2 Grothendieck–Riemann-Roch

Contents

2.1 Statement

Everything in this lecture is over an algebraically closed field $k = \overline{k}$ (e.g., $\mathbb{C}$, $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{F}}_p$, $\cup_{n\in\mathbb{N}}\mathbb{C}((t^{1/n}))$, ...).

Theorem 2.1 (Grothendieck–Riemann–Roch)**.** *Suppose X is a smooth quasi-projective variety. Then the Chern character induces an isomorphism*

$$\mathrm{ch} : G_0(X)_{\mathbb{Q}} \cong A_*(X)_{\mathbb{Q}}.$$

Moreover, if $X \to Y$ is a projective morphism between smooth quasi-projective varieties, we have

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathrm{td}\, T_X)).$$

Remark 2.2. When X is a smooth projective curve and $Y = \mathbb{A}^0$, this recovers the classical Riemann–Roch theorem from Lecture 1.

2.2 Morphisms of varieties

Recall that last time we define affine varieties $X \subseteq k^n$, projective varieties $X \subseteq \mathbb{P}^n = \frac{k^{n+1}\setminus\{0\}}{k^*}$, and basic opens $U \subseteq X \subseteq \mathbb{C}^n$. We also considered the rings

$$\mathcal{O}_X(U) = \{\phi : U \to k \mid \phi = f/g^n, \text{ for some } f \in k[x_1, \ldots, x_n], n \in \mathbb{N}\}$$

where $U = D(g) = \{x \in X \mid g(x) \neq 0\}$.

Definition 1. A *morphism* of basic opens $U \subseteq X \subseteq \mathbb{A}^n$, $V \subseteq Y \subseteq \mathbb{A}^m$ is a sequence $(\phi_1, \ldots, \phi_m) \in \mathcal{O}_X(U)^m$ such that the corresponding morphism $U \to k^m$ factors through $V \subseteq k^m$.

Example 2.3.

1. Any inclusion of basic opens is a morphism.
2. If $D(g) \subseteq V(f_1, \ldots, f_c) \subseteq \mathbb{A}^n$, then the canonical bijection

$$\underset{\subseteq \mathbb{A}^{n+1}}{V(f_1, \ldots, f_c, yg-1)} \to \underset{\subseteq V(f_1,\ldots,f_c) \subseteq \mathbb{A}^n}{D(g)}$$

is a morphism of basic opens. It has inverse given by $(x_1, x_2, \ldots, x_n, \frac{1}{g}) : D(g) \to k^{n+1}$. That is, in the (big) category of basic opens, we have

$$V(f_1, \ldots, f_c, yg-1) \cong D(g).$$

3. A composition of morphisms of basic opens is a morphism of basic opens. So we have a "big" category of basic opens. We don't need a notation for this because we won't often use it.

Remark 2.4. A morphism of basic opens $\underset{\subseteq X}{U} \to \underset{\subseteq Y}{V}$ induces a ring homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$. In particular, every point $\mathbb{A}^0 \to U$ induces a (surjective) ring homomorphism $\mathcal{O}_X(U) \to k$. In fact, *every* surjection $\mathcal{O}_X(U) \to k$ comes from a point. That is, there is a bijection

$$\hom_{\mathcal{A}\mathrm{lg}_k}(\mathcal{O}_X(U), k) \cong \hom(\mathbb{A}^0, U) \cong U.$$

This holds more generally,

$$\hom_{\mathcal{A}\mathrm{lg}_k}(\mathcal{O}_X(U), \mathcal{O}_Y(V)) \cong \hom(V, U).$$

Definition 2. A *quasi-projective variety* or just *variety* is a union of basic opens in some projective variety $\overline{X}$.

$$X = \cup_{\lambda \in \Lambda} U_\lambda \subseteq \overline{X}$$

We continue to write $\mathcal{B}(X)$ for the category of *all* basic opens (of $\overline{X}$) contained in X.

Example 2.5. The set $\mathbb{A}^2 \setminus \{0\}$ from last lecture is not (isomorphic to) a basic open, nor a projective variety, but it is a quasi-projective variety. Similarly, $\mathbb{P}^n \setminus \{(0{:}0{:}\ldots{:}0{:}1)\}$ is a quasi-projective variety which is neither affine, nor projective.

Remark 2.6. One should think of quasi-projective varieties as being covered by basic opens in the same way that a smooth manifold is covered by opens that are homeomorphic to an open in $\mathbb{R}^n$.

Definition 3. A *morphism* of quasi-projective varieties is a function

$$f : X \to Y$$

such that for every $x \in X$ there exists a commutative diagram

$$\begin{array}{ccc} x \in & U & \longrightarrow & V \\ & \cap\!| & & \cap\!| \\ & X & \longrightarrow & Y \end{array}$$

such that U, V are basic opens and $U \to V$ is a morphism of basic opens. The category of quasi-projective varieties will be denoted $\mathcal{Q}$Proj.

In other words, a morphism of quasi-projective varieties is a morphism defined by quotients of polynomials.

Example 2.7.

