Derived Algebraic Geometry
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Spring Semester 2025

Lecture 4: Limits and colimits

May 14th, 2025

References:
1. Bousfield, Kan - Homotopy limits, completions and localizations
2. Hirschorn - Model categories and their localizations
3. Lurie - Higher topos theory

In this section we discuss limits specifically in the case of homotopy types. These will be crucial for defining rings and modules in the derived setting, as well as the canonical adjunctions such as $\mathcal{S} \rightleftarrows R$-mod, A-mod $\rightleftarrows B$-mod, and A-mod $\rightleftarrows A$-alg.

1 Weighted limits in simplicial categories

Consider the following two diagrams of topological spaces.

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} & \{1\} \arrow[d] \\ \{0\} \arrow[r] & \mathbb{R} \end{tikzcd}$$

There is a (unique) natural transformation from the second diagram to the first which term-wise is a homotopy equivalence. However, if we take pullbacks in the 1-category of topological spaces, we get the punctual category $*$ in the first case and the empty category $\varnothing$ in the second. We need to account for the existence of paths. A similar phenomenon happens in the category of groupoids.

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} & \{1\} \arrow[d] \\ \{0\} \arrow[r] & \{0 \cong 1\} \end{tikzcd}$$

Here we have equivalence of groupoids instead of homotopy equivalence, and instead of paths, we should account for the existence of isomorphisms.

Example 1. Suppose that $X \xrightarrow{f} Z \xleftarrow{g} Y$ is a pair of morphisms of topological spaces.The *homotopy pullback* $X \overset{h}{\times}_Z Y$ can be defined as the limit

$$X \overset{h}{\times}_Z Y = X \times_Z \hom(\Delta^1_{\mathrm{top}}, Z) \times_Z Y$$

where $\hom(\Delta^1_{\mathrm{top}}, Z)$ is the set of continuous morphisms equipped with the compact open topology and the two morphisms $\hom(\Delta^1_{\mathrm{top}}, Z) \rightrightarrows Z$ are evaluation at 0 and 1 in $\Delta^1_{\mathrm{top}} = [0,1] \subseteq \mathbb{R}$. Explicitly, points of $X \overset{h}{\times}_Z Y$ are triples (x, y, σ) where x, resp. y, is point of X, resp. Y, and σ is a path from $f(x)$ to $g(y)$.

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Note that there are canonical projection functors forming a square which commutes up to homotopy.

$$\begin{tikzcd} X \overset{h}{\times}_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\nearrow"] & Z \end{tikzcd}$$

Example 2. We can do a similar construction when $X \to Z \leftarrow Y$ are two morphisms of groupoids (or more generally, Kan complexes). In this case we use Δ^1 instead of Δ^1_{top}, and this is called the *2-pullback*.

$$X \overset{2}{\times}_Z Y = X \times_Z \mathrm{Map}(\Delta^1, Z) \times_Z Y.$$

This construction appears when studying descent conditions. For example, if we let $\mathrm{Pic}(X)$ denote the Picard groupoid[1] of a scheme X, then we have

$$\begin{aligned} \mathrm{Pic}(\mathbb{A}^1) \overset{2}{\times}_{\mathrm{Pic}(\mathbb{G}_m)} \mathrm{Pic}(\mathbb{A}^1) &\cong \mathrm{Pic}(\mathbb{P}^1) \\ &\neq Bk^* = \mathrm{Pic}(\mathbb{A}^1) \times_{\mathrm{Pic}(\mathbb{G}_m)} \mathrm{Pic}(\mathbb{A}^1). \end{aligned}$$

Here, we are working with varieties over a field k, and BG means the group G considered as a category with one object.

To perform this kind of operation in a general simplicial category, we generalise $\hom(\Delta^1, -)$.

Definition 3. A simplicial category C is said to be *powered over* $\mathcal{S}\mathrm{et}_\Delta$ if for every $Y \in C$, $K \in \mathcal{S}\mathrm{et}_\Delta$, there exists an object $Y^K \in C$, and a natural transformation

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Map}_C(-, Y)) \cong \mathrm{Map}_C(-, Y^K)$$

Example 4. The simplicial category $\mathcal{S}\mathrm{et}_\Delta$ is powered over itself with $Y^K = \mathrm{Map}(K, Y)$.

