Algebraic K-theory was originally defined by Grothendieck in order to state his generalisation of the Riemann-Roch theorem.

Outline:

1. Lecture 1. Riemann–Roch Theorem (smooth projective curves / $\mathbb{C}$)
2. Lecture 2. Grothendieck–Riemann–Roch (quasi-projective varieties / $k = \overline{k}$)
3. Lecture 3. Exact sequences, K_1, $K_{<0}$ (affine schemes (= rings))
4. Lecture 4. K-theory as the universal localising invariant (stable ∞-categories)
5. Lecture 5. Recent advances.

Contents

1 Riemann–Roch

In this lecture, the base field is always the complex numbers $\mathbb{C}$.

1.1 Riemann–Roch Statement

The goal for today's talk is to understand the words in the following statement.

Theorem 1.1.1 (Riemann–Roch)**.** *Let X be a smooth projective curve of genus g. Then there exists a divisor K (the canonical divisor) such that for every divisor D on X, we have*

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^0(X, \mathcal{O}(K-D)) = \deg(D) + 1 - g.$$

1.2 Affine varieties

Definition 1.2.1 (Affine Variety over $\mathbb{C}$). *An* affine variety *is a subset* $X \subseteq \mathbb{C}^n$ *of the form*

$$X = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \begin{array}{r} f_1(z_1, \ldots, z_n) = 0 \\ f_2(z_1, \ldots, z_n) = 0 \\ \vdots \end{array} \right\}$$

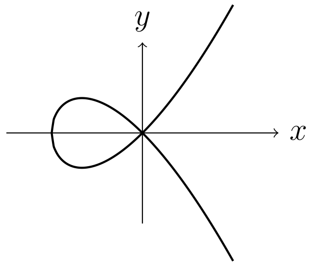
for some collection of polynomials $\{f_i\}_{i \in I} \subseteq \mathbb{C}[x_1, \ldots, x_n]$*. We say* X *is the* zero set *of the* f_i*.*

Remark 1.2.2. Since $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, we can assume the set is finite, but it is convenient to allow infinite sets.

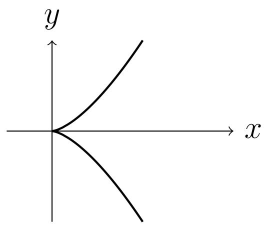
Example 1.2.3 (Examples of Affine Varieties)**.**

1. *Affine space:* $X = \mathbb{C}^n =: \mathbb{A}^n$ itself (i.e., taking $k = 0$).
2. *Node:* $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^2(x + 1)\}$.
3. *Cusp:* $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3\}$.

Node: $y^2 = x^2(x+1)$

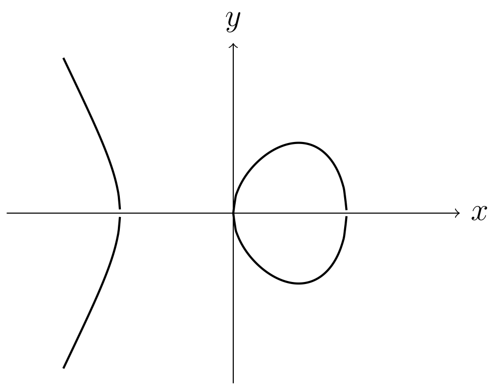


Cusp: $y^2 = x^3$



4. *Elliptic curve:* $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 + ax + b\}$ where $4a^3 + 27b^2 \neq 0$.

Elliptic curve: $y^2 = x^3 - x = x(x-1)(x+1)$



5. *Complement of a hypersurface:* Given an affine variety $X = V(\{f_i\}) \subseteq \mathbb{C}^n$ and polynomial g, the complement $U := X \setminus V(g)$ is not a closed subvariety of $\mathbb{C}^n$. However, the affine variety

$$U' = \{(x_1, \dots, x_n, y) \in \mathbb{C}^{n+1} : f_i(x_1, \dots, x_n) = 0, g(x_1, \dots, x_n) \cdot y = 1\}$$

projects bijectively to U. This gives the commutative diagram:

$$\begin{array}{ccc} U' & \longrightarrow & \mathbb{C}^{n+1} \\ \big\downarrow \sim & & \big\downarrow \\ U & \longrightarrow & \mathbb{C}^n \end{array}$$

