Derived Algebraic Geometry
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1 The classical version

Suppose that X is a topological space, and $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is an open covering. Then giving a continuous morphism

$$f : X \to \mathbb{R}$$

is the same thing as giving a collection of continuous morphisms $f_i : U_\lambda \to \mathbb{R}$ that agree on the intersections. That is, such that for every μ, λ we have

$$f_\lambda|_{U_\lambda \cap U_\mu} = f_\mu|_{U_\lambda \cap U_\mu}.$$

Said another way the collection of f_λ is in the equaliser of the two canonical restriction maps

$$\prod_{\lambda\in\Lambda} \hom_{cont.}(U_\lambda, \mathbb{R}) \rightrightarrows \prod_{\lambda,\mu\in\Lambda} \hom_{cont.}(U_\lambda \cap U_\mu, \mathbb{R}).$$

As we mentioned last week, we can also do this with local homeomorphisms. If $p : Y \to X$ is a continuous morphism of topological spaces such that for every $y \in Y$ there is an open neighbourhood $y \in V \subseteq Y$ such that $p(V) \subseteq X$ is open and $V \to p(V)$ is a homeomorphism, then to give a continuous morphism $f : X \to \mathbb{R}$ is the same thing as giving a continuous morphism $g : Y \to \mathbb{R}$ that is constant on fibres. That is, such that

$$\pi(y_1) = \pi(y_2) \Rightarrow g(y_1) = g(y_2).$$

Said another way, g is in the equaliser of the two maps

$$\hom_{cont.}(Y, \mathbb{R}) \rightrightarrows \hom_{cont.}(Y \times_X Y, \mathbb{R})$$

induced by the two projections $pr_i : Y \times_X Y \to Y$; $(y_1, y_2) \mapsto y_i$ where $i = 1$ or 2.

We could also have done this discussion in other settings. Instead of $\mathbb{R}$, we could have used any topological space. We could also have assumed Y, X were differential manifolds, or complex analytic varieties with the appropriate notion of local homeomorphism, and used some other $F(-)$ instead of $\hom_{cont.}(-, \mathbb{R})$.

Grothendieck topologies are an abstraction and generalisation of these.

Definition 1. Suppose that C is a classical category. A *topology*[1] T on C is a collection of families $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ of morphisms, called *coverings* satisfying the following conditions.

[1]We actually give the definition of a pretopology. But since pretopologies have a canonically associated topology which gives rise to the same category of sheaves, people often call pretopologies topologies.

1. Every singleton
$$\{Y \xrightarrow{\sim} X\}$$
containing an isomorphism is a covering.[2]
2. If $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ is a covering and $Y \to X$ is a morphism, then the pullbacks $Y \times_X U_\lambda$ exist[3] in C and
$$\{Y \times_X U_\lambda \to U_\lambda\}_{\lambda \in \Lambda}$$
is a covering.
3. If $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ is a covering and for each λ we have a covering $\{V_{\lambda\mu} \to U_\lambda\}_{\mu \in M_\lambda}$, the the family of compositions
$$\{V_{\lambda\mu} \to U_\lambda \to X\}_{\lambda \in \Lambda, \mu \in M_\lambda}$$
is a covering.

Exercise 2. Show that the following are topologies.

1. C is the category of topological spaces and T is the collection of families $\{p_\lambda : U_\lambda \to X\}_{\lambda \in \Lambda}$ such that each $U_\lambda \to X$ is an open immersion and $\sqcup_{\lambda \in \Lambda} p_\lambda(U_\lambda) \to X$ is surjective.
2. C is the category of topological spaces and T is the collection of families $\{p_\lambda : U_\lambda \to X\}_{\lambda \in \Lambda}$ such that each $U_\lambda \to X$ is a local homeomorphism and $\sqcup_{\lambda \in \Lambda} p_\lambda(U_\lambda) \to X$ is surjective.