Example 5. The category Top of topological spaces has a canonical structure of simplicial category with $\mathrm{Map}(X, Y)_n = \hom(X \times \Delta^n_{\mathrm{top}}, Y)$. It is powered over $\mathcal{S}\mathrm{et}_\Delta$ with $Y^K = \hom_{\mathrm{Top}}(|K|, Y)$ where $|K|$ is the geometric realisation and $\hom_{\mathrm{Top}}(|K|, Y)$ is equipped with the compact open topology.

[1] Objects are invertible $\mathcal{O}_X$-modules, and morphisms are isomorphisms of $\mathcal{O}_X$-modules.

Definition 6 (Cf.[Bousfield–Kan, XI.3.1, XII.2.1], [Hirschorn Def.18.1.2, Def.18.1.8])**.** Suppose that C is a $\mathcal{S}\mathrm{et}_\Delta$-powered simplicial category whose underlying category C_0[2] admits all small limits, let I be a small category and $p : I \to C_0$ a functor.

The *weighted limit*, with respect to a functor $W : I \to \mathcal{S}\mathrm{et}_\Delta$, is defined as

$$\varprojlim^W p = \mathrm{eq}\left(\prod_{i \in Ob\ I} p_i^{W_i} \rightrightarrows \prod_{\substack{i \xrightarrow{u} j \\ \in Arr\ I}} p_j^{W_i} \right)$$

where the two morphisms are induced by

$$\begin{aligned} p_u^{W_{\mathrm{id}}} &: p_i^{W_i} \to p_j^{W_i}, \\ p_{\mathrm{id}}^{W_u} &: p_i^{W_j} \to p_i^{W_i}. \end{aligned}$$

Exercise 7. Show that if W is the constant functor with value $* \in \mathcal{S}\mathrm{et}_\Delta$ then $\varprojlim^W = \varprojlim$. That is, in this case the weighted limit is the same as the usual (co)limit in the classical category $\mathcal{C}_0$.

Exercise 8. Suppose $C = \mathcal{S}\mathrm{et}_\Delta$ with the canonical powering, and each p_i is discrete, in the sense that $p_i : \Delta^{op} \to \mathcal{S}\mathrm{et}$ is constant. Show that $\varprojlim^W = \varprojlim$.

Exercise 9. Consider Top with the simplicial enrichment and powering from Example 5. Let $W : \Lambda^2_2 \to \mathcal{S}\mathrm{et}_\Delta$ be the diagram $\{0\} \to \Delta^1 \leftarrow \{1\}$. Show that for any $p : I \to \mathrm{Top}$ we have

$$\varprojlim^W p = p_0 \overset{h}{\times}_{p_1} p_2$$

from Example 1.

Example 10. The canonical choice for the weighting $W : I \to \mathcal{S}\mathrm{et}_\Delta$ is to take the nerve of the over categories

$$W(i) = N(I_{/i}).$$

Given $i \to j$, the morphism $N(I_{/i}) \to N(I_{/j})$ sends $(i_n \to \dots \to i_0 \to i)$ to $(i_n \to \dots \to i_0 \to j)$ where the last morphism is the composition $i_0 \to i \to j$.

Example 11. If I is a discrete category (no non-identity morphisms) then all $W(i)$ are $*$ so all $p_i^{W_j}$ are p_i. That is,

$$\prod^W = \prod$$

(for the standard W described in Example 10). This includes the case where I is empty. That is, the weighted terminal object is the terminal object.

[2]Recall that when C is a simplicial category, the notation C_0 means the classical category with $Ob\ C = Ob\ C_0$ and $\hom_{C_0}(X, Y) = \mathrm{Map}(X, Y)_0$.

Example 12. In the case of Λ_2^2 this gives the diagram $\{0\} \to \{0{\to}2{\leftarrow}1\} \leftarrow \{1\}$. So with this weighting, the weighted limit of a diagram $A \to B \leftarrow C$ is

$$\mathrm{eq}\left(A \times B^{\Lambda_2^2} \times C \rightrightarrows B \times B\right)$$

where one arrow is the combination of $A \to B$ and $B^{\Lambda_2^2} \to B^{\{0\}} \cong B$ and the other is the combination of $C \to B$ and $B^{\Lambda_2^2} \to B^{\{1\}} \cong B$.

