Derived Algebraic Geometry
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Spring Semester 2025

Lecture 6: Rings

May 21th, 2025

References:
1. Lawvere, “Functorial semantics of algebraic theories”, 1963.
2. Gabriel, Ulmer, Lokal präsentierbare Kategorien, 1971.
3. Adamek, Rosicky, “Locally presentable and accessible categories”, 1994.
4. Goerss, Jardine, Simplicial Homotopy Theory, 1997.
5. Lurie, “Spectral algebraic geometry”, Section 25.1.1.
6. Cesnavicius, Scholze, “Purity for flat cohomology”, Section 5.1.

1 Simplicial rings

A ring is a set R together with maps

$$\begin{aligned} 0 &: * \to R & \qquad add. &: R \times R \to R \\ 1 &: * \to R & \qquad mult. &: R \times R \to R \end{aligned}$$

satisfying various axioms. For example, distributivity, which can be expressed in the following commutative diagram.

$$\begin{tikzcd} R \times R \times R \arrow[r, "(\mathrm{id}, add.)"] \arrow[d, "(pr_1, pr_2, pr_1, pr_3)"'] & R \times R \arrow[dd, "mult."] \\ R \times R \times R \times R \arrow[d, "(mult., mult.)"'] & \\ R \times R \arrow[r, "add."] & R \end{tikzcd} \qquad \begin{tikzcd} (a,b,c) \arrow[r, mapsto] \arrow[d, mapsto] & & (a, b+c) \arrow[dd, mapsto] \\ (a,b,a,c) \arrow[d, mapsto] & & \\ (ab, ac) \arrow[r, mapsto] & ab + ac & a(b+c) \end{tikzcd}$$

In the derived setting we want to replace the set R with a homotopy type. One approach to this is to use Kan complexes equipped with the above structure (in a 1-categorical sense) as concrete models. In order for weak equivalence to agree with homotopy equivalence, we need to restrict our attention to *cofibrant* simplicial rings. This essentially means that each R_n should be a polynomial algebra in a way compatible with the degeneracy morphisms.[1,2]

[1] See [Goerss, Jardine, Simplicial Homotopy Theory, Cor.V.1.10] for the simplicial groups version.

[2] For any set K we can consider the polynomial algebra $\mathbb{Z}[K] := \mathbb{Z}[x_k]_{k \in K}$ with one variable for each element k of K. If $K_\bullet$ is a simplicial set, then $\mathbb{Z}[K_\bullet]$ is a cofibrant simplicial ring.

Calculations with Kan complexes are fiddly, unpleasant, and often uninsightful. Calculations with cofibrant simplicial rings tend to be worse. So, following references such as [Lurie, SAG, §25.1.1] and [Cesnavicius, Scholze, Purity for flat cohomology, §5.1], we will take a different approach based on the Lawvere theory approach to algebra. Since product preserving functors are sifted colimits of representables, this is essentially the same information, just organised in a different way.

This approach has the advantage that it is *model independent* in the sense that it doesn't see which particular theory of infinity categories we are working with.

Exercise 1. Let $\mathcal{P}\mathrm{oly}$ be the 1-category whose objects are rings of the form

$$P_n := \mathbb{Z}[x_1, \dots, x_n]$$

for $n \geq 0$ and morphisms are ring homomorphisms. Notice that this category admits finite coproducts and

$$\varnothing = P_0, \qquad P_n \sqcup P_m \cong P_{n+m}.$$

1. Let R be a usual ring. Show that

$$P_n \mapsto \hom(P_n, R)$$

defines a presheaf of sets on $\mathcal{P}\mathrm{oly}$ which sends coproducts to products.

2. Conversely, suppose that $F : \mathcal{P}\mathrm{oly}^{op} \to \mathcal{S}\mathrm{et}$ is a presheaf that sends coproducts to products. Using the maps

$$\begin{array}{ll} 0 : \mathbb{Z}[x] \overset{x \mapsto 0}{\to} \mathbb{Z} & \qquad add : \mathbb{Z}[x] \overset{x \mapsto y+z}{\to} \mathbb{Z}[y,z] \\ 1 : \mathbb{Z}[x] \overset{x \mapsto 1}{\to} \mathbb{Z} & \qquad mult : \mathbb{Z}[x] \overset{x \mapsto yz}{\to} \mathbb{Z}[y,z] \end{array}$$

show that the set $F(P_1)$ has a canonical structure of commutative ring (with unity).