6. *General linear group:* $\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathrm{Mat}_n(\mathbb{C}) \mid \det(A) \neq 0\}$ is an example of a U as in the previous point. That is,

$$GL_n(\mathbb{C}) \cong \{(A, t) \in \mathrm{Mat}_n(\mathbb{C}) \times \mathbb{C} \mid \det(A) \cdot t = 1\}$$

7. *Intersection:* If $X_1, X_2 \subseteq \mathbb{A}^n$ are affine varieties defined by sets of polynomials $\mathcal{F}_1, \mathcal{F}_2$ respectively, then $X_1 \cap X_2$ is the affine variety defined by $\mathcal{F}_1 \cup \mathcal{F}_2$.

8. *Union:* If $X_1, X_2 \subseteq \mathbb{A}^n$ are affine varieties defined by sets of polynomials $\mathcal{F}_1, \mathcal{F}_2$, then $X_1 \cup X_2$ is the affine variety defined by $\{fg : f \in \mathcal{F}_1, g \in \mathcal{F}_2\}$.

1.3 Projective varieties

Definition 1.3.1 (Complex Projective Space)**.** Complex projective space *is the set*

$$\mathbb{P}^n = \frac{\{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\}}{\sim}$$

of equivalence classes under the relation

$$(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n), \quad \lambda \in \mathbb{C}^\times.$$

One writes $(z_0 : \cdots : z_n) \in \mathbb{P}^n$ *for the equivalence class containing* $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}$.

Remark 1.3.2. For each $i = 0, \dots, n$ we have a bijection

$$\mathbb{C}^n \xrightarrow{\sim} U_i := \{(z_0 : \cdots : z_n) \mid z_i \neq 0\}$$
$$(x_1, \dots, x_n) \mapsto (x_1 : \cdots : x_i : 1 : x_{i+1} : \cdots : x_n)$$

These cover $\mathbb{P}^n$.

$$\mathbb{P}^n = \cup_{i=0}^n U_i.$$

Exercise 1.3.3. Describe the intersections $U_{i_1} \cap \cdots \cap U_{i_j}$ as subsets of $U_0 \cong \mathbb{C}^n$.

Definition 1.3.4 (Projective Variety)**.** *A* projective variety *is a subset $X \subseteq \mathbb{P}^n$ such that for each affine chart U_i, the intersection $X \cap U_i$ is an affine variety in $U_i \cong \mathbb{A}^n$.*

Example 1.3.5 (Homogeneous Polynomials)**.** If $\mathcal{F}$ is a set of homogeneous polynomials (i.e., polynomials of the form $\sum_{i_0+\cdots+i_n=d} a_{i_1,\ldots,i_k} z_0^{i_0} \ldots z_n^{i_n}$ for some d), then the zero set $V(\mathcal{F}) = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n : f(z_0, \ldots, z_n) = 0 \text{ for all } f \in \mathcal{F}\}$ is a projective variety. In fact, *every* projective variety is of this form.

Example 1.3.6 (Grassmannians)**.** The *Grassmannian* $\mathrm{Gr}(k, n)$ is the variety of k-dimensional subspaces of $\mathbb{C}^n$. For example: $\mathrm{Gr}(2, 4)$ (planes in $\mathbb{C}^4$), which can be embedded in $\mathbb{P}^5$ via Plücker coordinates.

$$\begin{aligned} \mathrm{Gr}(2,4) &\hookrightarrow \mathbb{P}^5 \\ \langle v_1, v_2 \rangle &\mapsto \langle v_1 \wedge v_2 \rangle \end{aligned}$$

If we use p_{ij} for the coordinate of $\mathbb{P}^5$ corresponding to $e_i \wedge e_j \in \mathbb{C}^4 \wedge \mathbb{C}^4$, then the image of $\mathrm{Gr}(2, 4)$ in $\mathbb{P}^5$ is defined by the *Plücker relation*:

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

Example 1.3.7 (Segre Embedding)**.** $\mathbb{P}^n \times \mathbb{P}^m$ has a structure of projective variety via the *Segre embedding*

$$\begin{gathered} \mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{(n+1)(m+1)-1} \\ (x_0 : \cdots : x_n), (y_0 : \cdots : y_m) \mapsto (x_0y_0 : x_0y_1 : \cdots : x_iy_i : \cdots : x_ny_m) \end{gathered}$$

The image is defined by the quadratic relations $z_{ij}z_{kl} - z_{il}z_{kj} = 0$ for all i, k and j, l. Consequently, if $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are projective varieties, then $X \times Y$ can canonically be identified with a subvariety of $\mathbb{P}^{(n+1)(m+1)-1}$.

