Derived Algebraic Geometry
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Lecture 9: Stable infinity categories

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References:
- Lurie, HA, §1
- Segal, "Categories and cohomology theories"
- Goerss, Jardine "Simplicial homotopy theory"

Generalised cohomology theories. The first goal in this lecture is to define *stabilisation* of a (pointed) quasi-category. Historically, the motivation to defining spectra and stabilisation was to represent cohomology theories. For example, to any abelian group A one can find a space $K(A, n)$ such that for CW complexes X, the set of morphisms up to homotopy are in canonical bijection with the singular cohomology groups

$$\hom(X, K(A, n)) \cong H^n_{sing}(X, A).$$

As n ranges over the natural numbers, the spaces $K(A, n)$ assemble into a *spectrum* $K(A)$ living in a category of *spectra* where all *generalised cohomology theories* are representable.

Fibre sequences. Generalised cohomology theories are not part of this course, so instead I propose the following motivation (hopefully more accesible). For any fibre product in the quasi-category[1] $\mathcal{S}_{*/}$ of the form $F = * \times_B E$ we have a long exact sequence

$$\cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \cdots \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B \qquad (1)$$

where the last three are pointed sets.[2] One question that stabilisation solves is:

Question 1. Can we continue this long exact sequence to the right?

Namely, can we build a category of objects behaving like spaces, but which have homotopy groups for all $n \in \mathbb{Z}$, and long exact sequences as above, but continuing into negative homotopy groups?

If $F = * \times_B E$ in $\mathcal{S}_{*/}$, then defining $\Omega B = * \times_B *$ we get a new fibre product $\Omega B = * \times_E F$. Using the equivalence $\pi_n \Omega B \cong \pi_{n+1} B$ one gets an isomorphism

[1] The category $\mathcal{S}_{*/}$ is the quasi-category of *pointed spaces*. Its objects are spaces $X \in \mathcal{S}$ equipped with a choice of base point $x : * \to X$. This will all be explained more concretely below.

[2] A sequence of morphisms of pointed sets $(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$ is *exact* if $f(X) = g^{-1}(\{z\})$.

between the long exact sequence associated to $\Omega B = * \times_E F$ and the shift of Eq.(1).

$$\cdots \to \underset{\cong \pi_{n+1}B}{\pi_n \Omega B} \to \pi_n F \to \pi_n E \to \underset{\cong \pi_n B}{\pi_{n-1}\Omega B} \to \cdots \to \pi_1 E \to \underset{\cong \pi_1 B}{\pi_0 \Omega B} \to \pi_0 F \to \pi_0 E \tag{2}$$

So a reasonable guess of how to proceed is to try and make Ω an equivalence of categories. This leads directly to the definition of the quasi-category of spectra

$$\mathrm{Sp} := \varprojlim(\cdots \xrightarrow{\Omega} \mathcal{S}_{*/} \xrightarrow{\Omega} \mathcal{S}_{*/} \xrightarrow{\Omega} \mathcal{S}_{*/}).$$

This will be described more carefully in Definition 32.

Stabilisation has many remarkable consequences. For example, a commutative square in Sp is a pullback square if and only if it is a pushout square. The second goal of this lecture is to discuss some of these (specifically, the one just mentioned).

Modules. The functor Ω has a left adjoint Σ. If Σ is fully faithful, then the original (pointed) quasi-category $\mathcal{M}$ embeds fully faithfully into its stabilisation $\mathcal{M} \subseteq \mathrm{Sp}(\mathcal{M})$. This happens with the categories of modules we have beens studying

$$R\text{-mod}^{\mathrm{cn}} \subseteq R\text{-mod} := \mathrm{Sp}(R\text{-mod}^{\mathrm{cn}}).$$

Our third goal in this lecture is to prove (or at least sketch a proof) that $\Sigma : R\text{-mod}^{\mathrm{cn}} \to R\text{-mod}^{\mathrm{cn}}$ is fully faithful. Using the adjunction

$$\Sigma^\infty : R\text{-mod}^{\mathrm{cn}} \rightleftarrows R\text{-mod} : \Omega^\infty,$$

the fact Σ^∞ is fully faithful, and the fact that in R-mod pushout and pullback squares are the same, one deduces that a *pushout* of modules gives rise to a long exact sequence of homotopy groups

$$\cdots \to \pi_n M \to \pi_n M' \oplus \pi_n M'' \to \pi_n(M' \sqcup_M M'') \to \pi_{n-1} M \to \dots$$

Recall that we used this long exact sequence in a previous lecture to describe the homotopy groups of the quotient ring $R/\!\!/f$.

