Derived Algebraic Geometry
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Lecture 7: Modules and Algebras

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In this lecture we expand last week's study of animated rings to animated modules and animated algebras. After introducing definitions of animated modules, animated algebras, R-modules and R-algebras, we compare various functors such as the symmetric algebra functor

$$\mathrm{Sym}^* : \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}),$$

the composition of the forgetful / free algebra functors

$$\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \xrightarrow{\mathcal{F}} \mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}) \xrightarrow{\mathcal{U}} \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}),$$

and the interaction of the forgetful functor $R\text{-alg} \to R\text{-mod}^{\mathrm{cn}}$ with base change $S \otimes_R - : R\text{-mod}^{\mathrm{cn}} \to S\text{-mod}^{\mathrm{cn}}$. This will allow us in the future to show that for any animated ring $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ and $f \in \pi_0 R$, there is a cofibre sequence of R-modules

$$R \xrightarrow{f} R \to R /\!\!/ f,$$

and in particular, a long exact sequence of homotopy groups of $\pi_0 R$-modules

$$\cdots \to \pi_n R \xrightarrow{f} \pi_n R \to \pi_n(R /\!\!/ f) \to \pi_{n-1} R \xrightarrow{f} \pi_{n-1} R \to \dots.$$

1 Rings, Modules, Algebras

The way we set up $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ in the last lecture can be done more generally. We don't want to develop the full generality, the following situations should cover everything we need.

Consider the following situations.

$\mathcal{C}$	$\supseteq$	$\mathcal{C}^{\mathrm{sfp}}$		
$\mathcal{R}\mathrm{ing}$	$\supseteq$	$\mathcal{P}\mathrm{oly}$	$=$	$\{P_n := \mathbb{Z}[x_1, \dots, x_n] \mid n \in \mathbb{N}\}$
$\mathcal{A}\mathrm{b}$	$\supseteq$	$\mathcal{F}\mathrm{ree}$	$=$	$\{\mathbb{Z}^{\oplus n} \mid n \in \mathbb{N}\}$
$\mathcal{M}\mathrm{od}$	$\supseteq$	$\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$	$=$	$\{(P_n, P_n^{\oplus m}) \mid n, m \in \mathbb{N}\}$
$\mathcal{A}\mathrm{lg}$	$\supseteq$	$\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$	$=$	$\{P_n \to P_n \otimes P_m \mid n, m \in \mathbb{N}\}$
R-mod	$\supseteq$	R-$\mathcal{F}\mathrm{ree}$	$=$	$\{R^{\oplus n} \mid n \in \mathbb{N}\}$

Here, $\mathcal{M}$od is the category whose objects are pairs (R, M) with R a (usual) ring and M a (usual) R-module. Morphisms are pairs $(R \to R', M \to M')$ consisting of a ring homomorphism and an R-module homomorphism. The category $\mathcal{A}$lg is the category $\mathcal{R}\mathrm{ing}^{\Delta^1}$ whose objects are morphisms in $\mathcal{R}$ing and morphisms are commutative squares. In the last line, R is a ring and R-mod the usual category of R-modules. Note that all of the above $\mathcal{C}^{\mathrm{sfp}}$ admit finite coproducts.

Definition 1. The *animation* of $\mathcal{C}$ is the category

$$\mathcal{A}\mathrm{ni}(\mathcal{C}) := \mathrm{PSh}_\Sigma(\mathcal{C}^{\mathrm{sfp}})$$

of presheaves (of spaces) which sends coproducts to products.

Remark 2. As in the case with $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$, we have underlying spaces and various structural morphisms. For example, $F \in \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}^{\mathrm{cn}})$ has underlying ring $R = F((P_1, 0))$, underlying space $M = F((\mathbb{Z}, \mathbb{Z}))$ and a canonical "multiplication"

$$\underset{\substack{=F((P_1,0))\times F((\mathbb{Z},\mathbb{Z})) \\ \cong F((P_1,0)\sqcup(\mathbb{Z},\mathbb{Z})) \\ \cong F((P_1,P_1))}}{R \times M} \to \underset{\cong F((\mathbb{Z},\mathbb{Z}))}{M}$$

associated to the map $(\mathbb{Z}, \mathbb{Z}) \to (P_1, P_1)$; $(1, 1) \mapsto (1, x)$.