1. For $X \in \mathcal{Q}$Proj and $\{U_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{B}(X)$ then $U := \cup_\Lambda U_\lambda \in \mathcal{Q}$Proj and the inclusion $U \to X$ is a morphism. In this case U is called an *open subvariety* of X.

2. In the previous notation, we also have $Z = X \setminus U \in \mathcal{Q}$Proj and $Z \to X$ is a morphism. In this case Z is called a *closed subvariety* of X.

3. For $X, Y \in \mathcal{Q}$Proj, the product $X \times Y$ has a canonical structure of quasi-projective variety (via the Segre embedding). The two projections $X \leftarrow X \times Y \to Y$ are morphisms.

2.3 Quasi-coherent $\mathcal{O}_X$-modules

Now we have a nice category of quasi-projective varieties. We are going to fix a quasi-projective variety X and study certain families of vector spaces parameterised by X.

Definition 4 (Quasi-coherent $\mathcal{O}_X$-module)**.** A *quasi-coherent $\mathcal{O}_X$-module* on a quasi-projective variety X is a functor $F : \mathcal{B}(X)^{op} \to \mathcal{A}$b such that:

1. Each $F(U)$ is an $\mathcal{O}_X(U)$-module

2. Each restriction map $F(U) \to F(V)$ (for $V \subseteq U$) is a morphism of $\mathcal{O}_X(U)$-modules

3. For every inclusion $V \subseteq U$ of basic opens, the natural map

$$F(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to F(V) \tag{$*$}$$

is an isomorphism

A *morphism* of quasi-coherent $\mathcal{O}_X$-modules is a natural transformation $\phi : F \to G$ such that each component $\phi_U : F(U) \to G(U)$ is a morphism of $\mathcal{O}_X(U)$-modules. If each $F(U)$ is a *finitely generated* $\mathcal{O}_X(U)$-module, then we say that F is *coherent*.

Write $\mathcal{Q}\mathrm{Coh}(X)$ and $\mathcal{C}\mathrm{oh}(X)$ for the categories of quasi-coherent and coherent $\mathcal{O}_X$-modules.

Remark 2.8. One can check that if $U = U_0 \cup U_1$ with $U, U_0, U_1 \in \mathcal{B}(X)$ then for any quasi-coherent $\mathcal{O}_X$-module F we have $F(U) = F(U_0) \times_{F(U_0 \cap U_1)} F(U_1)$.[1] Consequently, there is a unique sheaf F' on the X (considered as a topological space via open subvarieties) such that $F'|_{\mathcal{B}(X)} = F$. However, I don't want to talk about sheaves in this series of lectures.

Remark 2.9. For every point $x \in U$ and $F \in \mathcal{Q}\mathrm{Coh}(X)$ we get an associated k-vector space

$$F_x := F(U) \otimes_{\mathcal{O}_X(U)} k$$

where $\mathcal{O}_X(U) \to k$ is the homomorphism associated to $x \to U$. The condition $(*)$ ensures that this is independent of U. In this way you can/should think of F as a family of vector spaces parameterised by X, at least if F is coherent.

Example 2.10 (Examples in $\mathcal{Q}\mathrm{Coh}(X)$)**.**

1. The functor $\mathcal{O}_X$, and more generally the $\mathcal{O}(D)$ (for $D \in \mathrm{Div}(X)$) are in $\mathcal{C}\mathrm{oh}(X)$.
2. The functor $\mathcal{K}_X : U \mapsto \{\begin{smallmatrix} K_X & U \neq \varnothing \\ 0 & U = \varnothing \end{smallmatrix}$ is in $\mathcal{Q}\mathrm{Coh}(X)$ but not in $\mathcal{C}\mathrm{oh}(X)$ in general.
3. On projective space $\mathbb{P}^n$, the $\mathcal{O}(d)$ for $d \in \mathbb{Z}$ are in $\mathcal{C}\mathrm{oh}(\mathbb{P}^n)$. These are defined via the canonical projection $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ as follows: for basic opens $U \subseteq \mathbb{P}^n$, we have

$$\mathcal{O}(d)(U) = \left\{ \phi : \pi^{-1}(U) \to k \;\middle|\; \begin{array}{c} \phi(\lambda x) = \lambda^d \phi(x) \\ \text{for all } \lambda \in k^*, x \in \pi^{-1}(U) \end{array} \right\}$$

4. Direct sums and products: If $\{F_\lambda\}_{\lambda \in \Lambda}$ is a family in $\mathcal{Q}\mathrm{Coh}(X)$, then $\bigoplus_{\lambda \in \Lambda} F_\lambda$ and $\prod_{\lambda \in \Lambda} F_\lambda$ are in $\mathcal{Q}\mathrm{Coh}(X)$ where $(\bigoplus_{\lambda \in \Lambda} F_\lambda)(U) = \bigoplus_{\lambda \in \Lambda} F_\lambda(U)$ and $(\prod_{\lambda \in \Lambda} F_\lambda)(U) = \prod_{\lambda \in \Lambda} F_\lambda(U)$.

[1] Basically, if $U_0 = D(f)$ and $U_1 = D(g)$ then $U = U_0 \cup U_1$ implies that there are $a, b \in \mathcal{O}_X(U)$ with $1 = af + bg$ in $\mathcal{O}_X(U)$. The claim $F(U) = F(U_0) \times_{F(U_0 \cap U_1)} F(U_1)$ follows from $1 = af + bg$ and the condition $(*)$.