Here is the outline:

1. Pointed quasi-categories.
2. Stable quasi-categories.
3. Stabilisation.
4. Stabilisation of quasi-categories of modules.

1 Pointed quasi-categories

Definition 2. Let C be a quasi-category and $X \in C$ an object.

1. X is *initial* if $\mathrm{Map}(X, Y)$ is contractible for all Y.

2. X is *final* if $\mathrm{Map}(W, X)$ is contractible for all W.
3. C is *pointed* if it admits both an initial object $\varnothing$ and a final object $*$, and an equivalence $\varnothing \cong *$.

Example 3.
1. If C is (the nerve of) a 1-category then an object is initial / final in the quasi-categorical sense if and only if it is initial / final in the usual sense.
2. $* \in \mathcal{S}$ is a final object (one can see this using the fact that $\mathrm{Map}_{\mathcal{S}}(X, *) \cong \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X, *) = *$).
3. For any $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$, the zero object $\iota R \in R\text{-mod}^{\mathrm{cn}}$ is both initial and final. (One can see this using the adjunctions (ι, π) and (π, ι).)[3]

Exercise 4. Show that the full subcategory of initial objects (resp. final objects) is either empty, or a contractible Kan complex. Hint.[4] Hint.[5] Hint.[6]

Notation 5. We will sometimes use

$$\varnothing, \qquad \text{(resp. } *)$$

to represent a chosen initial, resp. final, object. If they are equivalent (i.e., C is pointed) we sometimes write

$$0$$

and call it a *zero object*. We will also sometimes write 0 for any morphism which factors through a 0 object and call it a *zero morphism*

Remark 6. Something missing[7] from the lecture on adjunctions is the following characterisation:

Suppose that C, D are quasi-categories and $F : C \rightleftarrows D : G$ are functors. Then (F, G) form an adjunction if and only if there exists a map $: \mathrm{id}_C \to GF$ such that for all $X \in C, Y \in D$, the morphism

$$\mathrm{Map}(FX, Y) \to \mathrm{Map}(GFX, GY) \to \mathrm{Map}(X, GY)$$

[3]Note that $R\text{-mod}^{\mathrm{cn}} \subseteq \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$ is not full. We only take morphisms which project to id_R. In other words,

$$\mathrm{Map}_{R\text{-mod}^{\mathrm{cn}}}(M, N) = \{\mathrm{id}_R\} \times_{\mathrm{Map}_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})}(R,R)} \mathrm{Map}_{\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})}(M, N).$$

So if, for example, $N = \iota R$, under the adjunction $\mathrm{Map}(M, \iota R) \overset{\pi}{\to} \mathrm{Map}(\pi M, R) = \mathrm{Map}(R, R)$, this becomes $\{\mathrm{id}_R\} \times_{\mathrm{Map}(R,R)} \mathrm{Map}(R, R) \cong *$.

[4]Recall that the homotopy category hC of a quasi-category C has morphisms $\pi_0 \mathrm{Map}(X, Y)$.

[5]Recall further that a quasi-category is a Kan complex if and only if all morphisms are equivalences.

[6]Finally recall that a Kan complex is contractible if and only if any two objects are equivalent, and all mapping spaces are contractible.

[7]We did not include it because the "composition" $\mathrm{Map}(X, Y) \to \mathrm{Map}(X, Y')$ map (associated to a morphism $Y \to Y'$) is not so straight-forward in quasi-categories and we did not want to go into that level of subtlety at that time.

induced by composition with $\ : X \to GFX$ is an equivalence of spaces, [HTT, Prop.5.2.2.8]. Dually, if and only if there exists a map $\varepsilon : FG \to \mathrm{id}_D$ such that the corresponding morphism of mapping spaces is an equivalence.

Using this criterion, one sees that for any final object Y, the constant functor $\mathrm{Fun}(\varnothing, C) \cong * \xrightarrow{Y} C$ corresponding to Y is right adjoint to the canonical functor $C \to *$. That is, any final object is a limit of the empty diagram. Dually, any initial object is a colimit of the empty diagram.

$$\varnothing = \varinjlim_{\varnothing}, \qquad * = \varprojlim_{\varnothing}$$

It is easy to manufacture categories with initial and final objects.

Definition 7. Let C be a quasi-category and X an object. The n-simplices

$$(C_{X/})_n \subseteq C_{n+1}$$

of the *under category* $C_{X/}$ are those $(n+1)$-simplices of C whose initial vertex is X. The simplicial structure is induced by the map $\Delta \to \Delta$; $[n] \mapsto \{* < 0 < 1 < \cdots < n\}$; $\hom([n], [m]) \to \hom([n+1], [m+1])$. The *over category* is[8]

$$((C^{op})_{X/})^{op}.$$

Exercise 8. Describe the simplices and simplicial structure of $C_{/X}$ explicitly.