Example 3.

1. For every $X \in \mathcal{C}$, we have $\hom(-, X) \in \mathrm{PSh}_\Sigma(\mathcal{C}^{\mathrm{sfp}}, \mathcal{S}\mathrm{et})$.
2. More generally, for every $X \in \mathcal{C}_\Delta = \mathrm{Fun}(\Delta, \mathcal{C})$, each

$$[n] \mapsto \hom(Y, X_n)$$

is automatically a Kan complex, [Goerss, Jardine, Lem.I.3.4]. This defines a functor

$$\mathcal{C}_\Delta \to \mathrm{PSh}_\Sigma(\mathcal{C}^{\mathrm{sfp}})$$

which identifies $\mathcal{A}\mathrm{ni}(\mathcal{C})$ with the localisation of $\mathcal{C}_\Delta$ along *weak equivalences*, [HTT, Corollary 5.5.9.3], Quillen [HTT, Prop.5.5.9.1], Bergner [HTT, Prop.5.5.9.2].

Exercise 4. Show that Yoneda induces an equivalence of categories

$$\mathcal{C} \xrightarrow{\sim} \mathrm{PSh}_\Sigma(\mathcal{C}^{\mathrm{sfp}}, \mathcal{S}\mathrm{et}).$$

Exercise 5. Show that the $\mathcal{A}\mathrm{ni}(\mathcal{C})$ are presentable categories in the sense of Definition 6, Lecture 6. Hint.[1]

Remark 6. In particular, the categories $\mathcal{A}\mathrm{ni}(\mathcal{C})$ admit all small limits and small colimits, and we can apply the adjoint functor theorem to them.

[1] Consider set R of morphisms of the form $\hom(-, P \sqcup Q) \to \hom(-, P) \sqcup \hom(-, Q)$ for $P, Q \in \mathcal{C}^{\mathrm{sfp}}$.

2 R-modules and R-algebras

Definition 7. Let $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Define

$$R\text{-mod}^{\mathrm{cn}} = \pi^{-1}(R), \qquad R\text{-alg} = \pi^{-1}(R)$$

where π are the canonical projection functors

$$\pi : \mathcal{M}\mathrm{od} \to \mathcal{R}\mathrm{ing}, \qquad \pi : \mathcal{A}\mathrm{lg} \to \mathcal{R}\mathrm{ing}$$

induces by restriction along the full inclusions

$$\begin{aligned} \mathcal{P}\mathrm{oly} &\to \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree} \\ P_n &\mapsto (P_n, 0) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{P}\mathrm{oly} &\to \mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} \\ P_n &\mapsto (P_n \xrightarrow{\mathrm{id}} P_n) \end{aligned}$$

Exercise 8. Show that if R is a usual ring, and we restrict to presheaves of sets, then R-mod$^{\mathrm{cn}}$, resp., R-alg are equivalent to the usual categories of R-modules and R-algebras.

Example 9 (Free/forget adjunctions)**.** The categories R-mod$^{\mathrm{cn}}$ and R-alg come equipped with canonical "underlying space" functors

$$\begin{aligned} \mathcal{U} : R\text{-mod}^{\mathrm{cn}} &\to \mathcal{S}, \\ F &\to F(\mathbb{Z}, \mathbb{Z}) \end{aligned} \qquad\qquad \begin{aligned} \mathcal{U} : R\text{-alg} &\to \mathcal{S} \\ F &\mapsto F(\mathbb{Z} \to P_1) \end{aligned}$$

As these are limits and filtered colimits preserving functors between presentable categories, they admit left adjoint "free" functors

$$R[-] : \mathcal{S} \to R\text{-mod}^{\mathrm{cn}}, \qquad\qquad R[-] : \mathcal{S} \to R\text{-alg}.$$

