Derived Algebraic Geometry
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Lecture 10: Cotangent complex

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References:
- Quillen, On the (co-)homology of commutative rings.
- Illusie, Complexe cotangent et deformations I, II
- Lurie, Derived Algebraic Geometry.

Outline:
1. Classical Kähler differentials
2. Cotangent complex
3. Finiteness conditions
4. Deformation theory

1 Classical Kähler differentials

In this section everything is classical. That is, all categories are 1-categories. To emphasize that our modules are classical we write $R\text{-mod}^{\heartsuit}$. More precisely,

$$R\text{-mod}^{\heartsuit} \subseteq R\text{-mod}^{\mathrm{cn}}$$

is the fully subcategory of those modules whose underlying space is a set.

Definition 1. For $A \to B \in \mathcal{A}\mathrm{lg}$ and $M \in B\text{-mod}^{\heartsuit}$, a *derivation* is an A-linear map

$$\partial : B \to M$$

satisfying the Leibniz rule:

$$\partial(ab) = a\partial b + b\partial a.$$

Write $\mathrm{Der}_A(B, M)$ for the set of A-linear derivations towards the B-module.

Exercise 2. Show that for any $a \in A$, we have $\partial a = 0$. Hint.[1]

Example 3. Suppose that $A \to B \in \mathcal{A}\mathrm{lg}$ and $M \in B\text{-mod}^{\heartsuit}$.
1. If $\partial, \partial' \in \mathrm{Der}_A(B, M)$ then $\partial + \partial' \in \mathrm{Der}_A(B, M)$.
2. If $\partial \in \mathrm{Der}_A(B, M)$, $b \in B$, then $b\partial \in \mathrm{Der}_A(B, M)$.

[1] Use A-linearity.

3. If $\partial \in \mathrm{Der}_A(B, M)$, $\phi : M \to M' \in B\text{-mod}^{\heartsuit}$ then $\phi \circ \partial \in \mathrm{Der}_A(B, M')$.
4. If $\partial \in \mathrm{Der}_A(B, M)$ and $B' \xrightarrow{\phi} B \in A\text{-}\mathcal{A}\mathrm{lg}$ then $\partial \circ \phi \in \mathrm{Der}_A(B', M|_{B'})$.

Definition 4. For $A \to B \in \mathcal{A}\mathrm{lg}$ the module of (classical) *Kähler differentials* is the universal derivation. That is, a derivation $B \xrightarrow{d} \Omega^{\mathrm{cl}}_{B/A}$ such that for any B-module M the induced map

$$\hom_{B\text{-mod}}(\Omega^{\mathrm{cl}}_{B/A}, M) \xrightarrow{-\circ d} \mathrm{Der}_A(B, M)$$

is an isomorphism.

Exercise 5.
1. Suppose $B = A[x_1, \dots, x_n]$. Show that the map

$$\mathrm{Der}_A(B, M) \to M^{\oplus n}; \qquad \partial \mapsto (\partial x_1, \dots, \partial x_n)$$

is an isomorphism of B-modules. Deduce that $B \xrightarrow{d} \Omega^{\mathrm{cl}}_{B/A}$ exists and is isomorphic to $\oplus_{i=1}^n B dx_i$.
2. Note that any $A \to B$ can be written as a colimit (in $\mathcal{A}\mathrm{lg}$) of polynomial algebras $(A{\to}B) = \varinjlim(P_\lambda{\to}Q_\lambda)$. Here $P_\lambda = \mathbb{Z}[x_1, \dots, x_{n_\lambda}]$ and $Q_\lambda = \mathbb{Z}[x_1, \dots, x_{n_\lambda}, y_1, \dots, y_{m_\lambda}]$. Show that the morphism

$$\mathrm{Der}_A(B, M) \to \varprojlim \mathrm{Der}_{P_\lambda}(Q_\lambda, M|_{P_\lambda})$$

associated to such a colimit is an isomorphism.
3. Using the previous two parts, deduce that $\Omega^{\mathrm{cl}}_{B/A}$ exists in general. Hint.[2]

2 The cotangent complex

Now we animate Ω^{cl} from the previous section. Recall that $\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg})$ is the sifted colimit completion of $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$. In particular, to define a functor $\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}) \to \mathcal{C}$ to any category admitting sifted colimits, it suffices to define a functor $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} \to \mathcal{C}$.