Example 13. In the case $I = \{a \rightrightarrows b\}$ we get the diagram $\{a\} \overset{0}{\underset{1}{\rightrightarrows}} \left\{\begin{smallmatrix} a_0 \searrow & \\ & b \\ a_1 \nearrow & \end{smallmatrix}\right\}$. So with this weighting, the weighted limit of a diagram $A \rightrightarrows B$ is

$$\mathrm{eq}\left(A \times B^{\Lambda_2^2} \rightrightarrows B \times B\right)$$

where one arrow is projection composed with the diagonal $A \to B \times B$ and the other is projection composed with the "endpoints" map $B^{\Lambda_2^2} \to B^{\{0\}} \times B^{\{1\}}$.

Example 14. In the case $I = \mathbb{N}^{op} = \{\ldots{\to}2{\to}1{\to}0\}$ the over categories are $\mathbb{N}^{op}_{\geq n} = \{\ldots{\to}n{+}2{\to}n{+}1{\to}n\}$ so the weighted limit of a diagram $\cdots \to X_2 \to X_1 \to X_0$ with the standard weighting is

$$\mathrm{eq}\left(\prod_{n\in\mathbb{N}} X_n^{N\mathbb{N}_{\geq n}} \rightrightarrows \prod_{m\geq n} X_n^{N\mathbb{N}_{\geq m}}\right).$$

2 Limits in quasi-categories

Definition 15. Let C be a quasi-category and I a small quasi-category. We say C *admits I-limits* if the constant functor $\gamma : C \to \mathrm{Fun}(I, C)$ (induced by composing with $I \to *$) admits a right adjoint $\varprojlim$. That is, we ask for a functor

$$\varprojlim : \mathrm{Fun}(I, C) \to C$$

and two natural transformations $\varepsilon : \mathrm{id} \to \varprojlim \gamma$ and $\eta : \gamma \varprojlim \to \mathrm{id}$ such that there exist 2-cells

$$\begin{tikzcd} \varprojlim \arrow[rr, "\mathrm{id}"] \arrow[dr] & & \varprojlim \\ & \varprojlim \gamma \varprojlim \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & \gamma \varprojlim \gamma \arrow[dr] & \\ \gamma \arrow[ur] \arrow[rr, "\mathrm{id}"'] & & \gamma \end{tikzcd}$$

in C, resp. $\mathrm{Fun}(I, C)$.

Remark 16. The data we asked for above was the triple $(\varprojlim, \varepsilon, \eta)$ but in fact, ε and η determine each other (up to homotopy). Since $\gamma : C \to \mathrm{Fun}(I, C)$ is fully faithful, we could have asked for only a pair $(\varprojlim, \varepsilon)$ such that ε is an equivalence, and $\varprojlim$ induces equivalences $\mathrm{Map}(\gamma X, Y) \to \mathrm{Map}(\varprojlim \gamma X, \varprojlim Y)$ for all $X \in C$, $Y \in \mathrm{Fun}(I, C)$. However, I want to use η later, and I don't want to have to construct it.

Exercise 17. Suppose that C is a 1-category. Show that C has I-limits in the above sense if and only if it has I-limits in the usual sense.

Unwrapping the definitions, using the counit η, every diagram $X : I \to C$ determines a (strictly) commutative diagram of simplicial sets

$$\eta_X : \qquad \begin{tikzcd} I \times \{0\} \arrow[d] \arrow[r] & * \arrow[dr, "\varprojlim X"] & \\ I \times \Delta^1 \arrow[rr] & & C \\ I \times \{1\} \arrow[u] \arrow[urr, "X"'] & & \end{tikzcd}$$

More explicitly, the counit $\eta : \gamma \varprojlim \to \mathrm{id}$ is a morphism in the functor quasi-category $\mathrm{Fun}(\mathrm{Fun}(I, C), \mathrm{Fun}(I, C))$, or in other words, a morphism of quasi-categories $\mathrm{Fun}(I, C) \times \Delta^1 \to \mathrm{Fun}(I, C)$. If we evaluate on a diagram $X : I \to C$ (i.e., compose with the corresponding morphism $* \overset{X}{\to} \mathrm{Fun}(I, C)$) we obtain a morphism $\Delta^1 \to \mathrm{Fun}(I, C)$, or equivalently, a functor

$$\eta_X : I \times \Delta^1 \to C.$$

Note that η_X factors through the quotient $* \coprod_{I \times \{0\}} (I \times \Delta^1) \to C$ because the source is a constant diagram.