Exercise 9. Suppose C is a quasi-category and $X \in C$ an object.

1. Show that the identity map $(X \xrightarrow{\mathrm{id}} X) \in C_{X/}$ is an initial object. Hint.[9] Hint.[10]
2. Show that if C admits a final object $*$ and $X \to *$ is any morphism, the object $(X \to *)$ is a final object in $C_{X/}$.

Corollary 10. *The under category $\mathcal{S}_{*/}$ is pointed.*

Notation 11. If C is a quasi-category with a final object $*$, the under category $C_{*/}$ is usually written

$$C_{*/} := C_*$$

to simplify the notation.

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[8]Recall that if K is a simplicial set, then K^{op} is the simplicial set $\Delta \xrightarrow{op} \Delta \xrightarrow{K} \mathcal{S}\mathrm{et}$ where $op : \{x_0 < x_1 < \cdots < x_1\} \mapsto \{x_n < \cdots < x_1 < x_0\}$.

[9]Recall that if C is a quasi-category, then each $\mathrm{Map}^L(X, Y)$ is a Kan complex.

[10]In particular, $\mathrm{Map}^L(X, Y)$ is contractible if and only if every $\partial\Delta^n \to \mathrm{Map}^L(X, Y)$ extends to some $\Delta^n \to \mathrm{Map}^L(X, Y)$.

2 Stable quasi-categories

2.1 Definition

Definition 12. An *additive category* is a 1-category $\mathcal{A}$ such that:

1. $\mathcal{A}$ admits all finite coproducts and finite products (including empty ones),
2. the canonical comparisons (see Remark 14)

$$\sqcup_{i=1}^{n} X_i \to \sqcap_{i=1}^{n} X_i$$

   are isomorphisms (for $n \in \mathbb{N}_{\geq 0}$), and
3. each $\hom(X, Y)$ equipped with the canonical abelian monoid structure (see Exercise 15)

$$* \to \hom(X,Y), \qquad \hom(X,Y) \times \hom(X,Y) \to \hom(X,Y)$$

   is a group.

Notation 13. When $\sqcup = \sqcap$ we will sometimes write $\oplus$ for this operation.

Remark 14. The case $n = 0$ says that $\varnothing \cong *$. Using this isomorphism, we obtain maps $0_{i,j} : X_i \to * \stackrel{\sim}{\leftarrow} \varnothing \to X_j$. Combining these with the $\mathrm{id}_i : X_i \to X_i$ we obtain the map $\sqcup_{i=1}^{n} X_i \to \sqcap_{i=1}^{n} X_i$ in the above definition.

Exercise 15. For objects A, B in an additive category $\mathcal{A}$, consider the map

$$\begin{array}{ccc} \hom(A,B) \times \hom(A,B) & \overset{+}{\dashrightarrow} & \hom(A,B) \\ \downarrow{\scriptstyle \times} & & \uparrow{\scriptstyle \pi \circ (-) \circ \delta} \\ \hom(A \times A, B \times B) & \overset{\cong}{\longleftarrow} & \hom(A \times A, B \sqcup B) \end{array}$$

where $\delta : A \to A \times A$ is the diagonal and $\pi : B \sqcup B \to B$ is the folding map. Show that $+$ gives $\hom(A, B)$ the structure of an abelian monoid. Hint.[11]

Exercise 16 (Harder)**.** Suppose that A is a small additive category and consider the category $\mathrm{PSh}_\Sigma(A, \mathcal{S}\mathrm{et})$ of functors $F : A^{op} \to \mathcal{S}\mathrm{et}$ which send finite coproducts to finite products.

1. Show that the final object $* \in \mathrm{PSh}_\Sigma(A, \mathcal{S}\mathrm{et})$ is also initial.
2. Suppose that $X, Y \in \mathrm{PSh}_\Sigma(A, \mathcal{S}\mathrm{et})$ are representable. Show that $X \times Y \cong X \sqcup Y$ in $\mathrm{PSh}_\Sigma(A, \mathcal{S}\mathrm{et})$. Hint.[12]
3. Show that $\mathrm{PSh}_\Sigma(A, \mathcal{S}\mathrm{et})$ is also additive. Hint.[13] Hint.[14]

[11] To define the zero morphism, consider the canonical maps $0 = \varnothing \to B$ and $A \to * = 0$.

[12] By Yoneda, it suffices to show that $\mathrm{Map}(X \times Y, G) \cong \mathrm{Map}(X, G) \times \mathrm{Map}(Y, G)$ for any $G \in \mathrm{PSh}_\Sigma(A, \mathcal{S})$.