Definition 6. The *cotangent complex* is the unique colimit preserving functor

$$\Omega : \mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}) \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$$

restricting to

$$\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} \to \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}; \qquad (P \to Q) \mapsto (Q, \Omega^{\mathrm{cl}}_{Q/P}).$$

In particular, $\Omega_{Q/P} = \Omega^{\mathrm{cl}}_{Q/P}$ for $(P{\to}Q) \in \mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$.

[2] Use the adjunctions $- \otimes_{P_\lambda} B\text{-mod}^{\heartsuit} \rightleftarrows P_\lambda\text{-mod}^{\heartsuit} : (-)|_{P_\lambda}$.

Exercise 7. Suppose that $A \to B \in \mathcal{A}\mathrm{lg}$ is a classical algebra. Show that we have

$$\pi_0 \Omega_{B/A} \cong \Omega^{\mathrm{cl}}_{B/A}.$$

Hint.[3]

Theorem 8 (First cofibre sequence)**.** *Suppose that $A \to B \to C$ is a composable pair of ring homomorphisms. Then there is a cofibre sequence*

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B}$$

in C-mod.

Proof. Consider the category $\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}\mathcal{A}\mathrm{lg})$ which is the sifted completion of the category $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$ whose objects are composable pairs of morphisms $P{\to}Q{\to}R$ of the form $P = \mathbb{Z}[x_1, \ldots, x_n]$, $Q = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_m]$, $R = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_\ell]$. Morphisms in $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$ are pairs of commutative squares

$$\begin{array}{ccccc} P & \longrightarrow & Q & \longrightarrow & R \\ \downarrow & & \downarrow & & \downarrow \\ P' & \longrightarrow & Q' & \longrightarrow & R' \end{array}$$

in $\mathcal{P}\mathrm{oly}$. The assignment

$$\begin{aligned} \mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} &\to \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree} \\ (P{\to}Q{\to}R) &\mapsto (R, \Omega_{Q/P} \otimes_Q R) \end{aligned}$$

extends to a colimit preserving functor

$$\Phi : \mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}\mathcal{A}\mathrm{lg}) \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}).$$

In the category $\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}\mathcal{A}\mathrm{lg})$ we have a canonical pushout square, informally described as

$$\begin{array}{ccc} (A{\to}B{\to}C) & \longrightarrow & (A{\to}C\overset{=}{\to}C) \\ \downarrow & & \downarrow \\ (B\overset{=}{\to}B{\to}C) & \longrightarrow & (B{\to}C\overset{=}{\to}C) \end{array}$$

More formally, we have such pushout squares defined when $(A{\to}B{\to}C)$ is in $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$, but this subcategory generates $\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}\mathcal{A}\mathrm{lg})$ under sifted colimits, and colimits of pushout squares are pushouts squares. We claim that the image of this square in $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$ is a pushout square of the form

$$\begin{array}{ccc} \Omega_{B/A} \otimes_B C & \longrightarrow & \Omega_{C/A} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega_{C/B} \end{array}$$

[3] Cf. Exercise 5.