Remark 1.3.8. One can consider $\mathbb{P}^n$ as a *compactification* of $\mathbb{A}^n \cong U_0$ where we have adjoined one point for every line through the origin in such a way that if a curve approaches that line "at infinity" then it will actually intersect at that new point "at infinity".

For example, consider the affine curves

$$C_1 : x = 0 \text{ (the } y\text{-axis)}, \qquad C_2 : xy = 1 \text{ (a hyperbola)}$$

in $\mathbb{C}^2 = \{(x, y)\} \cong \{(x : y : 1)\} = U_0$. These curves do not intersect in the affine plane. However, these curves are the intersection of U_0 with the projective curves

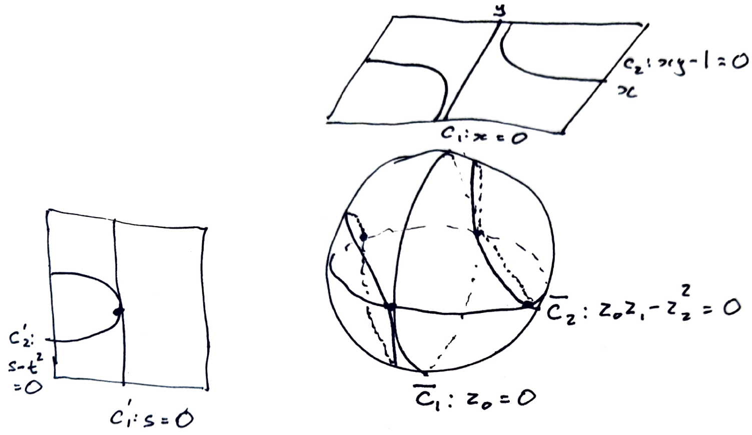
$$\overline{C}_1 : \{(z_0 : z_1 : z_2) \mid z_0 = 0\} \tag{1}$$

$$\overline{C}_2 : \{(z_0 : z_1 : z_2) \mid z_1z_2 = z_0^2\} \tag{2}$$

Intersecting with the chart $U_1 = \{(s : 1 : t)\} \cong \{(s, t)\}$, they become the curves

$$C_1' : s = 0 \text{ (the } t\text{-axis) }, \qquad C_2' : t = s^2 \text{ (a quadric)}$$

They intersect at the point $(s, t) = (0, 0) \leftrightarrow (0 : 1 : 0)$, the point at infinity corresponding to the line $\{(0, y) \mid y \in \mathbb{C}\} \subseteq U_0$.



1.4 Smooth Complex Projective Curves

Definition 1.4.1 (Smooth Point). *Let $X \subseteq \mathbb{A}^n$ be an affine variety and $x \in X$. We say X is* smooth of dimension d *at x if there exists an open ball $B \ni x$ (in the analytic topology, i.e., $B = \{z \mid ||z-x|| < \varepsilon$ for some $\varepsilon > 0)$ and a biholomorphic map $\phi : B \tilde{\to} B' \subseteq \mathbb{C}^n$ to an open subset B' of $\mathbb{C}^n$ such that*

$$B \cap X = \phi^{-1}\{(z_1, \ldots, z_d, 0, \ldots, 0) \in B'\}.$$

Example 1.4.2 (Non-smooth Points).

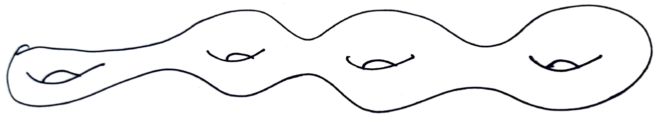
1. *Node:* The affine curve $y^2 = x^2(x + 1)$ has a node at the origin.