5. Kernels and cokernels: if $\phi : F \to G$ is a morphism in $\mathcal{Q}\mathrm{Coh}(X)$, then $\ker(\phi), \mathrm{coker}(\phi) \in \mathcal{Q}\mathrm{Coh}(X)$ where $(\ker(\phi))(U) = \ker(\phi_U)$ and $(\mathrm{coker}(\phi))(U) = \mathrm{coker}(\phi_U)$.

6. Tensor products and Homs: If $F, G \in \mathcal{Q}\mathrm{Coh}(X)$, then $F \otimes_{\mathcal{O}_X} G \in \mathcal{Q}\mathrm{Coh}(X)$ where $(F \otimes_{\mathcal{O}_X} G)(U) = F(U) \otimes_{\mathcal{O}_X(U)} G(U)$. If $F, G \in \mathcal{Q}\mathrm{Coh}(X)$, then $\mathcal{H}om(F, G) \in \mathcal{Q}\mathrm{Coh}(X)$ where $\mathcal{H}om(F, G)(U) = \hom_{\mathcal{Q}\mathrm{Coh}(U)}(F|_U, G|_U)$.

7. For any closed subvariety $Z \subseteq X$, the ideal sheaf $\mathcal{I}_Z$ defined by $U \mapsto \{f \in \mathcal{O}_X(U) : f|_{Z\cap U} = 0\}$ is in $\mathcal{C}\mathrm{oh}(X)$.

The following proposition follows easily from the definitions.

Proposition 2.11. *Let U be a basic open (hence isomorphic to an affine). Then we have equivalences of categories:*

$$\{\ \mathcal{O}_U(U)\text{-}\mathit{modules}\ \} \cong \mathcal{Q}\mathrm{Coh}(U)$$
$$\left\{ \begin{matrix} \textit{finitely generated} \\ \mathcal{O}_U(U)\text{-}\mathit{modules} \end{matrix} \right\} \cong \mathcal{C}\mathrm{oh}(U)$$

The equivalences are given by:

$$M \mapsto (V \mapsto M \otimes_{\mathcal{O}_U(U)} \mathcal{O}_U(V))$$
$$F(U) \leftarrow\!\shortmid F$$

Definition 5 (Grothendieck group G_0)**.** Let $X \in \mathcal{Q}\mathrm{Proj}$. The *Grothendieck group*

$$G_0(X) = \frac{\mathbb{Z}[\text{iso. classes of } F \in \mathcal{C}\mathrm{oh}(X)]}{\langle [F] = [F'] + [F''] \mid 0 \to F' \to F \to F'' \to 0 \rangle}$$

is the abelian group generated by symbols $[F]$ for $F \in \mathcal{C}\mathrm{oh}(X)$, subject to the relation $[F] = [F'] + [F'']$ whenever there exists a short exact sequence $0 \to F' \xrightarrow{i} F \xrightarrow{p} F'' \to 0$ in $\mathcal{C}\mathrm{oh}(X)$. Here *exact* means that $F' = \ker(i)$ and $F'' = \mathrm{coker}(i)$.

Example 2.12 (Examples of Grothendieck groups)**.**

1. **Point:** $G_0(\mathbb{A}^0) \cong \mathbb{Z}$, since $\mathcal{C}\mathrm{oh}(\mathbb{A}^0)$ is equivalent to the category of finite dimensional k-vector spaces.

2. **Affine space:** $G_0(\mathbb{A}^n) \cong \mathbb{Z}$. Since $k[x_1, \ldots, x_n]$ has finite global dimension, every $F \in \mathcal{C}\mathrm{oh}(\mathbb{A}^n)$ has a finite free resolution. That is, a sequence of morphisms
$$0 \to \mathcal{O}_{\mathbb{A}^n}^{\oplus r_m} \xrightarrow{d_m} \cdots \xrightarrow{d_2} \mathcal{O}_{\mathbb{A}^n}^{\oplus r_1} \xrightarrow{d_1} \mathcal{O}_{\mathbb{A}^n}^{\oplus r_0} \xrightarrow{d_0} F \to 0,$$
for some r_i such that $\ker(d_i) = \mathrm{im}(d_{i+1})$ for all i. By induction, it follows that $[F] = \sum_i (-1)^i [\mathcal{O}_{\mathbb{A}^n}^{\oplus r_i}] = \left(\sum_i (-1)^i r_i\right) [\mathcal{O}_{\mathbb{A}^n}]$.