Indeed, we can factor Φ as a composition of colimit preserving functors

$$\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}\mathcal{A}\mathrm{lg}) \to \mathcal{C} \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$$

$$(P{\to}Q{\to}R) \mapsto (Q{\to}R, \Omega_{Q/P}) \qquad (Q{\to}R, F) \mapsto (R, F \otimes_Q R)$$

where $\mathcal{C}$ is the sifted completion of the category $\mathcal{C}_0$ whose objects are triple $(P{\to}R, F)$ with $P{\to}R$ in $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$ and $F \cong P^{\oplus n}$. We leave it to the reader to describe the morphisms of $\mathcal{C}_0$. Note that our functors fit into a commutative diagram of colimit preserving functors (for any $B \to C \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$)

$$\begin{array}{ccccc}
 & & B\text{-mod} & \xrightarrow{-\otimes_B C} & C\text{-mod} \\
 & & \downarrow & & \downarrow \\
\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}\mathcal{A}\mathrm{lg}) & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \\
{\scriptstyle (P\to R\to Q)\mapsto(P\to R)}\downarrow & & \downarrow{\scriptstyle (Q\to R,F)\mapsto(Q,F)} & & \\
\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}) & \xrightarrow[\Omega]{} & \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) & &
\end{array}$$

Here, the vertical functor B-mod $\to \mathcal{C}$ is the inclusion of the full subcategory of those presheaves $\mathcal{C}_0^{op} \to \mathcal{S}$ whose restriction to $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} \subseteq \mathcal{C}_0$ is $(B \to C)$. □

Exercise 9.

1. Suppose that $A \to B$ and $A \to C$ are morphisms in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Show that there is an equivalence

$$\Omega_{B/A} \otimes_B (B \otimes_A C) \xrightarrow{\sim} \Omega_{B \otimes_A C/C}$$

   in $(B \otimes_A C)$-mod.

2. Show that the above equivalence can be promoted to a functor

$$\mathrm{Fun}(\Lambda_0^2, \mathcal{R}\mathrm{ing}) \to \mathrm{Fun}(\Delta^1, \mathcal{M}\mathrm{od}).$$

   Hint.[4]

Exercise 10. Suppose that $A \in \mathcal{R}\mathrm{ing}$ and $A \to A[x_1, \ldots, x_n]$ is the free A-algebra on n generators. Show that

$$\Omega_{A[x_1,\ldots,x_n]/A} \cong A^{\oplus n}$$

is the free A-module on n generators.

Exercise 11.

1. Consider the algebra $\mathbb{Z}[x] \to \mathbb{Z}$; $x \mapsto 0$. Show that

$$\Omega_{\mathbb{Z}/\mathbb{Z}[x]} \cong \mathbb{Z}[1]$$

   in $\mathbb{Z}$-mod.

[4]Cf. the proof of Proposition 8.

2. Given $A \in \mathcal{R}\text{ing}$ and $f \in \pi_0 A$ show that

$$\Omega_{(A/\!\!/f)/A} \cong (A/\!\!/f)[1]$$

in $A/\!\!/f$-mod.

3. Deduce that there is a cofibre sequence

$$(A/\!\!/f) \to \Omega_A \otimes_A (A/\!\!/f) \to \Omega_{A/\!\!/f}$$

in $A/\!\!/f$-mod where the first morphism is induced by $df \in \pi_0\Omega_A \cong \hom_{A\text{-mod}}(A, \Omega_A)$.

Exercise 12. For $A \to B \in \mathcal{A}\text{ni}(\mathcal{A}\text{lg})$ and $f \in \pi_0 B$ show that we have equivalences

$$\Omega_{B/A} \otimes_B B[{}^1\!/\!_f] \xrightarrow{\sim} \Omega_{B[{}^1\!/\!_f]/A}, \qquad\qquad \Omega_{B[{}^1\!/\!_f]/B} \cong 0.$$

Hint.[5] Hint.[6]

Corollary 13. *For any $A \in \mathcal{R}\text{ing}$ the assignment*

$$B \mapsto \Omega_{B/A}$$

is a Zariski sheaf on $\mathcal{A}\text{ff}_{/\operatorname{Spec}(A)}$.