3. Show that the above to assignments establish an equivalence of categories between the category of rings, and the category $\mathrm{PSh}^{\Pi}(\mathcal{P}\mathrm{oly})$ of functors $\mathcal{P}\mathrm{oly}^{op} \to \mathcal{S}\mathrm{et}$ which send coproducts to products.

$$\mathcal{R}\mathrm{ing} \cong \mathrm{PSh}_{\Sigma}(\mathcal{P}\mathrm{oly}, \mathcal{S}\mathrm{et}).$$

Remark 2. The notation PSh_{Σ} is based on [HTT, Definition 5.5.8.8].

Definition 3. The category

$$\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) := \mathrm{PSh}_{\Sigma}(\mathcal{P}\mathrm{oly}) \subseteq \mathrm{PSh}(\mathcal{P}\mathrm{oly}, \mathcal{S})$$

is the full subcategory of those presheaves (of spaces) $R \in \mathrm{PSh}(\mathcal{P}\mathrm{oly})$ which take coproducts to products. That is, such that the canonical comparison morphisms

$$R(P_n) \to R(P_1) \times \cdots \times R(P_1)$$

are equivalences.

Remark 4. The space $R(P_1) \in \mathcal{S}$ is called the *underlying space* of R. This defines a functor of quasi-categories.

$$\begin{aligned} \mathcal{U} : \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) &\to \mathcal{S} \\ R &\mapsto \mathcal{U}(R) := R(P_1) \end{aligned}$$

As in the above exercise, the maps $0, 1, add, mult$ define maps

$$\begin{aligned} 0 &: * \to \mathcal{U}(R) & add &: \mathcal{U}(R) \times \mathcal{U}(R) \to \mathcal{U}(R) \\ 1 &: * \to \mathcal{U}(R) & mult &: \mathcal{U}(R) \times \mathcal{U}(R) \to \mathcal{U}(R) \end{aligned}$$

in $\mathcal{S}$. Since limits are only defined up to homotopy, these maps are only defined up to homotopy. Similarly, the associativity, commutativity, identity, additive inverse, distributivity axioms only hold up to homotopy.

Example 5.

1. Any usual ring R defines a product preserving functor

$$P_n \mapsto \hom(P_n, R) = \underbrace{R \times \cdots \times R}_{n \text{ copies}}$$

These are called *discrete rings* or sometimes *static rings*. This defines a fully faithful embedding of the category of usual rings

$$\mathcal{R}\mathrm{ing} \subseteq \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}).$$

Since π_0 preserves products, this inclusion has a left adjoint

$$\begin{aligned} \pi_0 : \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) &\to \mathcal{R}\mathrm{ing} \\ R &\mapsto \pi_0 R \end{aligned}$$

Explicitly, π_0 sends a functor $R : \mathcal{P}\mathrm{oly}^{op} \to \mathcal{S}$ to the composition $\mathcal{P}\mathrm{oly}^{op} \to \mathcal{S} \xrightarrow{\pi_0} \mathcal{S}\mathrm{et}$

2. More generally, suppose that $R \in \mathcal{R}\mathrm{ing}_\Delta$ is commutative ring object in the 1-category of simplicial sets. That is, a functor $R : \Delta^{op} \to \mathcal{R}\mathrm{ing}$. Then R is automatically a Kan complex [Goerss, Jardine, Lemma I.3.4] and the assignment

$$P_n \mapsto \underbrace{R \times \cdots \times R}_{n \text{ times}}$$

defines a product preserving functor between the 1-categories $\mathcal{P}\mathrm{oly}$ and $\mathcal{K}\mathrm{an}$. Taking the nerve of this defines a product preserving functor between the corresponding quasi-categories, i.e., an object of $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$.

Warning: The functor

$$\mathcal{R}\mathrm{ing}_\Delta \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$$

we have just defined is not fully faithful! In fact, it identifies $\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ with the localisation of $\mathcal{R}\mathrm{ing}_\Delta$ along weak equivalences.

$$\mathcal{R}\mathrm{ing}_\Delta[w.e.^{-1}] \xrightarrow{\sim} \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$$

This is [HTT, Corollary 5.5.9.3], following Quillen [HTT, Prop.5.5.9.1], and Bergner [HTT, Prop.5.5.9.2].

3. We will see later that for any animated ring R there is a canonical graded ring structure on $\prod_{n\in\mathbb{N}} \pi_n(R, 0)$, giving a functor

$$\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) \to \mathcal{G}r\mathcal{R}\mathrm{ing}$$

towards the 1-category of $\mathbb{N}$-graded rings.

