}(R) \longrightarrow \dots$$