Proof. By a claim in the topology lecture, it suffices to show that

$$\begin{array}{ccc} \Omega_{B/A} & \longrightarrow & \Omega_{B[{}^1\!/\!_f]/A} \\ \downarrow & & \downarrow \\ \Omega_{B[{}^1\!/\!_g]/A} & \longrightarrow & \Omega_{B[{}^1\!/\!_{fg}]/A} \end{array}$$

is a cartesian square (in $\mathcal{S}$) for any $f, g \in \pi_0 B$ generating the unit ideal. Under the equivalences of Exericise 12 this becomes

$$\begin{array}{ccc} \Omega_{B/A} & \longrightarrow & \Omega_{B/A} \otimes B[{}^1\!/\!_f] \\ \downarrow & & \downarrow \\ \Omega_{B/A} \otimes B[{}^1\!/\!_g] & \longrightarrow & \Omega_{B/A} \otimes B[{}^1\!/\!_{gf}] \end{array}$$

so it suffices to show that

$$\begin{array}{ccc} B & \longrightarrow & B[{}^1\!/\!_f] \\ \downarrow & & \downarrow \\ B[{}^1\!/\!_g] & \longrightarrow & B[{}^1\!/\!_{fg}] \end{array}$$

is cartesian. We did this in an earlier lecture. ∎

[5] Recall that $B[{}^1\!/\!_f] \otimes_B B[{}^1\!/\!_f] \cong B[{}^1\!/\!_f]$.

[6] Use Exercise 9.

3 Finiteness conditions

Consider the following classes of objects in $\mathcal{C}\mathrm{plx}(R)_{\geq 0}$ for R a classical ring.

1. Chain complexes of the form
$$(\cdots \to 0 \to 0 \to R^{\oplus n_i} \to \cdots \to R^{\oplus n_1} \to R^{\oplus n_0}) \tag{1}$$
where $R^{\oplus n_0}$ is in degree zero. Starting with $R = (\ldots \to 0 \to R)$ concentrated in degree zero, and repeatedly applying finite sum and cone (i.e., repeatedly taking finite colimits in the derived category), we can construct any chain complex of the form (1).
2. Chain complex of the form
$$P = (\cdots \to 0 \to 0 \to P_i \to \cdots \to P_1 \to P_0) \tag{2}$$
where P_0 is in degree zero, and there is a finite set of elements $f_1, \ldots, f_n \in R$ such that $\{\mathrm{Spec}(R[f_i^{-1}]) \to \mathrm{Spec}(R)\}_{i \in I}$ is a Zariski covering and each localisation $P \otimes_R R[f_i^{-1}] \in \mathcal{C}\mathrm{plx}_{\geq 0}(R[f_i^{-1}])$ is of the form (1). In other words, each P_i is a projective R-module of finite rank [Stacks Project, 00NX]. Note, for any complex of the form (2) there exists a complex C of the form (1) such that $C = P \oplus Q$ for some Q. That is, we can build any P using finite sums, cones, and direct summands starting from $R = (\ldots \to 0 \to 0 \to R)$.
3. Chain complexes C that we can approximate by ones of the form (2). Concretely, C such that for all n there is P of the form (2) and an equivalence $\tau_{\leq n} C \cong \tau_{\leq n} P$. That is, chain complexes quasi-isomorphic to one of the form
$$P = (\cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0) \tag{3}$$
where each P_i is a projective R-module of finite rank.

These three classes all have their uses. The second two are related to compact objects in the following way.

Proposition 14 ([DAG, pg.19]). *Let $A \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ and $M \in A$-mod. The following are equivalent.*

1. *$M \in A$-mod is compact. That is,* $\mathrm{Map}(M, -)$ *commutes with filtered colimits.*
2. *M is in the smallest* stable *subcategory of A-mod which:*
   (a) *contains A, and*
   (b) *is closed under direct summand.*

Exercise 15. Prove the classical version of Proposition 14. That is, prove that if A is a classical ring and $M \in A\text{-mod}^{\heartsuit}$ is a classical A-module, then $\hom_{A\text{-mod}^{\heartsuit}}(M, -)$ commutes with filtered colimits if and only if M is a direct summand of an A-module of the form $\mathrm{coker}(A^{\oplus m} \to A^{\oplus n})$.