Definition 18 (Joyal, [HTT, Def.4.2.1.1])**.** Suppose K is a simplicial set. Define

$$K^{\blacktriangleleft} := * \coprod_{K \times \{0\}} (K \times \Delta^1)$$

Warning 19. The above notation is non-standard! What we have defined above corresponds to $K^{\blacktriangleleft} = \Delta^0 \diamond K$ in [HTT].

Exercise 20. Define an isomorphism between the quasi-category $\mathrm{Fun}(I^{\blacktriangleleft}, C)$ and the subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(I, C))$ consisting of those morphisms of diagrams $X_0 \to X_1$ such that X_0 is constant.

Exercise 21. Suppose that $X : I \to C$ is a morphism between small 1-categories, considered as quasi-categories. Show that a cone on the diagram X is the same thing as a functor

$$I^{\blacktriangleleft} \to C$$

which restricts to X along the canonical inclusion $I \subseteq I^{\blacktriangleleft}$.

Exercise 22. Describe all non-degenerate simplicies in $(\Delta^n)^{\blacktriangleleft}$ and $(\Lambda^2_2)^{\blacktriangleleft}$. In particular, show that $(\Lambda^2_2)^{\blacktriangleleft}$ is not *isomorphic* to $(\Delta^1)^{\blacktriangleleft}$.

While the construction $(-)^{\blacktriangleleft}$ is closely related to our counit above, and recovers the notion of cone in the 1-category case, in general it contains a lot of degenerate information. There is an alternative construction which is smoother in many settings.

Definition 23 ([HTT, Def.1.2.8.1]). Suppose K is a simplicial set. Given a finite linearly ordered set J, define

$$K^{\triangleleft}_J := \coprod_{J = I \sqcup I'} K_{I'}$$

where the coproduct is over decompositions $J = \{j_0 < j_1 < \cdots < j_n\} = \{j_0 < \cdots < j_i\} \sqcup \{j_{i+1} < \cdots < j_n\}$. Here we allow I and I' to be empty, and declare $K_\varnothing := *$.

Given a morphism of finite linearly ordered sets $\phi : J' \to J$ and a decomposition $J = I \sqcup I'$ we obtain an induced decomposition $J' = \phi^{-1} I \sqcup \phi^{-1} I'$ and maps $K_{I'} \to K_{\phi^{-1} I'}$. This makes $K^{\triangleleft}$ functorial in J. That is, we have a simplicial set $K^{\triangleleft}$.

Exercise 24. Show that $(\Delta^n)^{\triangleleft}$ is isomorphic to Δ^{n+1}. Show that $(\Lambda^2_2)^{\triangleleft}$ is isomorphic to $\Delta^1 \times \Delta^1$. In particular, if C is a quasi-category, $\mathrm{Fun}((\Lambda^2_2)^{\triangleleft}, C)$ is the category of commutative squares in C.

Exercise 25. Let K be a simplicial set. Show that there is a unique dashed morphism making (strictly) commutative triangles

$$\begin{tikzcd} K \arrow[r] \arrow[d] & K^{\triangleleft} \arrow[d] \\ K^{\blacktriangleleft} \arrow[r] \arrow[ur, dashed] & \Delta^1 \end{tikzcd}$$

where the solid arrows are the canonical inclusions, resp. projections.

The following proposition is surprisingly complicated to prove in general.

Proposition 26 ([HTT, Prop.4.2.1.2]). *Let K be a simplicial set. The canonical morphism $K^{\blacktriangleleft} \to K^{\triangleleft}$ is a categorical equivalence. That is, for every quasi-category C, it induces an equivalence of quasi-categories*

$$\mathrm{Fun}(K^{\triangleleft}, C) \xrightarrow{\sim} \mathrm{Fun}(K^{\blacktriangleleft}, C).$$

Definition 27 ([HTT, Not.1.2.8.4]). Let I be a simplicial set, C a quasi-category and $X : I \to C$ a functor. A *left cone* is a functor $I^{\triangleleft} \to C$ such that the composition $I \subseteq I^{\triangleleft} \to C$ is X.

Definition 28. We say that a left cone is a *limit diagram* if it corresponds to $\eta_X : I^{\blacktriangleleft} \to C$ under the equivalence of Proposition 26. In particular, the image of the apex $\{0\} \subseteq I^{\blacktriangleleft}$ is equivalent to $\varprojlim X$, and the restriction to $I \subseteq I^{\blacktriangleleft}$ is equivalent to X.