We also have a functor

$$\begin{aligned} \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree} &\to \mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} \\ (P_n, P_n^{\oplus m}) &\mapsto (P_n \to P_n \otimes P_m) \end{aligned}$$

For functoriality, identify $P_n^{\oplus m}$ with homogeneous degree one polynomials in $P_n \otimes P_m$. Composition with this functor induces an underlying module functor

$$\mathcal{U} : R\text{-alg} \to R\text{-mod}$$

which preserves finite limits and filtered colimits, so it has a left adjoint "free algebra" functor

$$\mathcal{F} : R\text{-mod} \to R\text{-alg}.$$

Putting everything together, we have constructed two adjunctions (and the composite)

$$\mathcal{S} \rightleftarrows R\text{-mod} \rightleftarrows R\text{-alg}$$

Remark 10. Categorifying these "free" module/algebra constructions was actually one of the main motivations for the study of the categories PSh_Σ in [Lawvere, Functorial semantics of algebraic theories].

3 Change of scalars

In this section, we fix a morphism $R \to S$ in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ and want to associate to it an adjunction

$$R\text{-mod} \rightleftarrows S\text{-mod}.$$

Heuristically, the left adjoint will be $M \mapsto \iota S \sqcup_{\iota R} M$ and the right adjoint will be $N \mapsto \iota R \times_{\iota S} N$ where

$$\iota : \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$$

is the "zero section", induced by composition with $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree} \to \mathcal{P}\mathrm{oly}$; $(P, F) \mapsto P$.

The formulas $\iota S \sqcup_{\iota R} M$ and $\iota R \times_{\iota S} N$ are common shorthand when working quasi-categories, but need some comment the first time you see them.

Pullback functors in a general quasi-category. Suppose that C is a quasi-category admitting pullbacks. So inside the quasi-category of commutative squares

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, C)^{\mathrm{cart.}} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, C)$$

we have the full subcategory of those squares which are cartesian. That is, for which $(\Lambda^2_2)^{\triangleleft} = \Delta^1 \times \Delta^1 \to C$ is a limit diagram. Given a morphism $f : X \to Y$ in C, we can consider the further subcategory of those squares whose lower horizontal morphism is f.

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, C)^{\mathrm{cart.}}_{f:X \to Y} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, C)^{\mathrm{cart.}} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, C).$$

One way of saying that limits are uniquely defined is the following. It is not so hard to prove but we omit the proof for now.

Lemma 11. *Suppose that I is a small simplicial set and C is a quasi-category admitting I-limits. Then the right adjoint to composition with $I \subseteq I^{\triangleleft}$.*

$$\mathrm{Fun}(I^{\triangleleft}, C) \rightleftarrows \mathrm{Fun}(I, C)$$

exists, is full, and the image is identified with those diagrams which are limit cones.

So we have an equivalence

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, C)^{cart.}_f \xrightarrow{\sim} \mathrm{Fun}(\Lambda^2_2, C)_f$$

Finally, since $\Lambda^2_2 = \Delta^1 \sqcup_{\Delta^0} \Delta^1$, one sees immediately that

$$\mathrm{Fun}(\Lambda^2_2, C)_{X \to Y} = \mathrm{Fun}(\Delta^1, C)_Y$$

is a bijection. The notation

$$(Z \to Y) \mapsto (Y \times_X Z \to X)$$

is shorthand for a composition

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, C)_Y & \overset{=}{\leftarrow} & \mathrm{Fun}(\Lambda^2_2, C)_{X \to Y} \\ & & \uparrow\cong \\ & & \mathrm{Fun}(\Delta^1 \times \Delta^1, C)^{cart.}_{X \to Y} & \to & \mathrm{Fun}(\Delta^1, C)_X \end{array}$$

Remark 12. Note that in constructing this pullback functor, we choose an inverse to the vertical equivalence. In general, the pullback functor is only defined up to this choice of inverse, however, the quasi-category of such inverses which is contractible. So it is unique up to homotopy, which is unique up to homotopy, which is unique up to etc.