3. **Closed-open decomposition:** If $U \subseteq X$ is open and $Z = X \setminus U$ then there is an exact sequence

$$G_0(Z) \to G_0(X) \to G_0(U) \to 0.$$

This sequence is exact on the left if $Z \subseteq X$ is a *regular embedding.*[2]

4. **Projective space:** $G_0(\mathbb{P}^n) \cong \mathbb{Z}^{\oplus n+1}$ with generators $[\mathcal{O}], [\mathcal{O}(1)], \ldots, [\mathcal{O}(n)]$. More generally, if X is a smooth variety then

$$G_0(\mathbb{P}^n \times X) \overset{\cong}{\leftarrow} \oplus_{i=0}^n G_0(X)$$
$$\sum_{i=0}^n [E_i \otimes \mathcal{O}(i)] \leftarrow\!\shortmid ([E_0], \ldots, [E_n])$$

5. **Grassmannian:**

$$G_0(Gr(2,4)) \cong \mathbb{Z}^{\oplus 6}.$$

This comes from the decomposition $G_0(Gr(2,4)) \cong \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus (\mathbb{A}_2 \oplus \mathbb{A}_2) \oplus \mathbb{A}_3 \oplus \mathbb{A}_4$ determined by a choice of *flag.*[3]

6. **Elliptic curve:** For an elliptic curve E, we have $G_0(E) \cong \mathbb{Z} \oplus \mathrm{Pic}(E)$ where $\mathbb{Z} \cong \{n[\mathcal{O}]\}$ and $\mathrm{Pic}(E) \cong \{[\mathcal{O}(D)] - [\mathcal{O}]\}$. There is an explicit bijection

$$\mathbb{Z} \oplus E \xrightarrow{\sim} \mathrm{Pic}(E)$$
$$(n, x) \mapsto \mathcal{O}(x + (n-1)x_0)$$

for some fixed point x_0.

7. **Smooth curves:** More generally, for a smooth projective curve C we have $G_0(C) \cong \mathbb{Z} \oplus \mathrm{Pic}(C)$. The subgroup $\mathrm{Pic}^0(C) = \{\mathcal{O}(D) \mid \deg D = 0\}$ has a canonical structure of smooth projective variety of dimension g = the genus of C.

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[2]If X is an affine variety then $Z \subseteq X$ is globally a regular embedding if there exists $f_1, \ldots, f_c \in \mathcal{O}_X(X)$ such that $Z = V(f_1, \ldots, f_c)$ and each f_{i+1} is a nonzero divisor in $\mathcal{O}_X(X)/\langle f_1, \ldots, f_i \rangle$. In general, $Z \subseteq X$ is a regular embedding if $Z \cap V \to V$ is globally a regular embedding for every basic open $V \subseteq X$.

[3]A *flag* is a sequence of subspaces $\{0\} = V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = V$ with $\dim V_i = i$. In the case of $Gr(2,4)$ we have $d = 4$ and:

(a) $\mathbb{A}_0$ is $\{V_2\}$,

(b) $\mathbb{A}_0 \cup \mathbb{A}_1 =$ is the set of planes W with $V_1 \subset W \subset V_3$,

(c) One $\mathbb{A}_0 \cup \mathbb{A}_1 \cup \mathbb{A}_2$ is $\{W \mid V_1 \subset W\}$,

(d) The other $\mathbb{A}_0 \cup \mathbb{A}_1 \cup \mathbb{A}_2$ is $\{W \mid W \subset V_3\}$,

(e) $\mathbb{A}_0 \cup \mathbb{A}_1 \cup (\mathbb{A}_2 \cup \mathbb{A}_2) \cup \mathbb{A}_3 = \{W \mid W \cap V_2 \neq \{0\}\}$,

(f) $\mathbb{A}_0 \cup \mathbb{A}_1 \cup (\mathbb{A}_2 \cup \mathbb{A}_2) \cup \mathbb{A}_3 \cup \mathbb{A}_4 = Gr(2,4)$.

2.4 Pushforward

Definition 6. Suppose that $f : X \to Y$ is in $\mathcal{Q}$Proj, $F \in \mathcal{Q}\mathrm{Coh}(X)$. We define f_*F via

$$(f_*F)(V) = \varprojlim_{f(U)\subseteq V} F(U)$$

where the limit is over basic opens U contained in $f^{-1}V$. That is, an element of $(f_*F)(V)$ is a sequence $(s_U)_{f(U)\subseteq V}$ of $s_U \in F(U)$, such that for each $U' \subseteq U$, the transition function sends s_U to $s_{U'}$.

Example 2.13.

1. Let X be a smooth curve, D a divisor, and $p : X \to \mathbb{A}^0$ the canonical projection to the base. Then $\mathcal{Q}\mathrm{Coh}(\mathbb{A}^0) \cong \mathcal{V}ec_k$ and

$$p_*\mathcal{O}(D) \cong H^0(X, \mathcal{O}(D)).$$

2. Let $\iota : Z \subseteq X$ be a closed subvariety. Then

$$\iota_*\mathcal{O}_Z \cong \mathcal{O}_X/\mathcal{I}_Z.$$

Definition 7 (Projective morphism)**.** A morphism $f : X \to Y$ of quasi-projective varieties is called *projective* if it factors as

$$X \overset{\iota}{\hookrightarrow} \mathbb{P}^n \times Y \overset{\text{proj}}{\to} Y$$

where ι is a closed embedding and proj is the projection to the second factor.