The main theorem relating limits in quasi-categories and weighted limits in simplicial categories is the following.

Theorem 29. *The category $\mathcal{S}$ of spaces admits all small limits (and colimts). Moreover, if $X : I \to \mathcal{K}\mathrm{an}$ is a functor of 1-categories with corresponding functor $NX : NI \to N\mathcal{K}\mathrm{an}$ of quasi-categories, then there exists an equivalence*

$$\varprojlim NX \cong \varprojlim{}^W X.$$

Remark 30. The weighted limit $\varprojlim^W X$ is often, but not always a Kan complex. (For nice indexing categories such as Λ^2_2 it is always a Kan complexes). If $\varprojlim^W X$ is not a Kan complex, in Theorem 29 we implicitly replace it with the Kan complex $\operatorname{Sing}|\varprojlim^W X|$.

Remark 31. See HTT, Prop.5.2.4.6] for a much stronger version of Theorem 29, which can be applied to many quasi-categories arising in practice, such as the category $\mathrm{PSh}(C)$ of presheaves of spaces on some small quasi-category C, and the category of simplicial rings.

Remark 32. The easiest way to convert the theory of limits in quasi-categories into the theory of colimits is to replace C with C^{op}. This is the simplicial set $C^{op} : \Delta \xrightarrow{\sigma} \Delta \xrightarrow{C} \mathcal{S}\mathrm{et}$ where σ sends $\{i_0 < \ i_1 < \ \cdots < i_n\}$ to $\{i_n < \ i_{n-1} < \ \cdots < i_0\}$. So a *colimit diagram* in C is a limit diagram in C^{op}.

Alternatively:

1. We consider the *left* adjoint instead of the right adjoint. That is, a functor $\varinjlim : \mathrm{Fun}(I, C) \to C$ equipped with natural transformations $\eta : \mathrm{id} \to \gamma \varinjlim$ and $\epsilon : \gamma \varinjlim \to \mathrm{id}$ such that there exists 2-cells

$$\begin{tikzcd}
\varinjlim \arrow[rr, "\mathrm{id}"] \arrow[dr] & & \varinjlim \\
& \varinjlim \gamma \varinjlim \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \gamma \varinjlim \gamma \arrow[dr] & \\
\gamma \arrow[ur] \arrow[rr, "\mathrm{id}"'] & & \gamma
\end{tikzcd}$$

   in C, resp. $\mathrm{Fun}(I, C)$.

2. $K^{\blacktriangleleft}$ is replaced with

$$K^{\blacktriangleright} = (K \times \Delta^1) \coprod_{K \times \{1\}} *.$$

3. $K^{\triangleleft}$ is replaced with $K^{\triangleright}$ defined via

$$K^{\triangleright}_J = \coprod_{J = I \sqcup I'} K_I.$$

We will discuss weighted colimits next week. Instead of simplicial categories *powered* over $\mathcal{S}\mathrm{et}_\Delta$ they use simplicial category *tensored* over $\mathcal{S}\mathrm{et}_\Delta$.

3 Main properties

We now summarise the main properties of (co)limits. All proofs are omitted but we give references to [HTT] for the interested reader.

Proposition 33 ([HTT, Lem.4.4.2.1] 2-out-of-3 for Cartesian squares)**.** *Let $C \in \mathcal{Q}\mathrm{Cat}$ and $X : \Delta^2 \times \Delta^1 \to C$ a diagram:*

$$\begin{tikzcd}
X_{00} \arrow[r] \arrow[d] \arrow[rr, bend left, dotted] \arrow[dr, dotted] \arrow[drr, dotted] & X_{10} \arrow[r] \arrow[d] \arrow[dr, dotted] & X_{20} \arrow[d] \\
X_{01} \arrow[r] \arrow[rr, bend right, dotted] & X_{11} \arrow[r] & X_{21}
\end{tikzcd}$$

Suppose that the right square is a pullback in C. Then the left square is a pullback if and only if the outer square is a pullback.

Definition 34. We say a diagram $p : K \to C$ is *finite* or $\aleph_0$*-small* if the simplicial set K has finitely many non-degenerate[3] simplicies. More generally, if κ is an uncountable regular cardinal[4] a diagram is called κ*-small* if each K_n is in $\mathcal{S}\mathrm{et}_{<\kappa}$.