In the case of $S\text{-mod}^{\mathrm{cn}} \to R\text{-mod}^{\mathrm{cn}}$, the counit of the adjunction $N \to \iota\pi N$ gives us a functor

$$\begin{tikzcd} S\text{-mod}^{\mathrm{cn}} \arrow[r] \arrow[d, phantom, "\cap|"] & \mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))_{\iota S} \arrow[d, phantom, "\cap|"] \\ \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \arrow[r, "\varepsilon"'] & \mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})) \end{tikzcd}$$

Exercise 13. Show that for any $N \in S\text{-mod}^{\mathrm{cn}}$, the image in $\mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))_{\iota R}$ is sent to an isomorphism under $\pi : \mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))_{\iota R} \to \mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}))_R$.

Remark 14. Hopefully you noticed that since pullbacks are only defined up to equivalence, the image of $\iota R \times_{\iota S} N \to \iota R$ under π is not necessarily $R \overset{\mathrm{id}}{\to} R$, but only some equivalence $R' \overset{\sim}{\to} R$. That is, it doesn't land in the strict fibre $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \times_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \{R\}$, but in the 2-fibre

$$\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \overset{2}{\times}_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \{R\} = \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \times_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} Iso(\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})) \times_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \{R\}.$$

Here, $Iso(\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}))$ is the full sub-quasi-category of those morphisms which are equivalences, and the fibre product uses the two canonical source/target projections $\mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})) \rightrightarrows \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$.

Since $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ is a *cocartesian fibration*, [HTT, Def.2.4.2.1], the canonical inclusion

$$R\text{-mod}^{\mathrm{cn}} = \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \times_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \{R\} \subseteq \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \overset{2}{\times}_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \{R\} \tag{1}$$

is an equivalence of quasi-categories.[2] Cocartesian fibrations deserve a whole lecture to be developed properly, so for now, I will just claim without proof or reference that Eq.(1) is an equivalence. Alternatively, one could have defined $R\text{-mod}^{\mathrm{cn}}$ as the 2-fibre, but this would introduce other complications.

[2]One way of proving this would be to use the fact that the functor of 1-categories $\mathcal{M}\mathrm{od}_\Delta \to \mathcal{R}\mathrm{ing}_\Delta$ is a cocartesian fibration.

Taking the remark into account, we have a commutative diagram

$$\begin{tikzcd}
S\text{-mod}^{\mathrm{cn}} \arrow[rrr, dashed] \arrow[dd] & & & R\text{-mod}^{\mathrm{cn}} \arrow[d] \\
& & & \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \overset{2}{\times}_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \{R\} \arrow[dd] \\
\mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))_{\iota S} & \overset{=}{\leftarrow} & \mathrm{Fun}(\Lambda^2_2, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))_{\iota R \to \iota S} & \\
& & \uparrow\cong & \\
& & \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))^{cart.}_{\iota R \to \iota S} \quad \to & \mathrm{Fun}(\Delta^1, \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}))_{\iota R}
\end{tikzcd}$$

Exercise 15. Construct the dual functor

$$\begin{aligned} R\text{-mod} &\to S\text{-mod} \\ M &\mapsto \iota S \sqcup_{\iota R} M \end{aligned}$$

Hint.[3]

Exercise 16 (Harder.)**.** Show that the pair

$$R\text{-mod} \rightleftarrows S\text{-mod}$$

form an adjunction.

4 Symmetric algebars and sifted completions

Recall that filtered colimits commute with finite limits in $\mathcal{S}$. A similar dichotomy holds with finite products.

Definition 17. A simplicial set K is called *sifted* it is non-empty and for every pair of diagrams $X, Y : K \to \mathcal{S}$ the natural transformation

$$\varinjlim_{k \in K} (X_k \times Y_k) \to (\varinjlim_{k \in K} X_k) \times (\varinjlim_{k \in K} Y_k)$$

is an equivalence.

