Derived Algebraic Geometry
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Lecture 2: Homotopy types

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References:
[Goerss, Jardine, Simplicial Homotopy Theory]
[Lurie, "Keredon", `https://kerodon.net/tag/00SY`]
[Haugseng, Yet another introduction to ∞-categories]

Last time I talked about Bezout's Theorem as motivation for the course. I gave four versions in settings that were progressively more general and examples demonstrating why we would want to make the following change of setting.

$\mathbb{R}$	$\rightsquigarrow$	$\mathbb{C}$
affine geometry	$\rightsquigarrow$	projective geometry
varieties	$\rightsquigarrow$	schemes
sets	$\rightsquigarrow$	homotopy types

This week is about homotopy types. The most urgent goal is to define what *space* and *weak equivalence* means.

In the future, the notion of a homotopy type will become a primitive. We will have axioms, analogous to ZFC describing how homotopy types should behave, and building everything from there. Unfortunately, we are not quite there yet, and so instead of giving an axiomatic description, we work with concrete models. Today we propose two such models: *simplicial sets* and *topological spaces* (or rather, Kan complexes and CW complexes).

More explicitly, the outline is the following.

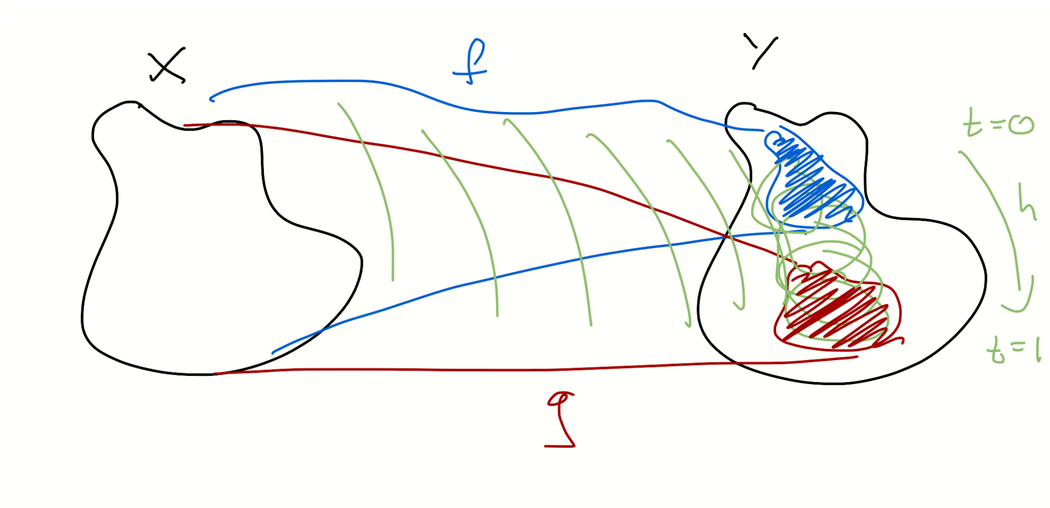
1. *homotopy groups* $\pi_n(X, x_0)$ and *weak equivalences* of topological spaces.
2. *simplicial sets* and the functor $\mathrm{Sing} : \mathrm{Top} \to \mathcal{S}\mathrm{et}_\Delta$.
3. *geometric realisation* $|-| : \mathcal{S}\mathrm{et}_\Delta \to \mathrm{Top}$ and *weak equivalence* of simplicial sets.

We also discuss:

4. *Kan complexes* (i.e,. ∞-groupoids) as models for homotopy types, and
5. *fibrant*, resp. *cofibrant*, replacement in $\mathcal{S}\mathrm{et}_\Delta$, resp. Top.

1 Topological spaces

Definition 1. Let $f, g : X \rightrightarrows Y$ be two continuous morphisms between topological spaces. A *homotopy* from f to g is a continuous morphism $h : X \times [0, 1] \to Y$ such that $h(-,0) = f(-)$ and $h(-,1) = g(-)$. If there exists a homotopy from f to g we write $f \sim g$.