2. *Cusp:* The affine curve $y^2 = x^3$ has a cusp at the origin.

In both cases all other points are smooth.

Definition 1.4.3 (Smooth Complex Projective Curve). *A* smooth projective curve *X is a projective variety which is smooth of dimension one at every point.*

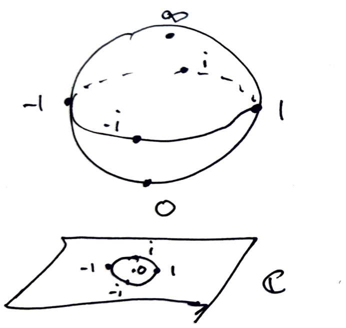
Remark 1.4.4 (Underlying Topological Space). We can consider smooth projective curves as compact real manifolds of real dimension 2. They are automatically oriented, so homeomorphic to a sphere with g handles. This g is called the *genus* of the curve. (There is also a purely algebraic description of genus, namely $\dim H^0(X, \mathcal{O}(K))$ where K is the canonical divisor mentioned in the statement of the Riemann–Roch theorem, and $H^0(X, \mathcal{O}(-))$ is defined below).

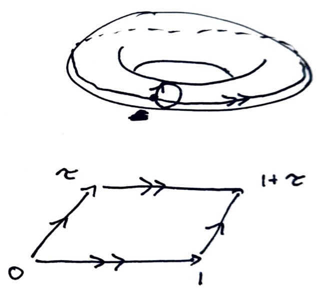


Example 1.4.5 (Genus Examples)**.**

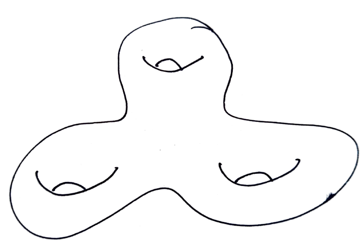
1. *Projective line:* The projective line $\mathbb{P}^1$ is topologically a sphere, hence has genus $g = 0$. Indeed, it is the one point compactification of $\mathbb{C} \cong \mathbb{R}^2$.

2. *Elliptic curve:* A smooth cubic curve in $\mathbb{P}^2$, such as $y^2z = x^3 + axz^2 + bz^3$ with $4a^3 + 27b^2 \neq 0$, is topologically a torus (i.e., the surface of a doughnut, or coffee mug) and has genus $g = 1$. Indeed, every elliptic curve is holomorphic to a quotient abelian group of the form $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with the canonical smooth complex manifold structure, where $\tau \notin \mathbb{R}$.





3. *Higher genus curves:* A smooth curve of degree d in $\mathbb{P}^2$ has genus $g = \frac{(d-1)(d-2)}{2}$. For example, the Klein quartic $x^3y + y^3z + z^3x = 0$ is a smooth degree 4 curve, so has genus $g = \frac{(4-1)(4-2)}{2} = 3$. Topologically it looks like the surface of a fidget spinner.



1.5 Divisors

Definition 1.5.1 (Basic Open)**.** *A* basic open *of an affine variety* $X \subseteq \mathbb{A}^n$ *is a subset of the form*

$$D(g) = \{x \in X : g(x) \neq 0\} \subseteq X$$

for some polynomial $g \in \mathbb{C}[x_1, \ldots, x_n]$. *The basic opens together with inclusion maps form a category which we denote* $\mathcal{B}(X)$. *If* X *is projective, then we define* $\mathcal{B}(X) = \bigcup_{i=0}^{n} \mathcal{B}(U_i \cap X)$ *to be the union of the basic opens of the* $n+1$ *standard affine varieties associated to* X.

Example 1.5.2.

1. If $g = 1$ (or more generally, if g is invertible on X) then $D(g) = X$.

2. If $g = 0$ (or more generally, if g vanises everywhere on X) then $D(g) = \varnothing$.

3. If $X = \mathbb{A}^1$ and $g = (x - a_1) \ldots (x - a_n)$ then $D(g) = X \setminus \{a_1, \ldots, a_n\}$. Similarly, every basic open of $\mathbb{P}^1$ is of the form $\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}$ for some nonempty set of points.