Remark 18 (Warning: this remark uses material not covered yet in the course)**.** It is more common to define sifted simplicial sets by asking that it is non-empty and the diagonal $K \to K \times K$ is cofinal, [HTT, Def.5.5.8.1 (Rosicky)], and then proving that Def.17 holds for such K, [HTT, Prop.5.5.8.6].[4]. Strangely, I could not find a

[3]Note that $\pi : \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ is both a left *and* a right adjoint to $\iota : \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$, since the same is true of the corresponding functors $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree} \leftrightarrow \mathcal{P}\mathrm{oly}$.

[4]More precisely, use $C = D = E = \mathcal{S}$ and $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$; $X, Y \mapsto X \times Y$ in [HTT, Prop.5.5.8.6], and the fact that colimits are universal in $\mathcal{S}$, [HTT, Lem.6.1.3.14]

reference for the converse, but here is a proof: Suppose that K is sifted in the sense of Def.17. To show that $K \to K \times K$ is cofinal, it suffices to show that for every pair of vertices $x, y \in K$ the simplicial set of spaces $K_{x/} \times_K K_{y/}$ is contractible, [HTT, Prop.4.1.3.1 (Jardine)]. Consider the two diagrams

$$\begin{aligned} X : K &\to \mathcal{S} \\ k &\mapsto \mathrm{Map}(x, k) \end{aligned} \qquad\qquad \begin{aligned} Y : K &\to \mathcal{S} \\ k &\mapsto \mathrm{Map}(y, k) \end{aligned}$$

Under straightening/unstraightening, these correspond to the fibrations $K_{x/} \to K$ and $K_{y/} \to K$ respectively, and the fibration $K_{x/} \times_K K_{y/} \to K$ corresponds to the functor $K \to \mathcal{S}$; $k \mapsto X_k \times Y_k$. In general, if $\pi : K' \to K$ corresponds to $F : K \to \mathcal{S}$, then $\varinjlim_{k \in K} F_k = \mathrm{Sing}\,|K'|$. But since $K_{x/}$ and $K_{y/}$ have initial objects, their associated Kan complexes are contractible. Putting it all together we have a proof that if K is sifted in the sense Def.17 then $K \to K \times K$ is cofinal:

$$\begin{aligned} \mathrm{Sing}\,|K_{x/} \times_K K_{y/}| &= \varinjlim_{k \in K} (X_k \times Y_k) \\ &= (\varinjlim_{k \in K} X_k) \times (\varinjlim_{k \in K} Y_k) && \text{Def.17} \\ &= \mathrm{Sing}\,|K_{x/}| \times \mathrm{Sing}\,|K_{y/}| \\ &= * \times * && \text{initial objects} \\ &= * \end{aligned}$$

So $K_{x/} \times_K K_{y/}$ is contractible, so $K \to K \times K$ is cofinal.

Example 19.
1. Any filtered category is sifted, [HTT, Exam.5.5.8.3].
2. $N(\Delta^{op})$ is sifted, [HTT, Lem.5.5.8.4].
3. In some sense, these are the only two examples we need to care about.[5]

The reason we care about sifted categories right now is that $\mathcal{A}\mathrm{ni}(\mathcal{C})$ is in fact the free completion of $\mathcal{C}^{\mathrm{sfp}}$ under sifted colimits.

Proposition 20 ([HTT, Prop.5.5.8.10, Lemm.5.5.8.14, Prop.5.5.8.15])**.** *For $\mathcal{C}^{\mathrm{sfp}} \subseteq \mathcal{C}$ as above, the smallest full sub-quasi-category of $\mathrm{PSh}(\mathcal{C})$ which:*
1. *contains all $\hom(-, X)$ for $X \in \mathcal{C}^{\mathrm{sfp}}$,*
2. *is closed under sifted colimits,*

is $\mathcal{A}\mathrm{ni}(\mathcal{C})$. Moreover, for any quasi-category $\mathcal{D}$ admitting sifted colimits, the canonical functor

$$\mathrm{Fun}_\Sigma(\mathcal{A}\mathrm{ni}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}^{\mathrm{sfp}}, \mathcal{D}) \tag{2}$$

is an equivalence, where Fun_Σ means the full subcategory of those functors preserving sifted colimits.