[13] Note that every object of $\mathrm{PSh}_\Sigma(A, \mathcal{S})$ can be written as a termwise (i.e., $(\varinjlim F_\gamma)(X) = \varinjlim F_\gamma(X)$) sifted colimit of representable objects.

[14] Recall also that in $\mathcal{S}$ we have $(\varinjlim_\gamma X_\gamma) \times (\varinjlim_\mu Y_\mu) \cong \varinjlim_\gamma \varinjlim_\mu (X_\gamma \times Y_\mu)$.

Definition 17. A quasi-category C is *stable* if it satisfies the following conditions:

(Sta0) It is pointed. That is, it admits both an initial object $\varnothing$ and a final object $*$ and an equivalence $\varnothing \cong *$.

(Sta1) It admits *fibres* and *cofibres*. That is, for every $f : X \to Y$, both $X \times_Y 0$ and $0 \sqcup_X Y$ exist.

(Sta2) A commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array} \tag{3}$$

is cartesian if and only if it is cocartesian.

Example 18. The category $*$ with a single object and single (identity) morphism is stable.

Notation 19.

1. If $X \overset{f}{\to} Y \overset{g}{\to} Z$ fits into a bicartesian square as in Eq.(3) we call it a *fibre sequence* or *cofibre sequence*, write

$$X =: \mathrm{fib}(Y \overset{g}{\to} Z), \qquad \mathrm{cof}(X \overset{f}{\to} Y) := Z$$

and call these respectively, the *fibre* and *cofibre*.

2. In the special case where Y is also a zero object we will write $\Omega Z = X$ and $\Sigma X = Z$. That is,

$$\Omega Z := 0 \times_Z 0 = \mathrm{fib}(0 \to Z), \qquad \Sigma X := 0 \sqcup_X 0 = \mathrm{cof}(X \to 0).$$

Remark 20. We will see two alternative set of axioms for a quasi-category to be stable, Exercise 22 and Remark 31. The axiom (Sta2) above has two advantages:

1. It highlights the connection between stable quasi-categories and *triangulated categories* (which don't appear in this course, but are the historical precedent of stable quasi-categories).
2. It highlights the analogy with abelian categories. Namely, (Sta2) is an analogue of the abelian category axiom: for any morphism $A \overset{f}{\to} B$ we have

$$\ker(B \to \mathrm{coker}(f)) \cong \mathrm{coker}(\ker(f) \to A).$$

2.2 Basic properties

In the following exercises we will show the following basic facts. Suppose that C is a stable quasi-category.

1. The functors $\Sigma : C \to C$ and $\Omega : C \to C$ are inverse equivalences. Moreover, in fact, we can replace (Sta2) with this axiom.

2. The *rotations* $\Omega Z \to X \to Y$ and $Y \to Z \to \Sigma X$ of fibre sequences are also fibre sequences.
3. C admits all finite colimits and finite limits.
4. Every cartesian square is cocartesian and vice versa.
5. A square

$$\begin{array}{ccccc} \mathrm{fib} f & \longrightarrow & X & \xrightarrow{f} & Y \\ {\scriptstyle \phi}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathrm{fib}(g) & \longrightarrow & Z & \xrightarrow[g]{} & W \end{array}$$

is bicartesian if and only if the comparison morphism ϕ is an equivalence (and similarly for cofibres).
6. Finite coproducts are equivalent to finite products.
7. The homotopy category hC is additive.

Exercise 21. Recall that if C is a category admitting I-shaped limits, the limit adjunction factors as

$$\delta : C \rightleftarrows \mathrm{Fun}(I^{\triangleleft}, C) \rightleftarrows \mathrm{Fun}(I, C) : \varprojlim$$

and similar for colimits.[15]

1. Show that for a general pointed quasi-category C, the pair

$$(\Sigma, \Omega)$$

form an adjunction. Hint.[16] Hint.[17] Hint.[18] Hint.[19]
2. Show that when C is stable, Σ and Ω are inverse equivalences. Hint.[20] Hint.[21]

Exercise 22.

1. Given a morphism $X \xrightarrow{f} Y$ in a pointed quasi-category C, show that there are factorisations

$$X \to \mathrm{fib}\Big(Y \to \mathrm{cof}(f)\Big) \to \Omega\Sigma X \to \mathrm{fib}\Big(\Omega\Sigma Y \to \Omega\Sigma\mathrm{cof}(f)\Big)$$

[15] Here, in the limit case, the left adjoints are induced by composition with $I \to I^{\triangleleft} \to *$; and the right adjoint $C \leftarrow \mathrm{Fun}(I^{\triangleleft}, C)$ is composition with the inclusion of the initial object $* \to I^{\triangleleft}$.