Exercise 16. Prove Proposition 14($2 \Rightarrow 1$). Note that the smallest *stable* subcategory of A-mod containing A is the same as the smallest subcategory of A-mod containing A and closed under:

1. finite colimits
2. desuspension $M \mapsto M[-1]$.

Exercise 17 (Harder)**.** Prove Proposition 14(1 $\Rightarrow$ 2).

The cotangent complex detects compact objects.

Proposition 18 ([DAG, Prop.3.2.14, Prop.3.2.18])**.** *Take $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Then B is compact as an object of $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$ if and only if $\Omega_{B/A}$ is compact as an object of B-mod and $\pi_0 A \to \pi_0 B$ is of finite presentation (in the classical sense, i.e., $\pi_0 B = (\pi_0 A)[x_1, \dots, x_n]/\langle f_1, \dots, f_c\rangle$ for some $f_i \in \pi_0 A[x_1, \dots, x_n]$.)*

The cotangent complex also detects finitely presented objects.

Definition 19. If $M \in A$-mod is in the smallest *stable* subcategory of A-mod which contains A (but is not necessarily closed under direct summands) then we say M is *finitely presented*. Similarly, if $B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$ is in the smallest subcategory which:

1. contains $A[x]$, and
2. is closed under finite colimits.

We say that B is *finitely presented*.

Proposition 20 ([DAG, Prop.3.2.14, Prop.3.2.18])**.** *Take $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Then B is finitely presented in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$ if and only if $\Omega_{B/A}$ is finitely presented in B-mod and $\pi_0 A \to \pi_0 B$ is of finite presentation in the classical sense.*

The final smallness condition we discuss needs the notion of truncation.

Definition 21. The full subcategory of objects $M \in A$-mod such that $\pi_i M = 0$ for all $i > n$ is denoted

$$A\text{-mod}_{\leq n}.$$

Lemma 22. *The category A-mod$_{\leq n}$ is presentable and the inclusion admits a left adjoint*

$$\tau_{\leq n} : A\text{-}mod \to A\text{-}mod_{\leq n}.$$

Proof. Repeatedly applying Σ (or $\Sigma^{-1} = \Omega$) induces an equivalence $\Sigma^n : A\text{-mod}_{\leq 0} \to A\text{-mod}_{\leq n}$, so for the first claim, it suffices to show that $A\text{-mod}_{\leq 0}$ is presentable. Using the adjunctions,

$$\mathcal{S} \rightleftarrows A\text{-mod}^{\mathrm{cn}} \rightleftarrows A\text{-mod}$$

one sees that $A\text{-mod}_{\leq 0}$ is the full subcategory of objects local with respect to the (image of the) map $S^1 \to *$ (in A-mod). That is, those objects M such that $\mathrm{Map}(*, M) \to \mathrm{Map}(S^1, M)$ is an equivalence. Therefore it is presentable. The second claim follows from the adjoint functor theorem since the inclusion $A\text{-mod}_{\leq 0} \to A$-mod preserves limits and filtered colimits. ∎

Definition 23 ([DAG, pg.19, pg.23])**.** Let $A \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ and $M \in A$-mod.

1. We say that M is *almost perfect* if for every n there exists a compact object $M' \in A$-mod and an equivalences $\tau_{\leq n} M' \cong \tau_{\leq n} M$.
2. Similarly, we say that $B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$ is *almost finitely presented* if for every n there exists a finitely presented A-algebra B' and a morphism $B' \to B$ inducing isomorphisms $\pi_m B' \to \pi_m B$ for all $m \leq n$.

Remark 24. The following are equivalent, [DAG, Prop.2.5.7].

1. The truncation $\tau_{\leq n} M$ is a compact object of $D(A)_{\leq n}$.
2. There exists a compact object $N \in D(A)_{\leq n}$ and an equivalence $\tau_{\leq n} N \cong \tau_{\leq n} M$.