Exercise 2.

1. Show that any two continuous morphisms $X \rightrightarrows \mathbb{R}$ are homotopic. Give an example of two continuous morphisms $X \rightrightarrows Y$ which are not homotopic.[1]
2. Show that $\sim$ is an equivalence relation on the set of continuous morphisms $\hom_{\mathrm{Top}}(X, Y)$ between two topological spaces.
3. Show that $\sim$ is preserved by pre- and post-composition. That is, if $f \sim g$ then $fa \sim ga$ and $bf \sim bg$ for any continuous $W \xrightarrow{a} X$, $X \xrightarrow{f} Y$, $X \xrightarrow{g} Y$, $Y \xrightarrow{b} Z$.

Definition 3. The *homotopy category* $h\mathrm{Top}$ has as objects topological spaces and hom sets $\hom_{h\mathrm{Top}}(X, Y) = \hom_{\mathrm{Top}}(X, Y)/\sim$.

A continuous morphism of topological spaces $f : X \to Y$ is a *homotopy equivalence* if it becomes an isomorphism in $h\mathrm{Top}$. If there exists a homotopy equivalence from X to Y we say that X and Y are *homotopy equivalent*.

If X is homotopy equivalent to a singleton $\{*\}$ then we say X is *contractible*.

Exercise 4.

1. Show that $\mathbb{R}^n$ is contractible.
2. Give an example of two topological spaces which are not homeomorphic, but which are homotopy equivalent.
3. Given an example of two topological spaces which are not homotopy equivalent.

Remark 5. There is also a pointed notion of homotopy. A *pointed space* is a pair (X, x_0) with X a topological space and $x_0 \in X$ a point. A *morphism* of pointed spaces $(X, x_0) \to (Y, y_0)$ is any continuous map $f : X \to Y$ such that $f(x_0) = y_0$. We write Top_* for the category of pointed topological spaces. A *homotopy* between morphisms $f, g : (X, x_0) \rightrightarrows (Y, y_0)$ of pointed spaces is a homotopy $h : X \times [0,1] \to Y$

[1] Hint: Try $X = \{0\}$ and $Y = \{\pm 1\}$

from $X \xrightarrow{f} Y$ to $X \xrightarrow{g} Y$ such that $h(x_0, t) = y_0$ for all $t \in [0,1]$. Exercise 2 can also be done in the pointed setting.

Definition 6. The set of path components of a topological space X is

$$\pi_0(X) = \hom_{\mathrm{Top}}(\{*\}, X)/\sim .$$

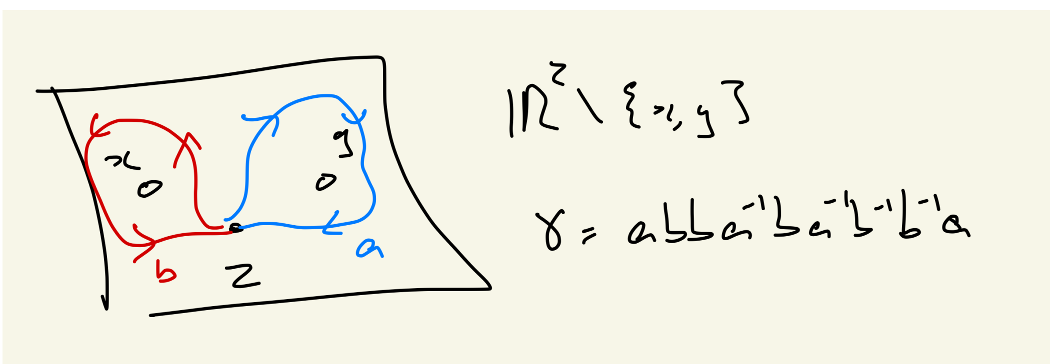
Let $S^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \ : \sum x_i^2 = 1\}$. Equipped with $e_0 := (1, 0, \dots, 0) \in S^n$ it becomes a pointed space. For $n \geq 0$, the nth *homotopy group* of a pointed space (X, x_0) is the set of morphisms of pointed spaces up to (pointed) homotopy equivalence

$$\pi_n(X, x_0) = \hom_{\mathrm{Top}_*}((S^n, e_0), (X, x_0))/\sim .$$

The homotopy groups $\pi_n(X, x_0)$ are a way of formalising how many "holes" are in a topological space.