4. More generally, if X is a projective or affine curve, then every basic open is of the form $X \setminus \{x_1, \ldots, x_n\}$. (But not conversely).

Definition 1.5.3 (Structure Sheaf on Basic Opens)**.** *Given an affine variety* $X \subseteq \mathbb{C}^n$ *and a basic open* $U = D(g) \subseteq X$, *write*

$$\mathcal{O}_X(U) = \left\{ \varphi : U \to \mathbb{C} \mid \varphi = \frac{f}{g^n} \text{ for some } f \in \mathbb{C}[x_1, \ldots, x_n], n \geq 0 \right\}$$

for the set of functions on U *of the form* f/g^n.

Remark 1.5.4. Note that if f' vanishes on X, then $f/g^n = (f + f')/g^n$ as a function on X. More precisely, one can show that the ring $\mathcal{O}_X(U)$ of functions, is isomorphic to the abstract ring

$$\mathcal{O}_X(U) \cong \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle f_1, \ldots, f_c \rangle}[g^{-1}]$$

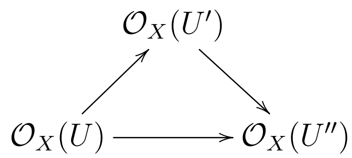
where $X = V(f_1, \ldots, f_c)$.

Remark 1.5.5. As U varies, the $\mathcal{O}_X(U)$ define a functor

$$\begin{aligned} \mathcal{B}(X)^{op} &\to \mathcal{R}\text{ing} \\ U &\mapsto \mathcal{O}_X(U). \end{aligned}$$

That is,

0. for every U we have a ring
$$\mathcal{O}_X(U),$$
1. for every inclusion $U' \subseteq U$, restriction gives a ring homomorphism
$$\mathcal{O}_X(U) \to \mathcal{O}_X(U'),$$
2. for every two inclusions $U'' \subseteq U' \subseteq U$ we have a commutative triangle of ring homomorphisms



Definition 1.5.6. *Suppose that $X \subseteq \mathbb{A}^n$ is* irreducible. *That is, X is not a union of two distinct nonempty varieties. Then each $\mathcal{O}_X(U) \to \mathcal{O}_X(U')$ (for $U' \neq \varnothing$) is injective, and we can define*
$$K_X := \bigcup_{U \neq \varnothing} \mathcal{O}_X(U).$$

Remark 1.5.7. If X is a smooth curve, then each $f \in K_X$ is a meromorphic function on the corresponding smooth complex manifold. In particular, the *order*
$$\mathrm{ord}_x(f)$$
of the pole (or zero) of f at $x \in X$ is well-defined.

Definition 1.5.8. *A* divisor *on a smooth projective curve X is a finite formal sum of points $D = \sum_{i=1}^d n_i x_i$. We write*
$$\mathrm{Div}(X) = \{\Sigma_{i=1}^d n_i x_i\}$$
for the (free) abelian group of divisors. The degree *of a divisor is*
$$\deg(\Sigma_{i=1}^d n_i x_i) = \Sigma_{i=1}^d n_i.$$

Example 1.5.9. The divisor associated to a rational function $f \in K_X$ is
$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f) \cdot x.$$

Definition 1.5.10. *Each divisor D determines a functor*

$$\begin{aligned}\mathcal{O}(D) : \mathcal{B}(X)^{op} &\to \mathcal{A}\mathrm{b} \\ U &\mapsto \{f \in K_X \mid \operatorname{div}(f) + D \geq 0 \text{ on } U\}\end{aligned}$$

where a divisor $E = \sum_x n_x \cdot x$ satisfies $E \geq 0$ if $n_x \geq 0$ for all x.

Example 1.5.11. We have $\mathcal{O}_X = \mathcal{O}(0)$ where 0 is the zero divisor.

Remark 1.5.12. Note that the assignement

$$\mathcal{K}_X : U \mapsto \begin{cases} K_X & \text{if } U \neq \varnothing \\ 0 & \text{if } U = \varnothing \end{cases}$$

also defines a functor $\mathcal{B}(X)^{op} \to \mathcal{A}\mathrm{b}$ for each each $\mathcal{K}_X(U)$ is a $\mathcal{O}_X(U)$-module and the transition morphisms are compatible with this structure.