[16] Recall that $(\Lambda^2_2)^{\triangleleft} = \Delta^1 \times \Delta^1 = (\Lambda^2_0)^{\triangleright}$.

[17] Consider the full subcategory $K \subseteq \mathrm{Fun}(\Lambda^2_0, C)$ consisting of diagrams of the form $\begin{array}{cc} X & \to 0 \\ \downarrow & \\ 0 & \end{array}$. Note that evaluation at 0 restricts to an equivalence $K \xrightarrow{\sim} C$. Indeed, since 0 is final, the forgetful functor $C_{/0} \to C$ is an equivalence, and $K = C_{/0} \times_C C_{/0}$. Do the analogous thing for $M \subseteq \mathrm{Fun}(\Lambda^2_2, C)$. is fully faithful.

[18] Consider the full subcategory $L \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, C)$ consisting of diagrams of the form $\begin{array}{ccc} X & \to & 0 \\ \downarrow & & \downarrow \\ 0 & \to & Y \end{array}$.

[19] Consider the canonical functors $K \leftarrow L \to M$.

[20] Use (Sta2).

[21] Consider the essential images of $K \to L$ and $L \leftarrow M$ from the previous hints.

2. Suppose that C satisfies (Sta0), (Sta1), and:

(Sta2′) Ω is an equivalence.

Show that C is stable. That is, show that C satisfies (Sta2). Hint.[22]

Exercise 23. Suppose that $X \to Y \to Z$ is a fibre sequence. Show that there are fibre sequences of the form $\Omega Z \to X \to Y$ and $Y \to Z \to \Sigma X$.

Exercise 24. Suppose that C is a stable quasi-category. Suppose that we have a fibre sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

Show that if $g = 0$, then $X \cong (\Omega Z) \times Y$. Similarly, show that if $f = 0$ then there is an equivalence $Z \cong Y \sqcup \Sigma X$. Hint.[23] Hint.[24]

Deduce that C admits finite products and finite coproducts. Hint.[25]

Exercise 25. Suppose that X, Y are two objects in a stable quasi-category.

1. Show that $X \sqcup_{X \sqcup Y} Y \cong 0$.
2. Deduce that

$$X \sqcup Y \cong X \times Y.$$

Hint.[26] Hint.[27]

Exercise 26. Suppose that C is a stable quasi-category. Using the fact that $C \to hC$ preserves finite products and finite coproducts, show that hC is additive.

Exercise 27. Recall that, for morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in a general quasi-category, if it exists, the following pullback P is equivalent to the equaliser $\mathrm{eq}(X \underset{g}{\overset{f}{\rightrightarrows}} Y)$

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \delta \\ X & \xrightarrow[(f,g)]{} & Y \times Y \end{array} \qquad\qquad P \cong \mathrm{eq}(X \underset{g}{\overset{f}{\rightrightarrows}} Y)$$

That is, if the pullback square on the left exists, then the equaliser exists. Show that if C is a stable quasi-category, then C admits equalisers. Dually, show that C admits coequalisers. Hint.[28]

[22] Use Exercise 21.
[23] Note that in general, $A \times B = A \times_* B$ and $A \sqcup B = A \sqcup_\varnothing B$.
[24] Use the 2-out-of-3 property for (co)cartesian squares.
[25] Use Exercise 21
[26] Note that in general, $A \times B = A \times_* B$ and $A \sqcup B = A \sqcup_\varnothing B$.
[27] Use 2-out-of-3 for (co)cartesian squares.
[28] Recall that we have seen in a previous exercise that there is a fibre sequence of the form $Y \to Y \times Y \to Y$.

Remark 28. Since a quasi-category admits all finite limits if and only if it admits finite products are equalisers, [HTT, Prop.4.4.3.2], we deduce that stable quasi-categories admit all finite limits. Dually, they admit all finite colimits.

In particular, stable quasi-categories admit all pullbacks and pushouts. (Alternatively, we could have deduced this directly since $X \times_Z Y \cong \mathrm{eq}(X \times Z \times Y \rightrightarrows Z \times Z)$ in general).

Exercise 29 (Harder. [HTT, Prop.1.1.3.4])**.** We will show that all pushout squares are pullback squares and vice versa. By duality it suffices to prove only the former. E.g., pullback squares in C^{op} correspond to pushout squares in C.