Definition 3. Let C be a category equipped with a topology T. A *presheaf* is a functor $F : C^{op} \to \mathcal{S}\mathrm{et}$. A presheaf is a *sheaf* if for every covering $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ we have
$$F(X) = \mathrm{eq}\left(\prod_{\lambda \in \Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda, \mu} F(U_\lambda \times_X U_\mu)\right).$$

Example 4. For any topological space E, the presheaf $\hom_{cont.}(-, E)$ on the category of topological spaces with the canonical topology is a sheaf.

Definition 5. A *topos* is a category of the form $\mathrm{Shv}(C)$ for some category C equipped with some topology T.

Remark 6. For any category equipped with a topology, the canonical inclusion $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ admits a left adjoint, called *sheafification*. There are a number of explicit descriptions of this adjoint. Here is one. Given a presheaf F, define $F^+(X) = \varinjlim \mathrm{eq}(\prod_{\lambda \in \Lambda} F(U_\lambda) \to \prod_{\lambda, \mu} F(U_\lambda \times_X U_\mu))$ where the colimit is over coverings. This is functorial in X, as well as F, so defines a functor $\mathrm{PSh}(C) \to \mathrm{PSh}(C)$. Then it turns out that applying this twice gives the left adjoint to inclusion. That is, for any presheaf F and sheaf G, the presheaf F^{++} is a sheaf, and we have $\hom(F, G) = \hom(F^{++}, G)$.

[2] By the next axiom, only assuming that identities are coverings gives the same notion, since pullbacks are only defined up to isomorphism.

[3] One can easily avoid assuming that these pullbacks exists, but it is standard to assume their existence, and all our examples will satisfy this, so we do.

2 Higher topoi

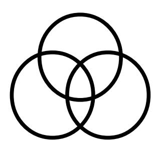
The notion of topology on a quasi-category is the same as that on a classical category.

Definition 7. Suppose that C is a quasi-category. A *topology*[4] T on C is a collection of families $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ of morphisms, called *coverings* satisfying the following conditions.

1. Every singleton
$$\{Y \xrightarrow{\sim} X\}$$
containing an equivalence is a covering.[5]
2. If $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering and $Y \to X$ is a morphism, then the pullbacks $Y \times_X U_\lambda$ exist[6] in C and
$$\{Y \times_X U_\lambda \to U_\lambda\}_{\lambda\in\Lambda}$$
is a covering.
3. If $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering and for each λ we have a covering $\{V_{\lambda\mu} \to U_\lambda\}_{\mu\in M_\lambda}$, the the family of compositions
$$\{V_{\lambda\mu} \to U_\lambda \to X\}_{\lambda\in\Lambda,\mu\in M_\lambda}$$
is a covering.

The notion of sheaf is more subtle. To see why, let's go back to a classical site.

Example 8. Consider a topological space X equipped with an open covering U_0, U_1, U_2 such that all of X, U_λ, $U_\lambda \cap U_\mu$ and $U_\lambda \cap U_\mu \cap U_\nu$ are contractible for distinct λ, μ, ν.



Consider the sheaf of complexes of abelian groups concentrated in degree zero $F : U \mapsto \hom(U, \mathbb{Q})$ where $\mathbb{Q}$ is given the discrete topology. Let's try and imitate Def.3 for the covering U_0, U_1, U_2. In the quasi-category $Cplx_{\mathbb{Q}}$, the equaliser

$$\mathrm{eq}\left(\prod_{i=0,1,2} F(U_i) \rightrightarrows \prod_{i,j=0,1,2} F(U_i \cap U_j)\right)$$

[4]We actually give the definition of a pretopology. But since pretopologies have a canonically associated topology which gives rise to the same category of sheaves, people often call pretopologies topologies.

[5]By the next axiom, only assuming that identities are coverings gives the same notion, since pullbacks are only defined up to isomorphism.