Remark 7. Note that $S^0 = \{\pm 1\}$ and $S^n = \varnothing$ for $n < 0$.

Example 8.
1. If X is contractible, then $\pi_j(X, x_0)$ has one element for all $0 \leq j$ and $x_0 \in X$.
2. If $x_0, x_1, \dots, x_n \in \mathbb{R}^2$ are $n+1$ distinct points, then $\pi_1(\mathbb{R}^2 \setminus \{x_1, \dots, x_n\}, x_0)$ is the free group on n generators.



3.
$$\pi_j(S^n, e_0) = \begin{cases} \{*\} & j = 0 < n \\ \{0\} & 0 < j < n \\ \mathbb{Z} & 0 < j = n \\ \text{major open problem} & 1 < n \ll j \end{cases}$$

4. $\pi_j\Big(X \times Y, (x_0, y_0)\Big) \cong \pi_j(X, x_0) \times \pi_j(Y, y_0)$.

Definition 9 ([HTT, Def.1.1.3.4])**.** A continuous morphism $X \to Y$ of topological spaces is a *weak equivalence* if
1. $\pi_0(X) \to \pi_0(Y)$ is an isomorphism, and
2. $\pi_j(X, x_0) \to \pi_j(Y, f x_0)$ is an isomorphism for all $x_0 \in X$.

Exercise 10. Show that any homotopy equivalence is a weak equivalence.

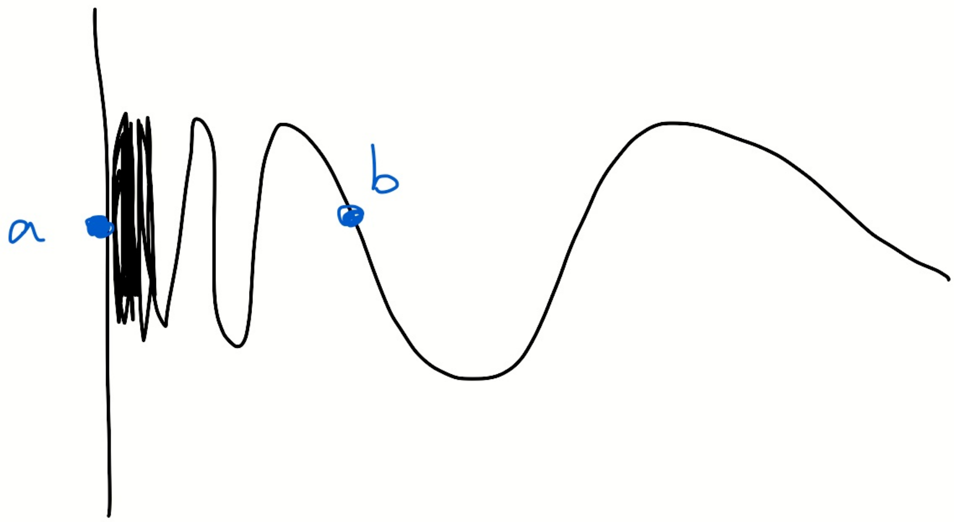
Example 11. There are weak equivalences which are not homotopy equivalences. Consider the topologists sin curve

$$X = \left\{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\right\} \sqcup \left\{(t, \sin \tfrac{\pi}{t}) \in \mathbb{R}^2 \mid t > 0\right\}$$

with points $a = (0, 0)$ and $b = (1, 0)$. There are no continuous surjections $X \to \{a, b\}$ so the inclusion

$$\{a, b\} \to X$$

is a weak equivalence which is not a homotopy equivalence; it has no inverse in $h\mathrm{Top}$.