Proposition 2.14. *If $f : X \to Y \in \mathcal{Q}$Proj is projective, then $f_* : \mathcal{Q}\mathrm{Coh}(X) \to \mathcal{Q}\mathrm{Coh}(Y)$ sends coherent sheaves to coherent sheaves.*

Proposition 2.15. *There is a unique collection of morphisms of abelian groups $f_* : G_0(X) \to G_0(Y)$ associated to projective morphisms $f : X \to Y$ satisfying the following properties.*

1. *For closed immersions $\iota : Z \hookrightarrow X$, we have $\iota_*([F]) = [\iota_*F]$.*
2. *For projections $\pi : \mathbb{P}^n \times Y \to Y$ we have $\pi_*([\mathcal{O}(i) \otimes \pi^*F]) = [F]$ for $i = 0, \dots, n$.*
3. *Functoriality: $(g \circ f)_* = g_* \circ f_*$ for composable projective morphisms.*

2.5 Pullbacks

Proposition 2.16. *Suppose that $f : X \to Y$ is in $\mathcal{Q}\mathrm{Proj}$ and $G \in \mathcal{Q}\mathrm{Coh}(Y)$. Then there exists a unique $f^*G \in \mathcal{Q}\mathrm{Coh}(X)$ such that:*

1. *If $U \in \mathcal{B}(X)$ and $f(U) \subseteq V$ for some $V \in \mathcal{B}(Y)$, then*

$$(f^*G)(U) = G(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)$$

   where we use the induced ring morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$.

2. *If $\{U_\lambda\}_{\lambda\in\Lambda}$ is a family of basic opens, closed under intersection, and $U = \cup_{\lambda\in\Lambda} U_\lambda$ is also a basic open, then*

$$(f^*G)(U) = \varprojlim_{\lambda\in\Lambda} (f^*G)(U_\lambda)$$

Remark 2.17. The above proposition is a consequence of the sheaf property mentioned in Remark 2.8 and the fact that for any basic open $V \subseteq Y$ the preimage $f^{-1}V$ is a union of basic opens.

Example 2.18 (Pullback examples)**.**

1. For any morphism $f : X \to Y$, we have

$$f^*\mathcal{O}_Y = \mathcal{O}_X.$$

2. If $\iota : U \to X$ is an open subvariety and $F \in \mathcal{Q}\mathrm{Coh}(X)$, then

$$\iota^* F \cong F|_U$$

   where $F|_U$ is simply the functor F restricted to basic opens contained in U.

3. If $p : X \to \mathbb{A}^0$ is the canonical projection and $V \cong k^{\oplus I} \in \mathcal{V}ec_k$ is a vector space with basis or cardinality I. Then

$$p^*V \cong \mathcal{O}_X^{\oplus I}.$$

Recall that there is a very clean description for finitely generated abelian groups up to isomorphism. Namely, they are of the form $\mathbb{Z}^r \oplus \mathbb{Z}/n_1 \oplus \cdots \oplus \mathbb{Z}/n_k$. Coherent sheaves are slightly more complicated, but still quite accessible.

Remark 2.19 (Flat pullback)**.** If $j : U \to X$ is an open subvariety, there is an induced group homomorphism $G_0(X) \to G_0(U)$; $[F] \mapsto [j^*F]$. More generally, if $f : Y \to X$ is *flat* in the sense that $f^* : \mathcal{C}\mathrm{oh}(X) \to \mathcal{C}\mathrm{oh}(Y)$ sends exact sequences to exact sequences, then we get a group homomorphism.

$$\begin{aligned} G_0(X) &\to G_0(Y) \\ [F] &\mapsto [j^*F]. \end{aligned}$$

Remark 2.20 (Stratification of coherent sheaves)**.** Suppose X is a quasi-projective variety and $F \in \mathcal{C}\mathrm{oh}(X)$. Then there exists a sequence of closed subvarieties $\varnothing = Z_{-1} \subset Z_0 \subset \cdots \subset Z_s = X$ such that if $\iota_i : W_i = Z_i \setminus Z_{i-1} \to X$ is the inclusion, we have

$$\iota_i^* F \cong \mathcal{O}_{W_i}^{\oplus r_i}$$

for some $r_0 \geq r_1 \geq \cdots \geq r_s \in \mathbb{N}$. Geometrically, $\mathcal{O}_X^{\oplus r}$ is the module of sections s of the projection

$$\begin{array}{c} X \times \mathbb{A}^r \\ s \left(\Big\uparrow \; \Big\downarrow \right) p \\ X \end{array}$$

So we can/should think of the coherent sheaf F as the varieties $W_i \times \mathbb{A}^{r_i}$ glued together in some way.

Next lecture we will be concerned with vector bundles, namely, coherent $\mathcal{O}_X$-modules where the rank is locally constant.

Definition 8 (Vector bundle)**.** A *vector bundle* on a quasi-projective variety X is a coherent $\mathcal{O}_X$-module E such that for every point $x \in X$, there exists a basic open $U \ni x$ and an isomorphism $E|_U \cong \mathcal{O}_U^{\oplus r}$ for some $r \geq 0$.