2 Adjoint functor theorem and free rings

Recall that last time we defined an adjunction as those quadruples $(F, G, \varepsilon, \eta)$ where $F : C \rightleftarrows D : G$ are functors, $\varepsilon : \mathrm{id}_C \to GF$, $\eta : FG \to \mathrm{id}_D$ are natural transformations, such that there exist 2-cells $\mathrm{id}_F \sim \eta F \circ F\varepsilon$ and $\mathrm{id}_G \sim \eta G \circ G\varepsilon$. We also saw that right adjoints necessarily preserve limits, and left adjoints necessarily preserve colimits. For a large class of categories—presentable categories—these conditions are also sufficient.

Definition 6 ([HTT, Thm.5.5.1.1, Prop.5.5.4.2])**.** A quasi-category C is *presentable* if there exists a small quasi-category $\mathcal{G}$ (of "generators") and a small set of morphisms $R \subseteq \mathrm{Fun}(\Delta^1, \mathrm{PSh}(\mathcal{G}))$ (the "relations") such that C is equivalent to the full subcategory of *R-local* presheaves F. That is, those presheaves F such that $\mathrm{Map}(f, F)$ is an equivalence for all $f \in R$.

$$C \cong \left\{ F \in \mathrm{PSh}(\mathcal{G}) \;\middle|\; \mathrm{Map}(f, F) \text{ is an equiv. for all } f \in R \right\}.$$

Remark 7. Since $\mathrm{PSh}(\mathcal{G})$ is the quasi-category obtained by freely adjoining small colimits to $\mathcal{G}$, [HTT, Prop.5.1.5.6], and there exists a left adjoint $\mathrm{PSh}(\mathcal{G}) \to C$ identifying C as the category obtained from $\mathrm{PSh}(\mathcal{G})$ by formally inverting elements of R, [HTT, Prop.5.5.4.2], C should be thought of as the category freely generated by $\mathcal{G}$ modulo the relations R,

$$\mathrm{PSh}(\mathcal{G})[R^{-1}] \xrightarrow{\sim} C.$$

Example 8.

1. $\mathrm{PSh}(K)$ is presentable for any small quasi-category K. In particular, the category of spaces $\mathcal{S} = \mathrm{PSh}(*)$ is presentable.
2. $\mathrm{Shv}_\tau(C)$ is presentable for any small quasi-category C equipped with a topology.

3. $\mathcal{A}$ni($\mathcal{R}$ing) is presentable.

Exercise 9. Using the adjunction

$$\mathrm{PSh}(\mathcal{G}) \rightleftarrows C.$$

Show that presentable categories admit all small limits and small colimits.

Theorem 10 (Left adjoint functor theorem [HTT, Cor.5.5.2.9])**.** *A functor $F : C \to D$ between presentable quasi-categories is a left adjoint if and only if it preserves colimits.*

To state the right adjoint functor theorem we need to develop the notion of κ-filtered colimits.

Definition 11 ([HTT, Def.5.3.1.7])**.** Let κ be a regular cardinal.[3]

1. A simplicial set K is *κ-small* if it has $< \kappa$ nondegenerate simplicies.
2. A quasi-category Λ is *κ-filtered* if for every κ-small simplicial set K and every functor $K \to \Lambda$ there exists an extension $K^{\triangleright} \to \Lambda$.
3. If $\kappa = \omega$ we say that Λ is *filtered*.

Example 12. Any quasi-category which admits κ-small colimits is κ-filtered, but not every κ-filtered quasi-category admits κ-small colimits.

Exercise 13. Suppose Λ is a small classical category. Show that Λ is filtered if and only if the quasi-category $N\Lambda$ is filtered.

Theorem 14 (Right adjoint functor theorem [HTT, Cor.5.5.2.9])**.** *A functor $G : D \to C$ between presentable quasi-categories is a right adjoint if and only if it preserves limits and κ-filtered colimits for some κ.*

Example 15. The canonical functor

$$\mathcal{U} : \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) \to \mathcal{S}$$

sending a ring to its underlying space admits a left adjoint

$$\mathbb{Z}[-] : \mathcal{S} \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}).$$

Indeed, the inclusion $\mathcal{A}$ni($\mathcal{R}$ing) $\subseteq$ PSh($\mathcal{P}$oly) preserves finite limits and filtered colimits because these commute with finite products. Evaluation functors $\mathrm{PSh}(K) \to \mathcal{S}$; $F \mapsto F(k)$ preserve all small limits and small colimits. (We saw these facts in the lecture on limits).

[3]That is, κ is a set such that for any subset $I \subseteq \kappa$, and I-indexed collection of subsets $\{J_i \subseteq \kappa \mid i \in I\}$, the coproduct $\coprod_{i \in I} J_i$ has cardinality $\leq \kappa$.