Proposition 25 ([DAG, Prop.3.2.14, Prop.3.2.18])**.** *Take $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Then B is almost finitely presented in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$ if and only if $\Omega_{B/A}$ is almost finitely presented in B-mod and $\pi_0 A \to \pi_0 B$ is of finite presentation in the classical sense.*

Remark 26. So to summarise, we have the following equivalences.

$$\begin{array}{rcl} f \text{ is finitely presented} & \iff & (\mathrm{FP}_0) \text{ holds and } \Omega_{B/A} \text{ is finitely presented.} \\ f \text{ is locally finitely presented} & \iff & (\mathrm{FP}_0) \text{ holds and } \Omega_{B/A} \text{ is perfect.} \\ f \text{ is almost finitely presented} & \iff & (\mathrm{FP}_0) \text{ holds and } \Omega_{B/A} \text{ is almost perfect.} \end{array}$$

Where (FP_0) means $\pi_0 A \to \pi_0 B$ is of finite presentation in the classical sense. I.e., $\pi_0 B = (\pi_0 A)[x_1, \dots, x_n]/\langle f_1, \dots, f_c \rangle$ for some $f_i \in \pi_0 A[x_1, \dots, x_n]$.

4 Deformation theory

It is often useful and natural to consider families of algebraic objects X indexed by some algebraic variety T. For example, projecting the algebraic variety $X = \{(x, y, t) \in \mathbb{C} \mid xy = t\}$ to the t component we can consider it as a family hyperbole $X_t \subseteq \mathbb{C}^2$ indexed by $t \in \mathbb{C} = T$ degenerating to the axes $xy = 0$ at $t = 0$.

Conversely, we can ask the following.

Question 27. Given an algebraic variety X_0, what kind of families $p : X \to B$ exist with $X_0 = p^{-1}(b_0)$ for some $b_0 \in B$.

The question is local around b_0, and the standard sequence of reductions is:

1. Replace B with the local scheme $\mathrm{Spec}(\mathcal{O}_{B,b_0})$.
2. Replace the local scheme $\mathrm{Spec}(\mathcal{O}_{B,b_0})$ with the formal scheme $\mathrm{Spec}(\mathcal{O}_{B,b_0})^{\wedge}_{b_0} = \mathrm{Spec}(\varprojlim_n \mathcal{O}_{B,b_0}/\mathfrak{m}^n)$.
3. Consider each $\mathcal{O}_{B,b_0}/\mathfrak{m}^n$ one at a time. That is, given a family over $\mathrm{Spec}(\mathcal{O}_{B,b_0}/\mathfrak{m}^n)$, when does it extend to a family over $\mathrm{Spec}(\mathcal{O}_{B,b_0}/\mathfrak{m}^{n+1})$.

Restricting to affine schemes, we are then lead to the following definition and question.

Definition 28. Suppose $A \to B \in \mathcal{R}\mathrm{ing}$ and $M \in B\text{-mod}^{\heartsuit}$. A *square zero extension* (of B by M) is an A-algebra $\tilde{B}$ and an ideal $I \subseteq \widetilde{B}$ such that:

1. $I^2 = 0$,
2. $\widetilde{B}/I \cong B$, and
3. $I \cong M$ as B-algebras.

Question 29. Classify square zero extensions of B by M.

The cotangent complex answers this question.

Warning 30. Both Example 31 and Example 32 are mean't as expositional tools. They contain content which did not appear in this course (e.g., modelling A-mod using chain complexes of A-modules via Dold-Kan, the theory of cdgas, using model categories of digarams to calculate (co)limits in quasi-cateories, ...).