2.6 Cotangent sheaf

Definition 9 (Cotangent sheaf)**.** Let X be a quasi-projective variety. Consider the diagonal morphism $\Delta : X \to X \times X$; $x \mapsto (x, x)$ and let $\mathcal{I}_\Delta \subseteq \mathcal{O}_{X \times X}$ be the ideal sheaf of the diagonal. The *cotangent bundle* of X is defined as

$$\Omega_X := \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

where Δ^* denotes pullback along the diagonal morphism. The *tangent bundle* is the dual

$$\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X).$$

Remark 2.21. More explicitly, for any basic open $U \subseteq X$ we can find a basic open $V \subseteq X \times X$ such that $V \cap \Delta(X) = U$. In this case,

$$\Omega_X(U) = I/I^2$$

where $I = \{\phi : V \to k \mid \phi(U) = 0\}$.

Remark 2.22 (Geometric interpretation)**.** Intuitively, if $Z \subseteq Y$ is a closed subvariety with sheaf of ideals $\mathcal{I}_Z$, then $\mathcal{I}_Z/\mathcal{I}_Z^2$ captures the linear part of functions vanishing along Z. This controls tangent information about the directions perpendicular to Z in Y. When $Z = X$ and $Y = X \times X$, this turns out to be the same as the cotangent bundle.

Example 2.23 (Examples of cotangent sheaves)**.**

1. **Affine space:** For $X = \mathbb{A}^n$ we have $\Omega_{\mathbb{A}^n} \cong \mathcal{O}_{\mathbb{A}^n}^{\oplus n}$.

2. **Projective line:** For $X = \mathbb{P}^1$, we have $\Omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. This can be computed using the Euler sequence:

$$0 \to \Omega_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

   The degree -2 reflects the fact that $\mathbb{P}^1$ has "negative curvature" in the sense that it has no global vector fields.

3. **Node curve:** Consider the curve $X = V(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2$ from Lecture 1. At smooth points x, $\dim(\Omega_X)_x = 1$. However, at the singular point 0, the fiber $(\Omega_X)_{(0,0)}$ has dimension 2.

Definition 10 (Smooth variety)**.** A quasi-projective variety X is called *smooth* of dimension d at a point x if there is a basic open $x \in U$ such that $\Omega_X(U) \cong \mathcal{O}_X(U)^{\oplus d}$. It is called *smooth* if it is smooth at every point.

2.7 Chow groups

Definition 11 (Dimension and cycles)**.** An irreducible variety Z has *dimension d* if, generically, there are d-linearly independent differential forms. That is, for any non-empty basic open U we have

$$\dim_{K_Z} K_Z \otimes_{\mathcal{O}_Z(U)} \Omega_Z(U) = d.$$

For a quasi-projective variety X, let $X_{(d)}$ denote the set of irreducible subvarieties of X of dimension d. The free abelian group generated by $X_{(d)}$ is denoted

$$\mathcal{Z}_d(X) = \{\Sigma_{i=1}^N n_i[W_i] \mid N, n_i \in \mathbb{N}, W_i \in X_{(d)}\} \cong \bigoplus_{W \in X_{(d)}} \mathbb{Z}$$

An element of $\mathcal{Z}_d(X)$ is called a *d-cycle*.

Example 2.24.

1. If X is a smooth curve we have $\mathcal{Z}_0(X) = \mathrm{Div}(X)$.

2. If $Z \to X$ is a closed subvariety, we have a canonical morphism

$$\mathcal{Z}_d(Z) \to \mathcal{Z}_d(X).$$

For a general projective morphism $f : X \to Y$, there is a pushforward $f_* : \mathcal{Z}_d(X) \to \mathcal{Z}_d(Y)$ determined by

$$f_*([Z]) = \begin{cases} [K_Z : K_{f(Z)}] \cdot [f(Z)] & \text{if } \dim Z = \dim f(Z) \\ 0 & \text{otherwise} \end{cases}$$

Here $[K_Z : K_{f(Z)}]$ is the degree of the finite extension of fields $K_{f(Z)} \subseteq K_Z$.

3. **Flat pullback:** If $f : Y \to X$ is a flat morphism between irreducible varieties (see Remark 2.19), then there is a pullback map $f^* : \mathcal{Z}_d(X) \to \mathcal{Z}_{d+\dim Y-\dim X}(Y)$. For an irreducible subvariety $Z \subseteq X$ of dimension d, the preimage $f^{-1}(Z)$ may have multiple irreducible components W_i. We define $f^*([Z]) = \sum_i m_i[W_i]$ where m_i are appropriate multiplicities to account for *ramification*. See [Stacks Project, Tag 0AZE] for more details.

4. **Divisors from functions:** If W is an irreducible variety of dimension $d+1$ and $f \in K_W^*$, then f defines a d-cycle $\operatorname{div}(f) \in \mathcal{Z}_d(W)$ given by

$$\operatorname{div}(f) = \sum_{Z \in W_{(d)}} \operatorname{ord}_Z(f) \cdot [Z]$$

where $\operatorname{ord}_Z(f)$ is the order of vanishing of f along Z. See [Stacks Project, Tag 02AR] for the algebraic definition of $\operatorname{ord}_Z(f)$.