Example 16. This is the higher categorical version of the *polynomial ring* functor $I \mapsto \mathbb{Z}[x_i]_{i \in I}$. Indeed, if $K \in \mathcal{S}\mathrm{et} \subseteq \mathcal{S}$ is a discrete space, then $\mathbb{Z}[K] \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ is the polynomial ring with one variable for each $k \in K$. In other words, the following square commutes

$$\begin{tikzcd} \mathcal{S}\mathrm{et} \arrow[r, "\mathbb{Z}[-]"] \arrow[d] & \mathcal{R}\mathrm{ing} \arrow[d] \\ \mathcal{S} \arrow[r, "\mathbb{Z}[-]"'] & \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}) \end{tikzcd}$$

Example 17. We will (hopefully) see next time that $\mathcal{U}\mathbb{Z}[S^1]$ has homotopy groups

$$\pi_n(\mathcal{U}\mathbb{Z}[S^1], 0) = \begin{cases} \mathbb{Z}[x] & n = 0 \\ \mathbb{Z}[x] & n = 1 \\ 0 & n \geq 2 \end{cases}$$

3 Modules

For modules we continue with the idea that algebraic categories are freely generated under sifted colimits by their subcategories of finite free objects. For $R \in \mathcal{R}\mathrm{ing}$, we could use the category $\mathcal{F}\mathrm{ree}_R$ of finite free R-modules as generators. However for a general $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$, this category $\mathcal{F}\mathrm{ree}_R$ is no longer a 1-category. So we define the category of modules over *all* rings at once. This has the advantage that it gives some nice control over the adjunctions R-mod $\rightleftarrows$ S-mod.

Definition 18. Let $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$ denote the category of pairs (P, F) such that $P \in \mathcal{P}\mathrm{oly}$ and F is a free R-module. Morphisms $(P, F) \to (P', F')$ are pairs consisting of a a morphism of rings $P \to P'$ and a morphism of P-modules $F \to F'$.

Exercise 19. Do Exercise 1 for $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$. That is, show that a pair (R, M) consisting of a (usual) ring R and R-module M is the same thing as a functor

$$\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}^{op} \to \mathcal{S}\mathrm{et}$$

which sends coproducts to products. So the category $\mathcal{M}\mathrm{od}$, whose objects are pairs (R, M) consisting of $R \in \mathcal{R}\mathrm{ing}$ and an R-module M, and morphisms $(R, M) \to (S, N)$ are pairs consisting of a ring homomorphism $R \to S$ and an R-module homomorphism $M \to N$ is equivalent to the category of presheaves which send coproducts to products

$$\mathcal{M}\mathrm{od} \cong \mathrm{PSh}_\Sigma(\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}, \mathcal{S}\mathrm{et}).$$

Note that coproduct in $\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$ are defined as

$$(P, F) \sqcup (Q, G) = (P \otimes Q, (F \otimes Q) \oplus (Q \otimes G))$$

Definition 20. The category

$$\mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) := \mathrm{PSh}_\Sigma(\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}, \mathcal{S}) \subset \mathrm{PSh}(\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree})$$

is the full subcategory of those presheaves (of spaces) $M \in \mathrm{PSh}(\mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree})$ which take coproducts to products.

Definition 21. Consider the functor $\mathcal{P}\mathrm{oly} \to \mathcal{P}\mathrm{oly}\mathcal{F}\mathrm{ree}$; $P \mapsto (P, 0)$. Since this preserves coproducts, composition induces a functor

$$\pi : \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od}) \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing}).$$

Give $R \in \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$, the category of *$R$-modules* is the fibre of R. That is, the quasi-category

$$R\text{-}\mathrm{mod}_{\geq 0} := * \times_{\mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})} \mathcal{A}\mathrm{ni}(\mathcal{M}\mathrm{od})$$

where the fibre product takes place in $\mathcal{S}\mathrm{et}_\Delta$ and $* \to \mathcal{A}\mathrm{ni}(\mathcal{R}\mathrm{ing})$ sends the unique object to R.

Remark 22. The $(-)_{\geq 0}$ refers to the fact that since we are working with spaces everywhere, there are no negative homotopy groups (yet).

Example 23. Even if R is a classical ring, the category R-$\mathrm{mod}_{\geq 0}$ is not a 1-category. We will see next time that $\mathbb{Z}$-$\mathrm{mod}_{\geq 0}$ is equivalent to the quasi-category $\mathcal{C}\mathrm{omp}_{\geq 0}$ of those chain complexes of abelian groups which are bounded below zero.

$$\mathbb{Z}\text{-}\mathrm{mod}_{\geq 0} \cong \mathcal{C}\mathrm{omp}_{\geq 0}.$$