Example 31. Let A be a classical ring, and suppose that $B' \to B$ is a square zero extension of classical A-algebras with kernel

$$I = \ker(B' \to B).$$

By rotation, this defines a cofibre sequence

$$(I \to) \qquad B' \to B \xrightarrow{\delta} I[1], \qquad\qquad \in A\text{-mod}.$$

in the category of A-modules. The morphism δ can be represented concretely by the weak equivalence[7] of chain complexes $[I{\to}B'] \xrightarrow{\sim} [0{\to}B]$. Then the cofibre sequence is as follows.

$$\left(\begin{bmatrix} 0 \\ \downarrow \\ I \end{bmatrix} \to \right) \qquad \begin{bmatrix} 0 \\ \downarrow \\ B' \end{bmatrix} \to \begin{bmatrix} I \\ \downarrow \\ B' \end{bmatrix} \xrightarrow{\delta} \begin{bmatrix} I \\ \downarrow \\ 0 \end{bmatrix}, \qquad\qquad \in A\text{-mod}.$$

Since A-mod is stable, one sees that as an A-module, B' can be reconstructed as the fibre of δ in A-mod,

$$B' \cong \mathrm{fib}(B \xrightarrow{\delta} I[1])$$

and this sets up a well-known bijection

$$\hom_{A\text{-mod}}(B, I[1]) \cong \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{extensions of the } A \text{ module } B \\ \text{by the } A \text{ module } I \end{array} \right\}.$$

Example 32. We can also recover the ring structure from Example 31. The following also works in the category $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$, but it is easier to see whats going on in cdgas (=commutative differential graded algebras, [HA, Def.7.1.4.1, Def.7.1.4.8]). In fact, 1-truncated[8] cdgas completely capture 1-truncated objects of $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$, [Drinfeld, On a notion of ring groupoid, 3.3.3].

Notice that since $I^2 = 0$, the chain complex $[I \overset{inc.}{\to} B']$, whose differential is inclusion, has a structure of commutative graded ring, where B' is in degree zero,

[7] Here, by $[M_1 \to M_0]$ we mean $[\cdots \to 0 \to 0 \to M_1 \to M_0]$.

[8] 1-truncated means all homotopy groups except π_0 and π_1 are zero.

and I is in degree one. Similarly, the complex of A-modules $[I \overset{0}{\to} B] = [I \to 0] \oplus [0 \to B]$ also has a commutative graded ring structure. Since the canonical morphism $[I \overset{inc.}{\to} B'] \to [I \overset{0}{\to} B]$ is surjective in each degree, it is a fibration in the model category of cdgas (cf.[HA, Prop.7.1.4.10]), so the cartesian square

$$\begin{array}{ccccc} B' = & \begin{bmatrix} 0 \\ \downarrow \\ B' \end{bmatrix} & \longrightarrow & \begin{bmatrix} 0 \\ \downarrow \\ B \end{bmatrix} & = B \\ & \searrow & & \searrow & \\ B \cong & \begin{bmatrix} inc. \ \begin{matrix} I \\ \downarrow \\ B' \end{matrix} \end{bmatrix} & \longrightarrow & \begin{bmatrix} \begin{matrix} I \\ \downarrow \\ B \end{matrix} \ 0 \end{bmatrix} & = B \oplus I[1] \end{array}$$

in the model category of cdgas is sent to a cartesian in the quasi-category of cdgas.[9] That is, we recover B', equipped with its ring structure, as the fibre product

$$B' = B \times_{B \oplus I[1]} B, \qquad \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$$

in the quasi-category $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$[10] As in the case of A-modules, this sets up a bijection

$$\hom_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A//B}}(B, B \oplus I[1]) \cong \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{square zero extensions} \\ \text{of } A\text{-algebras of } B \text{ by } I \end{array} \right\}, \tag{4}$$
$$\left(B \overset{s}{\to} B \oplus I[1] \right) \mapsto B \times_{B \oplus I[1]} B.$$

Definition 33 ([DAG, Def.3.3.1])**.** Let $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ and $M \in B\text{-mod}^{\mathrm{cn}}$. A *small extension of B by M* is a pullback in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A//B}$ of the form

$$\begin{array}{ccc} \tilde{B} & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow 0 \\ B & \xrightarrow[s]{} & B \oplus M[1] \end{array}$$

where $0 : B \to B \oplus M[1]$ is the morphism corresponding to $0 \to M[1]$ in $\mathcal{M}_B$ under the adjunction $B\text{-alg} \rightleftarrows B\text{-mod}^{\mathrm{cn}}$.