[6]One can easily avoid assuming that these pullbacks exists, but it is standard to assume their existence, and all our examples will satisfy this, so we do.

is the complex concentrated in homological degrees 0 and -1

$$\left[\prod_{i=0,1,2} \mathbb{Q} \to \prod_{i,j=0,1,2} \mathbb{Q}\right]$$

with morphism $(a_0, a_1, a_2) \mapsto \begin{pmatrix} 0 & a_1 - a_0 & a_2 - a_0 \\ a_0 - a_1 & 0 & a_2 - a_1 \\ a_0 - a_2 & a_1 - a_2 & 0 \end{pmatrix}$. The H_0 of this complex is $\{(a, a, a)\} \cong \mathbb{R}$, agreeing wth the H_0 of $F(X)$, but the H_{-1} has dimension seven. Three of these dimensions come from the diagonal, and some others come from the symmetry. So maybe the problem was that we were using $\{0, 1, 2\}^2$ as the indexing set instead of restricting to $0 \leq i < j \leq 2$. If we do this, we get the complex

$$\left[\prod_{i=0,1,2} \mathbb{Q} \to \prod_{0 \leq i < j \leq 2} \mathbb{Q}\right]$$

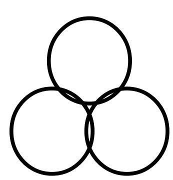
concentrated in homological degrees 0 and -1 with morphism $(a_0, a_1, a_2) \mapsto \begin{pmatrix} a_1 - a_0 & a_2 - a_0 \\ & a_2 - a_1 \end{pmatrix}$. This complex still has a nonzero H_{-1}, coming from the fact that the diagram

$$\begin{array}{ccccc} & & U_{01} & & \\ & \swarrow & & \searrow & \\ & U_0 & & U_1 & \\ \nearrow & & & & \nwarrow \\ U_{02} & \longrightarrow & U_2 & \longleftarrow & U_{12} \end{array}$$

is essentially an unfilled circle. To remove the unwanted factor we need to also take into account the triple intersection $U_0 \cap U_1 \cap U_2$. So we should really consider the complex

$$\left[\prod_{i=0,1,2} \mathbb{Q} \to \prod_{0 \leq i < j \leq 2} \mathbb{Q} \to \prod_{0 \leq i < j < k \leq 2} \mathbb{Q}\right].$$

Note that if this triple intersection was empty, we would want this extra factor in H_{-1}, since our X would be homotopic to a circle.



Example 9. A related example is the de Rham complex on $\mathbb{R}^2$ or on $\mathbb{R}^2 \setminus \{0\}$. By Poincaré's Lemma on contractible opens $U \subseteq \mathbb{R}^2$ the de Rham complex is quasi-isomorphic to $\mathbb{R}$ concentrated in degree zero $\mathbb{R} \cong \Omega^\bullet(U)$. On the other hand, $\Omega^\bullet(\mathbb{R}^2 \setminus \{0\})$ is quasi-isomorphic to $[\mathbb{R} \xrightarrow{0} \mathbb{R}]$. This is essentially the same as the example above.

Definition 10. Let C be an ∞-category equipped with a topology T and E an ∞-category admitting limits (the canonical choice is $E = \mathcal{S}$ the category of spaces). A *presheaf* with values in E is a functor $F : C^{op} \to E$. A presheaf is a *sheaf* if for every covering $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ we have

$$F(X) = \varprojlim_{n \in \Delta} \prod_{(\lambda_0, \dots, \lambda_n) \in \Lambda^{n+1}} F(U_{\lambda_0} \times_X \cdots \times_X U_{\lambda_n}).$$

The category of sheaves is the full subcategory

$$\mathrm{Shv}(C, E) \subseteq \mathrm{PSh}(C, E)$$

consisting of those presheaves which are sheaves.

Remark 11. It follows directly from the definition that if we have two topologies T_1, T_2 on an ∞-category, and all T_1 coverings are T_2 coverings, then any T_2-sheaf is a T_1-sheaf.

As in the classical case, the inclusion of sheaves into presheaves admits a left adjoint.

Proposition 12 ([HTT, Prop.6.2.2.7])**.** *Suppose C is an ∞-category equipped with a topology. Then the canonical inclusion* $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ *admits a left adjoint.*

Remark 13. The sheafification functor exists for abstract reasons but the adjoint functor theorem, but one can also give a more concrete description of it similar to Rem.6 above. One added complication is that instead of applying $(-)^+$ twice, one must apply it κ-many times for some ordinal κ which depends on the site, cf. the proof of [HTT, Prop.6.2.2.7].