[5] If C admits small colimits, then a functor $C \to D$ preserves sifted colimits if and only if it preserves filtered colimits and $N(\Delta)^{op}$-colimits, [HTT, Cor.5.5.8.17].

Following [Lurie, SAG, §25.2], we now use Proposition 20 to build the symmetric algebra functor.

Definition 21. The *symmetric algebra* functor

$$\mathrm{Sym}^* : \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \to \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$$

is the unique sifted colimit preserving functor which sends $(P, F) \in \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$ to $\varinjlim_{d \in \mathbb{N}} (P, \mathrm{Sym}_P^{\leq d} F)$.

Remark 22.

1. In the above definition, the colimit takes place in $\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$ while $(P, \mathrm{Sym}_P^{\leq d} F)$ lives in $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$. (This is necessary because we only allow finite free modules in $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$).
2. If we choose an isomorphism $F = P^{\oplus m}$ then $\mathrm{Sym}_P^{\leq d} F$ is identified with the P-module $P[x_1, \dots, x_m]_{\leq d}$ of polynomials of degree $\leq d$.

Recall that we have an adjunction

$$\mathcal{F} : \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \rightleftarrows \mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}) : \mathcal{U}$$

where $\mathcal{U}$ is composition with $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree} \to \mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly}$; $(P_n, P_n^{\oplus m}) \mapsto P_n \to P_n \otimes P_m$.[6]

Proposition 23. *There is a natural isomorphism*

$$\mathcal{U}\mathcal{F} \cong \mathrm{Sym}^* .$$

Proof. Since $\mathcal{U}, \mathcal{F}$, and Sym^* all preserves sifted colimits, by the equivalence Eq.(2) it suffices to construct a natural isomorphism on $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$. Choosing $(P, P^{\oplus m}) \in \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$, we have

$$\begin{aligned} \mathcal{U}\mathcal{F}(P, P^{\oplus m}) := \mathcal{U}(P, P[x_1, \dots, x_m]) &\overset{(*)}{\cong} \varinjlim_{d \in \mathbb{N}} (P, P[x_1, \dots, x_m]_{\leq d}) \\ &\overset{(**)}{\cong} \varinjlim_{d \in \mathbb{N}} \mathrm{Sym}_P^{\leq d}(P, P^{\oplus m}) =: \mathrm{Sym}^*(P, P^{\oplus m}) \end{aligned}$$

□

Exercise 24. Check that the two isomorphisms $(*)$ and $(**)$ in the above proof actually hold, and are natural in (P, F). (Feel free to assume that $\mathcal{S}\mathrm{et} \subseteq \mathcal{S}$ is closed under filtered colimits, since we have not developed enough machinery yet to prove this.)

[6] For functoriality note that $P_n^{\oplus m}$ is identified with the homogeneous degree one elements of the P_n-algebra $P_n \otimes P_m$.

Corollary 25. *Suppose $R \to S$ is a morphism in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. Then following square commutes.*

$$\begin{tikzcd} R\text{-}alg \arrow[r, "\mathcal{U}"] \arrow[d, "-\otimes_R S"'] & R\text{-}mod^{\mathrm{cn}} \arrow[d, "-\otimes_R S"] \\ S\text{-}alg \arrow[r, "\mathcal{U}"'] & S\text{-}mod^{\mathrm{cn}} \end{tikzcd}$$

More precisely, the canonical natural transformation

$$(\mathcal{U}-) \otimes_R S \to \mathcal{U}(- \otimes_R S) \tag{3}$$

is an equivalence.

Definition 26. In the proof and exercises below we make use the functor

$$\iota : \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) \to \mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg}),$$

which is composition with $\mathcal{P}\mathrm{oly}\mathcal{P}\mathrm{oly} \to \mathcal{P}\mathrm{oly}$; $(P_n {\to} P_n \otimes P_m) \mapsto P_n \otimes P_m$. We also use the functor

$$\begin{aligned} S\otimes_R, - : R\text{-alg} &\to S\text{-alg} \\ A &\mapsto \iota S \sqcup_{\iota R} A. \end{aligned}$$

Proof. Since every object in R-alg can be written as a sifted colimit of R-algebras of the form $R[x_1, \dots, x_n] := \iota R \sqcup (\mathbb{Z} \to P_n)$, Exercise 27, and all four functors commute with sifted colimits, it suffices to consider such R-algebras.