5. Let $D = \sum_i n_i[Z_i] \in \mathcal{Z}_{d-1}(X)$ where X is smooth of dimension d. As for smooth curves, we define the line bundle $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) = \{f \in K_X : \operatorname{div}(f)|_U + D|_U \geq 0\}$$

Definition 12 (Rational equivalence and Chow groups)**.** The *Chow group* $A_d(X)$ is defined by the exact sequence

$$\bigoplus_{W \in X_{(d+1)}} K_W^* \xrightarrow{\operatorname{div}} \overbrace{\bigoplus_{Z \in X_{(d)}} \mathbb{Z}}^{=\mathcal{Z}_d(X)} \to A_d(X) \to 0.$$

Remark 2.25 (Intersection product)**.** Suppose X is irreducible of dimension d. The graded abelian group $\oplus_{i \in \mathbb{N}} A_{d-i}(X)$ admits a structure of graded ring. (Note that we have placed A_i is in degree $d-i$. That is, we are grading by *codimension* codim $= d -$ dim not dimension). We would like to define a structure of graded ring on this graded abelian group using intersection $[V] \cdot [W]$" $=$ "$[V \cap W]$. There are a number of obstacles to this definition.

Firstly, $V \cap W$ may be a union of more than one irreducible subvariety $V \cap W = \cup_r T_r$. Worse, the T_r may not be of codimension $\operatorname{codim} V + \operatorname{codim} W$.

It is a quite technical classical theorem in intersection that for any classes $\alpha \in A_{d-i}(X)$, $\beta \in A_{d-j}(X)$ we *can* find representatives $\alpha = \sum n_k[V_k]$ and $\beta = \sum m_\ell[W_\ell]$ such that the irreducible components $T_{k\ell r}$ of the intersections $V_k \cap W_\ell$ have codimension $i + j$. Even then, we need to account for the fact that the intersections might have some multiplicity. For such cycles in *good position*, the defintion of the intersection product is

$$\alpha \cdot \beta = \sum_{k,\ell,m} n_k m_\ell \cdot i(V_k, W_\ell; T_{k\ell m})[T_{k\ell m}]$$

where the multiplicities come from *Serre's Tor formula.* See [Stacks Project, Tag 0B08] for more details.

Example 2.26 (Examples of Chow groups)**.**

1. For an irreducible variety X of dimension d, we have $A_d(X) \cong \mathbb{Z}$.

2. For a smooth variety X of dimension d the assignment $D \mapsto \mathcal{O}(D)$ induces an isomorphism
$$A_{d-1}(X) \xrightarrow{\sim} \operatorname{Pic}(X)$$
where $\operatorname{Pic}(X) = \{\mathcal{O}(D)\}/\cong$ is the set of isomorphism classes of $\mathcal{O}(D)$ equipped with $\otimes$. For any $L \cong \mathcal{O}(D)$ in $\operatorname{Pic}(X)$, the class $D \in A_{d-1}(X)$ is called the *first Chern class* of L and denoted
$$c_1(L).$$

Now we are going to extend the isomorphism $A_{d-1}(X) \cong \operatorname{Pic}(X)$ to the isomorphism in the GRR theorem. For an abelian group A we write

$$A_{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Theorem 2.27 (Universal property of Chern character)**.** *There exists a unique natural transformation*

$$G_0(X) \to A_*(X)_{\mathbb{Q}}$$
$$\alpha \mapsto ch(\alpha)$$

on smooth quasi-projective varieties X such that:

1. *For line bundles L, we have $ch([L]) = e^{c_1(L)} := \sum_{n \in \mathbb{N}} \frac{1}{n!} c_1(L)^n$.*

2. *For* $\alpha, \beta \in G_0(X)$*, we have* $ch(\alpha+\beta) = ch(\alpha) + ch(\beta)$*.*

3. *For flat morphisms* $f : Y \to X$ *(see Remark 2.19) and vector bundles* E *(see Definition 8), we have*

$$ch(f^*[E]) = f^*(ch([E])).$$

These morphisms induce isomorphisms

$$\mathrm{ch} : G_0(X)_{\mathbb{Q}} \cong A_*(X)_{\mathbb{Q}}$$

Remark 2.28. The groups $G_0(X)$ and $A_*(X)$ are contravariantly functorial for flat morphisms and ch is actually a natural transformation for this functoriality. That is, $\mathrm{ch}(f^*\alpha) = f^* \mathrm{ch}(\alpha)$ when f is flat. In this lecture we are interested in projective pushfowards. In order to make ch natural in projective pushfowards we need to use Todd classes.

Theorem 2.29 (Universal property of Todd classes)**.** *There exists a unique natural transformation*

$$\begin{aligned} G_0(X) &\to A_*(X)_{\mathbb{Q}} \\ \alpha &\mapsto \mathrm{td}(\alpha) \end{aligned}$$

on smooth quasi-projective varieties X *such that:*

1. *For line bundles* L*, we have* $\mathrm{td}([L]) = \frac{c_1(L)}{1-e^{-c_1(L)}}$.[4]

2. *For* $\alpha, \beta \in G_0(X)$*, we have* $\mathrm{td}(\alpha+\beta) = \mathrm{td}(\alpha) \cdot \mathrm{td}(\beta)$*.*

3. *For flat morphisms* $f : Y \to X$ *(see Remark 2.19) and vector bundles* E *(see Definition 8), we have*

$$\mathrm{td}(f^*[E]) = f^*(\mathrm{td}([E])).$$

Remark 2.30 (Splitting principle)**.** To prove existence and uniqueness of Chern and Todd classes, one uses the *splitting principle*: any vector bundle E of rank r on X can be pulled back to a sum of line bundles $L_1 \oplus \cdots \oplus L_r$ via some (flat projective surjective) $f : Y \to X$ that induces an injection $f^* : A_*(X) \to A_*(Y)$. This reduces the problem to line bundles, where the classes are explicitly defined.