Corollary 14. *Let C be an ∞-category equipped with a topology. The category* $\mathrm{Shv}(C)$ *admits all small colimits and small limits. The inclusion* $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ *preserves limits. That is, if $F_- : K \to \mathrm{Shv}(C)$; is a diagram of sheaves, then for any $X \in C$ the canonical morphism $(\varprojlim F_\lambda)(X) \to \varprojlim(F_\lambda(X))$ is an equivalence.*

3 Adjunctions

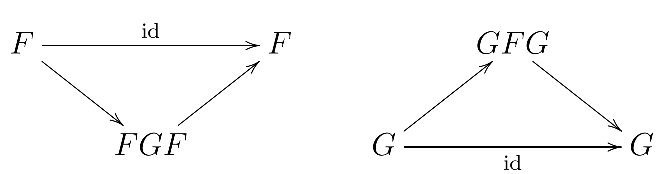
Definition 15. Let $C, D \in \mathcal{Q}\mathrm{Cat}$. An *adjunction* is a pair of functor

$$F : C \rightleftarrows D : G$$

equipped with natural transformations

$$\varepsilon : \mathrm{id}_C \to GF; \qquad \eta : FG \to \mathrm{id}_D$$

such that there exists 2-cells



Exercise 16. Show that if C, D are nerves of usual categories, then the above notion is the usual notion of adjunction.

Example 17. A limit functor $\mathrm{Fun}(I, C) \to C$ defined as we saw last time is a right adjoint.

Exercise 18. Suppose that $F : C \rightleftarrows D : G$ is an adjunction of quasi-categories and E is a third quasi-category. Show that there is an adjunction of quasi-categories

$$\mathrm{Fun}(C, E) \rightleftarrows \mathrm{Fun}(D, E)$$

and

$$\mathrm{Fun}(E, C) \rightleftarrows \mathrm{Fun}(E, D)$$

induced by composition.

Exercise 19. Suppose that we have two adjunctions

$$C \underset{G_1}{\overset{F_1}{\rightleftarrows}} D, \qquad D \underset{G_2}{\overset{F_2}{\rightleftarrows}} E$$

with units and counits respectively $\varepsilon_1, \eta_1, \varepsilon_2, \eta_2$. In the quasi-categories $\mathrm{Fun}(C, C)$ and $\mathrm{Fun}(E, E)$, choose compositions ε_3, η_3 for $\mathrm{id}_C \to G_1F_1 \to G_1G_2F_2F_1$ and $G_2G_1F_1F_2 \to G_2F_2 \to \mathrm{id}_D$. Show that $(F_2F_1, G_1G_2, \varepsilon_3, \eta_3)$ is an adjunction.

Proposition 20 ([HTT, 5.2.6.2])**.** *Let* $\mathrm{Fun}^L(C, D)$, *resp.* $\mathrm{Fun}^R(D, C)$, *denote the category of functors which are left, resp. right. adjoints. Then there is a canonical equivalence*

$$\mathrm{Fun}^L(C, D) \cong \mathrm{Fun}^R(D, C)^{op}$$

pairing up left and right adjoints.

Exercise 21. Suppose that $F : C \rightleftarrows D : G$ is an adjunction and both C and D admit I-shaped limits. Show that G preserves them. That is, there is a commutative diagram of quasi-categories

$$\begin{array}{ccc} \mathrm{Fun}(I, D) & \xrightarrow{\varprojlim} & D \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ \mathrm{Fun}(I, C) & \xrightarrow[\varprojlim]{} & C. \end{array}$$

Exercise 22. Recall that the category of presheaves of spaces admits all small limits and colimits. Using the adjunction

$$\mathrm{PSh}(C) \rightleftarrows \mathrm{Shv}(C)$$

show that $\mathrm{Shv}(C)$ admits all small limits and colimits.