Moreover, each $\mathcal{O}(D)(U)$ is a sub-$\mathcal{O}_X(U)$-module of $\mathcal{K}_X(U)$, and the transition morphisms $\mathcal{O}(D)(U) \to \mathcal{O}(D)(U')$ are compatible with this structure. In other words, we have an inclusion of quasi-coherent $\mathcal{O}_X$-modules

$$\mathcal{O}(D) \subseteq \mathcal{K}_X.$$

Remark 1.5.13 (Physical Interpretation)**.** In string theory, Riemann surfaces appear as worldsheets of strings. Line bundles $\mathcal{O}(D)$ on these surfaces can encode various physical properties:

1. Spin structures
2. Gauge field backgrounds
3. D-brane charges in type II string theory

The degree of a line bundle corresponds to quantized charges or fluxes.

1.6 Riemann–Roch restatement

Definition 1.6.1 (Global sections)**.** *Given a divisor D on an irreducible smooth curve X we define*

$$H^0(X, \mathcal{O}(D)) := \bigcap_{\varnothing \neq U \in \mathcal{B}(X)} \mathcal{O}(D)(U).$$

That is, an element of $H^0(X, \mathcal{O}(D))$ is an element of K_X which belongs to all $\mathcal{O}(D)(U)$.

Remark 1.6.2. We could also have directly defined

$$H^0(X, \mathcal{O}(D)) = \{f \in K_X \mid \operatorname{div}(f) + D \geq 0 \text{ on } X\}$$

but the above definition is warm-up for the definition of $H^0(X, F)$ that we will see next time when F is an arbitrary quasi-coherent $\mathcal{O}_X$-module.

Example 1.6.3. Consider the divisor $D = d \cdot \infty$ on $\mathbb{P}^1$ where $\infty = (1 : 0)$. Then $H^0(\mathbb{P}^1, \mathcal{O}(D))$ is identified with the set $\mathbb{C}[x, y]_d = \{\sum_{i=0}^d a_i x^i y^{d-i}\}$ of homogeneous polynomials of degree d. In particular, it is a complex vector space of dimension $d + 1$ (if $d \geq 0$ and 0 otherwise).

Theorem 1.6.4. *If X is a smooth projective curve, then each $H^0(X, \mathcal{O}(D))$ is a finite dimensional $\mathbb{C}$-vector space.*

Theorem 1.6.5 (Riemann–Roch)**.** *Let X be a smooth projective curve of genus g. Then there exists a unique divisor K (the canonical divisor) such that for every divisor D on X, we have*

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^0(X, \mathcal{O}(K - D)) = \deg(D) + 1 - g.$$

Example 1.6.6. For $X = \mathbb{P}^1$, we have $K = -2 \cdot \infty$ and $g = 0$. Then inputting everything we check that for $D = n \cdot \infty$ we have

$$\deg(D) + 1 - g = n + 1.$$

For the left side, we compute the dimensions case by case.
Case $n \geq 0$:

$$\dim H^0(X, \mathcal{O}(D)) = n + 1 \tag{3}$$
$$\dim H^0(X, \mathcal{O}(K - D)) = \dim H^0(X, \mathcal{O}((-n - 2)\infty)) = 0 \tag{4}$$

since $-n - 2 < 0$.
Case $n = -1$:

$$\dim H^0(X, \mathcal{O}(D)) = 0 \tag{5}$$
$$\dim H^0(X, \mathcal{O}(K - D)) = \dim H^0(X, \mathcal{O}(-\infty)) = 0. \tag{6}$$

Case $n \leq -2$:

$$\dim H^0(X, \mathcal{O}(D)) = 0 \tag{7}$$
$$\dim H^0(X, \mathcal{O}(K - D)) = \dim H^0(X, \mathcal{O}((-n - 2)\infty)) = -n - 1. \tag{8}$$

In all cases,

$$\dim H^0(X, \mathcal{O}(D)) - \dim H^0(X, \mathcal{O}(K - D)) = n + 1,$$

confirming Riemann-Roch.