Consider the following larger diagram.

$$\begin{tikzcd} \mathbb{Z}\text{-mod}^{\mathrm{cn}} \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{Sym}^*"] & \mathbb{Z}\text{-alg} \arrow[r] & \mathbb{Z}\text{-mod}^{\mathrm{cn}} \arrow[d] \\ R\text{-mod}^{\mathrm{cn}} \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{Sym}^*"] & R\text{-alg} \arrow[r] \arrow[d] & R\text{-mod}^{\mathrm{cn}} \arrow[d] \\ S\text{-mod}^{\mathrm{cn}} \arrow[r] \arrow[rr, bend right, "\mathrm{Sym}^*"'] & S\text{-alg} \arrow[r] & S\text{-mod}^{\mathrm{cn}} \end{tikzcd}$$

The upper rectangle and the outer square commute by Proposition 23 and the lower left square commutes by Exercise 28. It follows that the two paths $^{\downarrow} \to {}_{\downarrow\to}$ and $^{\downarrow} \to\to {}_{\downarrow}$ are equivalent. ☐

Exercise 27. Show that every object in R-alg can be written as a sifted colimit of R-algebras of the form $(R{\to}R[x_1, \dots, x_n]) := \iota R \sqcup (\mathbb{Z} \to P_n)$. Hint.[7]

Similarly, show that every object in R-mod$^{\mathrm{cn}}$ can be written as a sifted colimit of R-modules of the form $R^{\oplus m} := \iota R \sqcup_{\iota\mathbb{Z}} (\mathbb{Z}, \mathbb{Z}^{\oplus m})$.

[7] Note that any object of $\mathcal{A}\mathrm{ni}(\mathcal{A}\mathrm{lg})$ can be written as sifted colimit of algebras of the form $(P_n \to P_n \otimes P_m) = (P_n{\to}P_n) \sqcup_{(\mathbb{Z}\to\mathbb{Z})} (\mathbb{Z}{\to}P_m)$.

Exercise 28. Show that the following square is commutative.

$$\begin{tikzcd} R\text{-mod}^{\mathrm{cn}} \arrow[r] \arrow[d] & R\text{-alg} \arrow[d] \\ S\text{-mod}^{\mathrm{cn}} \arrow[r] & S\text{-alg} \end{tikzcd}$$

Hint.[8]

5 Quotients of rings

Definition 29. Take $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ and choose $f : \mathbb{Z}[x] \to R$ in $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$. (Note this is equivalent to choosing a base point $f \in \pi_0 \mathcal{U} R$ in the underlying space of R.) We define

$$R/\!\!/f := R \otimes_{\mathbb{Z}[x]} \mathbb{Z}$$

in R-alg, cf. Definition 26. When this procedure is iterated we write

$$R/\!\!/f_1/\!\!/f_2 \dots /\!\!/f_n := R/\!\!/f_1, \dots, f_n.$$

Proposition 30. *There is a pushout square in R-mod$^{\mathrm{cn}}$ of the form*

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d] & R \arrow[d] \\ 0 \arrow[r] & R/\!\!/f \end{tikzcd}$$

Proof. By Corollary 25 we have a commutative square

$$\begin{tikzcd} \mathbb{Z}[x]\text{-alg} \arrow[r] \arrow[d] & \mathbb{Z}[x]\text{-mod}^{\mathrm{cn}} \arrow[d] \\ R\text{-alg} \arrow[r] & R\text{-mod}^{\mathrm{cn}} \end{tikzcd}$$

where the vertical functors are left adjoints, and therefore preserve pushouts. So it suffices to consider the case $R = \mathbb{Z}[x]$ and $f = x$. We will deal with this square next week (or maybe the week after) using model categories. $\square$

[8] Use Exercise 27 to reduce to the case $\mathbb{Z} = R$.