2 Simplicial sets

Recall that a directed graph consists of a set G_0 of verticies a set G_1 of edges and two morphisms

$$d_0, d_1 : G_1 \rightrightarrows G_0$$

which associate to each edge $e \in G_1$ a source $d_1 e \in G_0$ and a target $d_0 e \in G_0$.

We can generalise this in higher dimensions by allowing "n-dimensional edges" between "$(n-1)$-dimensional edges" for all $n \in \mathbb{N}$. The information of all these higher edges and how they are related to each other is organised in the concept of a *simplicial set*.

Definition 12. We write $\Delta \subseteq \mathrm{LinOrdSet}$ for the full subcategory of the category LinOrdSet of linearly ordered sets whose objects are finite and non-empty. In other words, those linearly ordered sets which are isomorphic to $[n] = \{0 < 1 < \cdots < n\}$ for some $n \geq 0$. Morphisms are those morphisms of sets $p : [n] \to [m]$ such that $i \leq j \Rightarrow p(i) \leq p(j)$.

The category of simplicial sets $\mathcal{S}\mathrm{et}_\Delta$ is the category of functors $\Delta^{op} \to \mathcal{S}\mathrm{et}$, so

$$\mathcal{S}\mathrm{et}_\Delta := \mathrm{PSh}(\Delta)$$

Given such a functor $K : \Delta^{op} \to \mathcal{S}\mathrm{et}$ we write $K_n := K([n])$. Elements of K_n are called *n-simplicies* of K.

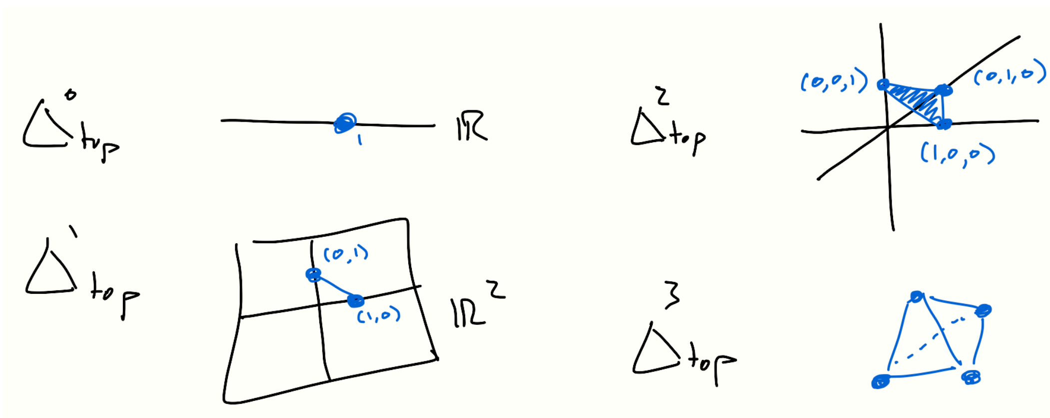
Example 13 (Δ^n)**.** For each n, the functor $\Delta^n := \hom_\Delta(-, [n]) : \Delta^{op} \to \mathcal{S}\mathrm{et}$ defines a simplicial set. By Yoneda's Lemma, for any $K \in \mathcal{S}\mathrm{et}_\Delta$,

$$\hom_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, K) \cong K_n.$$

Example 14 ($\mathrm{Sing}\, X$)**.** Define

$$\Delta^n_{\mathrm{top}} := \left\{ (x_0, \dots, x_n) \mid 0 \leq x_i \leq 1; \sum_{i=0}^n x_i = 1 \right\}$$

to be the convex hull of the standard basis vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. So Δ^0_{top} is a point, Δ^1_{top} is a line segment, Δ^2_{top} is a triangle, Δ^3_{top} is a tetrahedron, ...