[4] The power series $\frac{x}{1-e^{-x}} \in \mathbb{Q}[[x]]$ is defined to be the inverse of the power series $\frac{1-e^{-x}}{x} = 1 - \frac{x}{2} + \frac{x^2}{6} - \dots$.

2.8 Restatement

We can now restate the Grothendieck-Riemann-Roch theorem with all the machinery we've developed:

Theorem 2.31 (Grothendieck–Riemann–Roch, Restated)**.** *Suppose $f : X \to Y$ is a projective morphism of smooth quasi-projective varieties. Then the following square commutes, and the horizontal morphisms are isomorphisms.*

$$\begin{array}{ccccc} G_0(X)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & A_*(X)_{\mathbb{Q}} & \xrightarrow{\text{td}(\mathcal{T}_X)\cdot -} & A_*(X)_{\mathbb{Q}} \\ {\scriptstyle f_*}\downarrow & & & & \downarrow{\scriptstyle f_*} \\ G_0(Y)_{\mathbb{Q}} & \xrightarrow{\text{ch}} & A_*(Y)_{\mathbb{Q}} & \xrightarrow{\text{td}(\mathcal{T}_Y)\cdot -} & A_*(Y)_{\mathbb{Q}} \end{array}$$

Remark 2.32. When X is a smooth projective curve and $Y = \mathbb{A}^0$, this recovers the classical Riemann–Roch theorem from Lecture 1. In this case we have:

- $f_* : \underbrace{G_0(X)}_{\cong \mathbb{Z} \oplus \operatorname{Pic}(X)} \to \underbrace{G_0(\mathbb{A}^0)}_{\cong \mathbb{Z}}$ sends $L \in \operatorname{Pic}(X)$ to

  $$\dim H^0(X, L) - \dim H^0(X, \mathcal{H}om(L, \Omega_X)).$$

  This comes from *Serre duality*.
- For $D \in \operatorname{Div}(X)$ we have $f_*(D) = \deg D$. This follows from the definition.
- $\text{td}(\mathcal{T}_X) = 1 + \frac{1}{2}c_1(T_X) = 1 - \frac{1}{2}K$ where $K = \operatorname{div}(\Omega_X)$. This follows from the definitions.
- $\text{td}(\mathcal{T}_Y) = 1$.
- We have $\deg K = 2g - 2$. This can be obtained in various ways, but all of them involve some kind of theorem.

So for $L \cong \mathcal{O}(D)$, the square in the statement becomes

$$L \longmapsto 1 + D$$

$$\begin{array}{c} L \\ \downarrow \\ \dim H^0(X,L) \\ -\dim H^0(X,\mathcal{H}om(L,\Omega_X)) \end{array} \qquad \begin{array}{ccccc} \mathbb{Z} \oplus \operatorname{Pic}(X) & \xrightarrow{1+c_1} & \mathbb{Z} \oplus A_0(X) & \xrightarrow{(1-\frac{1}{2}K)\cdot -} & \mathbb{Z} \oplus A_0(X) \\ {\scriptstyle f_*}\downarrow & & & & \downarrow{\scriptstyle (0,\deg)} \\ \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} \end{array}$$

and the GRR formula becomes:

$$\begin{aligned}\dim H^0(X,L) - \dim H^0(X,\mathcal{H}om(L,\Omega_X)) \\ &= \mathrm{ch}(f_*[L])\cdot \mathrm{td}(\mathcal{T}_Y) \\ &\overset{[GRR]}{=} f_*(\mathrm{ch}([L])\cdot \mathrm{td}(\mathcal{T}_X)) \\ &= f_*((1+D)\cdot(1-\tfrac{1}{2}K)) \\ &= f_*(1+D-\tfrac{1}{2}K) \\ &= \deg D - \tfrac{1}{2}\deg K \\ &= \deg(D)+1-g\end{aligned}$$

Remark 2.33 (Sketch of proof)**.** The proof proceeds by:

1. Reducing to the case where f is a closed embedding or a projection using the factorization of projective morphisms
2. For closed embeddings, use *deformation to the normal cone* to reduce to the case of a regular closed immersion. That is, a closed immersion which locally looks like a zero section $Z \to Z \times \mathbb{A}^c$. In this case, one does a concrete calculation.
3. For projections $\mathbb{P}^n \times Y \to Y$, one uses the explicit description of $G_0(\mathbb{P}^n \times Y)$ and the fact that $\mathrm{td}(\Omega_{\mathbb{P}^n}) = (1 + H + H^2 + \ldots + H^n)$ where H is the class of a hyperplane.