1. If C is a stable quasi-category and I is a small quasi-category, show that $\mathrm{Fun}(I, C)$ is stable. In particular, it admits all finite limits and finite colimits.
2. Suppose that C is a stable quasi-category and $D \subseteq C$ a full sub-quasi-category which is closed under Σ and finite limits (calculated in C). Show that D is stable (and the inclusion preserves limits and colimits). Similarly, if $E \subseteq C$ is closed under Ω and finite colimits then show E is stable.
3. Let C be a stable quasi-category and consider the full subcategories $D, E \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, C)$ consisting of those squares which are pullbacks, resp. pushouts. Show that D and E are both stable. Deduce that their intersection $D \cap E$ is also stable. Hint.[29]
4. Finally, we will show that $D = D \cap E = E$. First note that any square of the form

$$\begin{array}{ccc} A & \xrightarrow{\cong} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow[\cong]{} & B' \end{array} \tag{4}$$

is in $D \cap E$. Given an arbitrary square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z \sqcup_X Y \end{array} \tag{5}$$

in E, write it as a pushout of a diagram in $D \cap E$ of shape $\swarrow\searrow\swarrow\searrow$. Deduce that (5) is in $D \cap E$, and therefore in D. That is, (5) is a pullback square.

Exercise 30. Show that a square $(*)$

$$\begin{array}{ccccc} \mathrm{fib} f & \longrightarrow & X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow & (*) & \downarrow \\ \mathrm{fib}(g) & \longrightarrow & Z & \xrightarrow[g]{} & W \end{array}$$

[29] Recall that Σ and Ω are both equivalences.

is bicartesian if and only if the comparison morphism ϕ is an equivalence (and similarly for cofibres). Hint.[30]

Remark 31. By Exercise 29 we obtain a third characterisation of stable quasi-categories: A quasi-category is stable if and only if it satisfies:

(Sta0) C is pointed.

(Sta1″) C admits all finite limits and finite colimits.

(Sta2″) All pullback squares are pushout square and vice versa.

3 Stabilisation

We haven't seen any examples yet (apart from the zero category), but we can manufacture stable quasi-categories relatively easily.

Definition 32 (HA, Def.1.4.2.8, Prop.1.4.2.21)**.** Suppose that C is a pointed quasi-category admitting pushouts and pullbacks of the form $0 \sqcup_X 0$ and $0 \times_Y 0$. Let $D \subseteq \mathbb{Z} \times \mathbb{Z}$ be the sub-poset

$$D = \{(i, i{-}1), (i, i), (i, i{+}1) \mid i \in \mathbb{Z}\}.$$

Write $\mathrm{Sp}(C) \subseteq \mathrm{Fun}(D, C)$ for the full subcategory of those diagrams $\mathcal{E} : D \to C$ such that

$$\mathcal{E}_{i,i-1} \cong 0 \cong \mathcal{E}_{i,i+1}$$

and all squares

$$\begin{array}{ccc} \mathcal{E}_{i,i} & \longrightarrow & \mathcal{E}_{i+1,i} (\cong 0) \\ \downarrow & & \downarrow \\ (0 \cong) \mathcal{E}_{i,i+1} & \longrightarrow & \mathcal{E}_{i+1,i+1} \end{array}$$

are cartesian.

Proposition 33 (HA, Cor.1.4.2.17)**.** *Suppose that C is a pointed quasi-category admitting finite limits. Then $\mathrm{Sp}(C)$ is stable.*

Sketch of proof. First suppose that C is presentable. Then $\mathrm{Fun}(D, C)$ is also presentable. Since there is a small set of morphisms[31] R of $\mathrm{Fun}(D, C)$ such that $\mathrm{Sp}(C)$ is the full subcategory of R-local objects,[32] we see that $\mathrm{Sp}(C)$ is also presentable, Lecture 6, Def.6, or [HTT, Thm.5.5.1.1, Prop.5.5.4.2]. In particular, $\mathrm{Sp}(C)$ admits all small colimits and small limits,[33] so we have (Sta1). Since C is pointed, so is

[30] Use Exercise 29.

[31] For $X \in C$, and $d \in D$ let $X_{\geq d}$ be the diagram with $X_{\geq d}(e) = \begin{cases} 0 & e<d \\ X & e \geq d \end{cases}$ and similarly, for $X_{>d}$. Then take R to be the set of morphisms of the form $0 \to X_{\geq(i,i+1)}$, $0 \to X_{\geq(i+1,i)}$, and $X_{>(i,i)} \to X_{\geq(i,i)}$ as X ranges over a set of generators for C.

[32] Recall that an object X is *R-local* if $\mathrm{Map}(f, X)$ is an equivalence for all $f \in R$.