[9] In general, if $X \to Y \leftarrow Z$ are fibrations with Y fibrant, then the diagram is injectively fibrant and therefore $X \times_Y Z$ models the pullback in the associated quasi-category. Moreover, one can show that in fact, it suffices that X, Y, Z are fibrant and one of the morphisms is a fibration. [Dugger, A primer on homotopy colimits, Proposition 14.5]

[10] This is the quasi-category associated to the model category of cdgas under A, [HA, Prop.7.1.4.11], at least when $\mathbb{Q} \subseteq A$.

Exercise 34 (Harder)**.** Continue with the notation from Def.33, so $\tilde{B} = B \times_{B \oplus M[1]} B$. Show that there is a long exact sequence

$$\dots \pi_{n+1}(B) \xrightarrow{\sigma} \pi_n(M) \to \pi_n(\tilde{B}) \to \pi_n(B) \xrightarrow{\sigma} \pi_{n-1}(M) \to \dots$$

where the morphisms σ are induced by $s : B \to B \oplus M[1]$. Hint.[11]

Proposition 35 ([DAG, Prop.3.3.5])**.** *Let $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ be such that $B \in \mathcal{R}\mathrm{ing}$ (that is B is a classical ring). The functor*

$$(B\text{-}mod^{\heartsuit}[1])_{\Omega_{B/A}/} \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A//B}$$
$$(\Omega_{B/A} \xrightarrow{s} M) \mapsto B \times_{B \oplus M} B$$

is fully faithful, and its essential image consists of those $\tilde{B} \to B$ in $\mathcal{R}\mathrm{ing}_{A//B}$ such that $\pi_0 \tilde{B}$ is a square zero extension of $\pi_0 B$ by I, Definition 28.

Proposition 36 ([DAG, Prop.3.3.5])**.** *Choose $k > 0$, let $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ be such that $\pi_n B = 0$ for $n > k$. The functor*

$$(B\text{-}mod^{\heartsuit}[k{+}1])_{L_{B/A}/} \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A//B}$$
$$(\Omega_{B/A} \xrightarrow{s} M) \mapsto B \times_{B \oplus M} B$$

is fully faithful, and its essential image consists of those $\tilde{B} \to B$ in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A//B}$ such that

1. *$\pi_i \tilde{B} \xrightarrow{\sim} \pi_i B$ for $i \neq k$, and,*
2. *there is a short exact sequence of $\pi_0 B$-modules*

$$0 \to \pi_k M \to \pi_k \tilde{B} \to \pi_k B \to 0$$

Remark 37. Suppose $A \to B \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. We can apply the above proposition to $\tau_{\leq k-1} B$ and $M = (\pi_k B)[k]$. In this case, the proposition tells us that we can build $\tau_{\leq k} B$ out of the triple

$$(\tau_{\leq k-1} B, \quad \pi_k B, \quad \Omega_{B/A} \to (\pi_k B)[k]).$$

In $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A/}$ we have $B = \varprojlim_{n \in \mathbb{N}} \tau_{\leq n} B$ so in fact, B is equivalent to the data of the *classical* A-algebra $\pi_0 B$, together with the sequence

$$\underbrace{\Omega_{(\tau_{\leq 0} B)/A} \to (\pi_1 B)[1]}_{\in \tau_{\leq 0} B\text{-mod}}, \quad \underbrace{\Omega_{(\tau_{\leq 1} B)/A} \to (\pi_2 B)[2]}_{\in \tau_{\leq 1} B\text{-mod}}, \quad \underbrace{\Omega_{(\tau_{\leq 2} B)/A} \to (\pi_3 B)[3]}_{\in \tau_{\leq 2} B\text{-mod}}, \quad \dots$$

That is, the A-algebra B is determined by the discrete A-algebra $\pi_0 B$ and purely "linear" data, where "linear" means contained in some category of modules.

[11]Note that the forgetful functor $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})_{A//B} \to \mathcal{S}$ commutes with fibre products.