Proposition 35 ([HTT, Prop.4.4.2.6, Prop.4.4.3.2])**.** *Let C be a quasi-category. The following are equivalent.*

1. *C has all κ-small limits.*
2. *C has equalisers and all κ-small products.*
3. *C has pullbacks and all κ-small products.*

Remark 36. The main tools in the above proposition are:

1. If $L' \to L$ is a monomorphism of simplicial sets, $L' \to K'$ any morphism, and $p : K' \sqcup_{L'} L \to C$ a diagram, then $\varprojlim p = \varprojlim p|_{K'} \times_{\varprojlim p|_{L'}} \varprojlim p|_L$, assuming all these limits exist, [HTT, Prop.4.4.2.2].
2. If $\{K_\alpha\}_{\alpha \in A}$ is a collection of simplicial sets and $p : \sqcup K_\alpha \to C$ is a diagram, then $\varprojlim p = \prod \varprojlim p|_{K_\alpha}$, assuming all those limits exist.

Proposition 37 ([HTT, Cor.5.1.2.3] Limits of presheaves are calculated object wise)**.** *Let $K, S \in \mathcal{S}\mathrm{et}_\Delta$ and suppose $C \in \mathcal{Q}\mathrm{Cat}$ admits K-indexed limits. Then*

1. *The quasi-category $\mathrm{Fun}(S, C)$ admits K-indexed limits.*
2. *A map $K^\triangleleft \to \mathrm{Fun}(S, C)$ is a limit diagram if and only if for each vertex $s \in S$, the induced map $K^\triangleleft \to C$ is a limit diagram.*

That is, for $F : K \to \mathrm{Fun}(S, C)$ and $s \in S_0$ we have

$$(\varprojlim_K F_k)(s) = \varprojlim_K (F_k(s)).$$

Proposition 38 ([HTT, Prop.5.3.3.3] Filtered colimits commute with finite limits)**.** *Suppose that I is a quasi-category. Then the following are equivalent.*

1. *K is* cofiltered*. That is, every finite diagram $D \to K$ admits a (not necessarily limit) cone $D^\triangleleft \to K$.*
2. *The limit functor $\varprojlim : \mathrm{Fun}(K, N\mathcal{K}\mathrm{an}) \to N\mathcal{K}\mathrm{an}$ preserves finite colimits.*

$$\varinjlim_D \varprojlim_K p = \varprojlim_K \varinjlim_D p.$$

Proposition 39 ([HTT, Lem.5.5.2.3] Limits commute with limits)**.** *Let K, L be simplicial sets, let $p : (K^\triangleleft) \times (L^\triangleleft) \to C$ be a diagram. Suppose that:*

1. *For every vertex $k \in K^\triangleleft$, the associated map $p_k : L^\triangleleft \to C$ is a limit diagram.*
2. *For every vertex $l \in L$, the associated map $p_l : K^\triangleleft \to C$ is a limit diagram.*

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[3]Recall a simplex $\sigma \in K_n$ is non-degenerate if it is not in the image of any $K_{n-1} \to K_n$.

[4]A cardinal κ is *regular* if $I \in \mathcal{S}\mathrm{et}_{<\kappa}$ and $\{K_i\}_{i \in I} \subseteq \mathcal{S}\mathrm{et}_{<\kappa}$ implies $\varinjlim_{i \in I} K_i \in \mathcal{S}\mathrm{et}_{<\kappa}$ where $\mathcal{S}\mathrm{et}_{<\kappa}$ is the category of sets of size $< \kappa$.

Then the restriction $p_0 : K^{\triangleleft} \to C$ *is a limit diagram, where* $0 \in K^{\triangleleft}$ *is the cone point. That is,*

$$\varprojlim_{k\in K} \varprojlim_{l\in L} p(k,l) = \varprojlim_{l\in L} \varprojlim_{k\in K} p(k,l).$$

Proposition 40 ([HTT, Def.6.1.1.2, Lem.6.1.3.14], Colimits are universal in $\mathcal{S}$)**.** *For any morphism* $X \to Y$ *in* $N\mathcal{K}\mathrm{an}$ *and any diagram* $p : K^{\triangleright} \to N\mathcal{K}\mathrm{an}$ *with vertex* Y*, we have*

$$X \times_Y \left(\varinjlim_{k\in K} p(k) \right) = \varinjlim_{k\in K} \left(X \times_Y p(k) \right)$$

where the colimits are taken in $N\mathcal{K}\mathrm{an}$*.*