[33] Lecture 6, Exercise 9

$\mathrm{Sp}(C)$, so we have (Sta0). Finally, we also have (Sta2'), since Ω (which is calculated objectwise) has inverse Ω^{-1} sending $\mathcal{E}_{\bullet,\bullet}$ to $\mathcal{E}_{\bullet+1,\bullet,+1}$.

If C is not presentable, then consider the Yoneda embedding $j : C \cong C_{*/} \to \mathrm{PSh}(C, \mathcal{S})_{*/}$. The latter is a pointed presentable quasi-category so $\mathrm{Sp}(\mathrm{PSh}(C, \mathcal{S})_{*/})$ is stable. Yoneda preserves limits, so we can identify $\mathrm{Sp}(C)$ with the full subcategory of $\mathrm{Sp}(\mathrm{PSh}(C, \mathcal{S})_{*/})$ consisting of objectwise representables. Inside $\mathrm{Sp}(\mathrm{PSh}(C, \mathcal{S})_{*/})$ the subcategory $\mathrm{Sp}(C)$ is closed under finite limits and Σ (which we previously identified with $\mathcal{E}_{\bullet,\bullet} \mapsto \mathcal{E}_{\bullet+1,\bullet,+1}$). So by Exercise 29(2), $\mathrm{Sp}(C)$ is stable. □

Spectra are the universal way to stabilise an quasi-category in the following sense.

Proposition 34 ([HA, Cor.1.4.2.23])**.** *Let C be a quasi-category which admits finite limits, and T a stable quasi-category. Then composition with the functor*

$$\begin{aligned} \Omega^\infty : \mathrm{Sp}(C) &\to C \\ \mathcal{E} &\mapsto \mathcal{E}_{0,0} \end{aligned}$$

induces an equivalence of quasi-categories

$$\mathrm{Fun}^{Lex}(T, \mathrm{Sp}(C)) \to \mathrm{Fun}^{Lex}(T, C)$$

where Fun^{Lex} *means the full subcategory of functors sending finite limits to finite limits, i.e.,* left exact *functors.*

Sketch of proof. In some (quite concrete) sense, our definition of $\mathrm{Sp}(C)$ is a model for the limit $\varprojlim(\dots \stackrel{\Omega}{\to} C \stackrel{\Omega}{\to} C \stackrel{\Omega}{\to} C)$. So we have $\mathrm{Fun}(T, \mathrm{Sp}(C)) \cong \varprojlim \mathrm{Fun}(T, C)$. If $\phi : T \to C$ preserves finite limits then we get a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\Omega} & T \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ C & \xrightarrow{\Omega} & C \end{array}$$

That is, the transition maps $\mathrm{Fun}^{Lex}(T, C) \to \mathrm{Fun}^{Lex}(T, C)$; $\phi \mapsto \Omega \circ \phi$ are equally well described by $\phi \mapsto \phi \circ \Omega$. But if T is stable, then $\Omega : T \to T$ is an equivalence, so the system is constant. Hence, the equivalence

$$\begin{aligned} \mathrm{Fun}^{Lex}(T, \mathrm{Sp}(C)) \cong \mathrm{Fun}^{Lex}(T, \varprojlim_{\Omega} C) &\stackrel{\sim}{\to} \varprojlim_{\Omega\circ-} \mathrm{Fun}^{Lex}(T, C) \\ &\stackrel{\sim}{\to} \varprojlim_{-\circ\Omega} \mathrm{Fun}^{Lex}(T, C) \stackrel{\sim}{\to} \mathrm{Fun}^{Lex}(T, C). \end{aligned}$$

□

Sometimes the stabilisation is trivial.

Exercise 35. Suppose that $\mathcal{F}\mathrm{in}_*$ is the (1-)category of finite pointed sets. Show that $\mathrm{Sp}(\mathcal{F}\mathrm{in}_*)$ is the zero category.

Exercise 36 (Harder)**.** More generally, suppose that C is a pointed n-category. That is, $\pi_i \operatorname{Map}(X, Y) = 0$ for all $i > n$, and $X, Y \in C$. Show that $\mathrm{Sp}(C)$ is the zero category. Hint. [34] Hint. [35]

Even when it is not trivial, stabilisation can alter the morphism spaces.

Example 37. In the category of pointed spaces $\mathcal{S}_*$ consider the pushout $S^1 \sqcup_* S^1$. The π_1 is the free group $\langle g_1, g_2 \rangle$ on two generators, and in particular, $\hom_{h\mathcal{S}_*}(S^0, S^1 \sqcup_* S^1)$ is not abelian. On the other hand since the homotopy category of any stable quasi-category is additive, the hom set

$$\hom_{h\mathcal{S}_*}(\Sigma^\infty S^0, \Sigma^\infty(S^1 \sqcup_* S^1))$$

of the image *is* abelian. In fact, this is the abelianisation $\mathbb{Z} \oplus \mathbb{Z}$ of $\langle g_1, g_2 \rangle$.