Any morphism $p : [n] \to [m]$ in Δ defines an $\mathbb{R}$-linear morphism $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$; $e_i \mapsto e_{p(i)}$, which restricts to a continuous morphism $\Delta^n_{\mathrm{top}} \to \Delta^m_{\mathrm{top}}$. In this way we get a functor

$$\Delta \to \mathrm{Top}; \qquad [n] \mapsto \Delta^n_{\mathrm{top}}$$

from Δ to the category of topological spaces. For any other topological space X, the assignment

$$\mathrm{Sing}\, X : [n] \mapsto \hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, X)$$

defines a simplicial set. Explicitly,

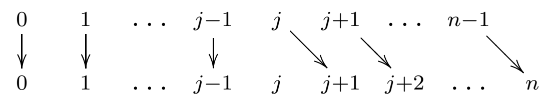
1. $\mathrm{Sing}_0\, X$ is the set of points of X,
2. $\mathrm{Sing}_1\, X$ is the set of paths in X,
3. $\mathrm{Sing}_2\, X$ is the set of triangles in X,
4. ...

Remark 15. The term "singular" refers to the fact that we might have restricted our attention to smooth manifolds X and smooth maps $\Delta^n_{\text{top}} \to X$. However, our maps are only required to be continuous, and we allow any topological space X.

Remark 16. Since the boundary $\partial\Delta^{n+1}_{\text{top}}$ of $\Delta^{n+1}_{\text{top}}$ is homeomorphic to S^n, all homotopy groups of X can be recovered from the simplicial set $\operatorname{Sing} X$. One can think of $\operatorname{Sing} X$ as a combinatorial model of X.

3 Kan complexes

Definition 17. For each $0 \leq j \leq n \neq 0$ the *face* morphism $\delta_j : [n-1] \to [n]$ are defined as the unique injection which does not have j in its image.



For any simplicial set $K : \Delta^{op} \to \mathcal{S}\text{et}$ we have a corresponding morphism

$$d_j : K_n \to K_{n-1}.$$

These (i.e., δ_j and d_j) are called *face* morphisms. For $\sigma \in K_n$ we call $d_j\sigma$ the *jth face of σ*.

Exercise 18. Show that every monomorphism in Δ is a composition of δ_j's.

Exercise 19. Consider the morphism $\Delta^n_{\text{top}} \to \Delta^{n+1}_{\text{top}}$ associated to δ_j. Draw this morphism for $0 \leq j \leq n \leq 2$.

Example 20 ($\partial\Delta^n$)**.** Consider the morphisms of simplicial sets $\delta_j : \Delta^{n-1} \to \Delta^n$. We define

$$\partial\Delta^n = \bigcup_{j=0}^{n} \delta_j(\Delta^{n-1})$$

as the union of these faces. Explicitly, $(\partial\Delta^n)_j \subseteq (\Delta^n)_j = \hom_\Delta([j],[n])$ is the set of morphisms $[j] \to [n]$ of linearly ordered sets which are not surjective.

Exercise 21. Show that $\partial\Delta^n_{\text{top}} = \cup_{j=0}^n \delta_j(\Delta^{n-1}_{\text{top}})$ is the boundary of $\Delta^n_{\text{top}} \subseteq \mathbb{R}^{n+1}$.

Exercise 22. Let K be a simplicial set.

1. Show that a morphism $f : \partial\Delta^n \to K$ of simplicial sets canonically determines a collection of simplicies $k_0, k_1, \ldots, k_n \in K_{n-1}$ such that we have $\delta_i^* k_j = \delta_{j-1}^* k_i$ for $i < j$.
2. (Harder) Conversely, show that a collection of simplicies $k_0, k_1, \ldots, k_n \in K_{n-1}$ such that we have $\delta_i^* k_j = \delta_{j-1}^* k_i$ for $i < j$ determines a morphism $f : \partial\Delta^n \to K$ of simplicial sets. Hint.[2]

[2] I would do this as follows. Consider the partially ordered set I consisting of those sub-linearly ordered sets $\sigma \subseteq [n]$ such that $\sigma \cong [n-1]$ or $\sigma \cong [n-2]$. This determines a diagram $I \to \mathcal{S}\text{et}_\Delta$; $\sigma \mapsto \Delta^\sigma$. Show that $\partial\Delta^n \cong \varinjlim_{\sigma\in I} \Delta^\sigma$, and therefore $\hom(\partial\Delta^n, K) = \varprojlim_{\sigma\in I} \hom(\Delta^\sigma, K)$. Now use Yoneda $\hom(\Delta^\sigma, K) \cong K_{|\sigma|-1}$.