Proposition 38. *Suppose that C is a presentable pointed quasi-category. If*

$$\Sigma : C \to C; \qquad X \mapsto 0 \sqcup_X 0$$

is fully faithful, then the left adjoint $\Sigma^\infty : C \to \mathrm{Sp}(C)$ to Ω^∞ is fully faithful.

Sketch of proof. A left adjoint L is fully faithful if and only if the unit $\mathrm{id} \to RL$ is an equivalence. In our case, this means all the morphisms $X \to \Omega\Sigma X \to \cdots \to \Omega^n \Sigma^n X \to \ldots$ are equivalences. Informally, the left adjoint

$$\Sigma^\infty : C \to \mathrm{Sp}(C)$$

sends X to the diagram with

$$(\Sigma^\infty X)_n = \begin{cases} \varinjlim_n \Omega^n \Sigma^{n+i} X & n \geq 0 \\ \varinjlim_n \Omega^{i+n} \Sigma^n X & n \leq 0 \end{cases}$$

In particular, $X \to \Omega^\infty \Sigma^\infty X$ is an equivalence. But this is the unit of the adjunction $\Sigma^\infty : C \rightleftarrows \mathrm{Sp}(C) : \Omega^\infty$, so the left adjoint $\Sigma^\infty : C \to \mathrm{Sp}(C)$ is fully faithful. $\square$

4 Stabilisation of categories of modules

Our goal is to show the following proposition.

Proposition 39. *Let $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Then every pushout square in R-mod^{cn} is also a pullback square (but not vice versa).*

Corollary 40. *Let $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. The canonical left adjoint*

$$R\text{-}mod^{\mathrm{cn}} \to \mathrm{Sp}(R\text{-}mod^{\mathrm{cn}}) =: R\text{-}mod$$

is fully faithful.

[34] Recall that $\operatorname{Map}(T, -)$ sends finite limits to finite limits.
[35] Use Yoneda to detect when an object is the zero object.

The proof we give uses model categories. These were developed in previous versions of this course. A reference is Goerss, Jardine "Simplicial homotopy theory". There should also be a model independent proof using the canonical classifying space fibration $G \to EG \to BG$ in a later version of these notes.

Sketch of proof of 39. Since $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$ is the quasi-category associated to the model category $\mathcal{M}\mathrm{od}_\Delta$, we perform the calculation in $\mathcal{M}\mathrm{od}_\Delta$ using appropriately fibrant-cofibrant diagrams. Up to equivalence in $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$, we can assume our diagram is a cocartesian square

$$\begin{array}{ccc} (R, A) & \longrightarrow & (R, B) \\ \downarrow & & \downarrow \\ (R, C) & \longrightarrow & (R, D) \end{array}$$

in $\mathcal{M}\mathrm{od}_\Delta$ with cofibrant top left arrows. In particular,

Suppose that $(R_\bullet, A_\bullet) \in \mathcal{M}\mathrm{od}_\Delta$ is any object and choose a cofibrant/trivial fibration factorisation $(R_\bullet, M_\bullet) \hookrightarrow (R_\bullet, M'_\bullet) \overset{w.e.}{\twoheadrightarrow} (R_\bullet, 0)$. In particular, each $M_n \to M'_n$ is injective.

$$\begin{array}{ccc} (R_\bullet, M_\bullet) & \xrightarrow{\ i\ } & (R_\bullet, M'_\bullet) \\ {\scriptstyle i}\downarrow & & \downarrow \\ (R_\bullet, M'_\bullet) & \longrightarrow & (R_\bullet, (M'_\bullet \oplus M'_\bullet)/M_\bullet) \end{array} \qquad \begin{array}{ccc} (R_\bullet, M_\bullet \oplus (M'_\bullet \oplus M'_\bullet)) & \xrightarrow{\ i\ } & (R_\bullet, M'_\bullet \oplus M'_\bullet) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ (R_\bullet, M'_\bullet \oplus M'_\bullet) & \xrightarrow[\ p\ ]{} & (R_\bullet, (M'_\bullet \oplus M'_\bullet)/M_\bullet) \end{array}$$

Since i is a cofibration, the square on the left is sent to a pushout square in $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$. On the other hand, since p is termise surjective, it is a fibration. So the square on the right is sent to a pullback square in $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$. Since $M'_\bullet$ is acyclic, the top two left corners are equivalent. Hence, $\mathrm{id} \to \Omega\Sigma$ is an equivalence. $\square$