Definition 23 (Λ^n_j)**.** For $0 \le j \le n$ we define the *jth horn* as the union

$$\Lambda^n_j = \bigcup_{i \neq j} \delta_i(\Delta^{n-1}).$$

Equivalently, $(\Lambda^n_j)_i \subseteq (\Delta^n)_i = \hom_\Delta([i],[n])$ is the set of those $[i] \to [n]$ whose image does *not contain* the subset $\{0, 1, \dots, j{-}1, j{+}1, \dots, n\}$.

Exercise 24. Define $\Lambda^{n+1}_{\text{top},j} = \bigcup_{i \neq j} \delta_i(\Delta^n_{\text{top}})$. Draw $\Lambda^n_{\text{top},j}$ for $0 \le j \le n \le 2$.

Exercise 25. Do the Λ^n_i analogue of Exercise 22.

Definition 26 (Kan fibration)**.** A morphism $f : X \to Y$ of simplicial sets is a *Kan fibration* if for every $0 \le j \le n$ with $0 \neq n$ and commutative square

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & K \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & L \end{array}$$

a dashed morphism exists making two triangles commutative. A simplicial set K is a *Kan complex* if the canonical morphism $K \to \Delta^0$ is a Kan fibration.

Remark 27. Note $\varnothing \to Y$ is a Kan fibration.

Exercise 28. Recall the topological space $\Lambda^n_{\text{top},j} = \bigcup_{i \neq j} \delta_i(\Delta^{n-1}_{\text{top}})$ from Exercise 24. Show that for any $0 \le j \le n$ there exists a continuous retraction[3] $\Delta^n_{\text{top}} \to \Lambda^n_{\text{top},j}$ to the inclusion $\Lambda^n_{\text{top},j} \subseteq \Delta^n_{\text{top}}$. Deduce that for any commutative square

$$\begin{array}{ccc} \Lambda^n_{\text{top},j} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n_{\text{top}} & \longrightarrow & \{*\} \end{array}$$

of topological spaces, there exists a dashed line making commutative triangles.

Example 29. Below we will see that Sing admits a left adjoint $|-|$ with $|\Delta^n| = \Delta^n_{\text{top}}$ and $|\Lambda^n_j| = \Lambda^n_{\text{top},j}$. Then it follows immediately from Exercise 28 that for any topological space X the simplicial set $\operatorname{Sing} X$ is a Kan complex.

Remark 30. One way of thinking about Kan fibrations is as an ∞-groupoid valued ∞-functor from a ∞-directed graph. We will see more about this next week.

1. Note that there is a pushout square

$$\begin{array}{ccc} \Delta^0 & \xrightarrow{d_1} & \Delta^1 \\ {\scriptstyle d_0}\downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & \Lambda^2_1 \end{array}$$

[3] That is, a continuous morphism such that the composition $\Lambda^n_{\text{top},j} \to \Delta^n_{\text{top}} \to \Lambda^n_{\text{top},j}$ is the identity.

So giving a map $\Lambda^2_1 \to K$ is the same as giving two edges $f, g \in K_1$ such that $d_0 f = d_1 g$. In this way, a map $\Lambda^2_1 \to K$ can be thought of as a a pair of composable edges $\xrightarrow{f}\xrightarrow{g}$. Then for any extension $\Lambda^2_1 \to \Delta^2 \dashrightarrow K$, the third edge $\Delta^1 \stackrel{\delta_1}{\to} \Delta^2 \dashrightarrow K$ can be thought of as a choice of composition $g \circ f$.

2. We can extend any edge $e : \Delta^1 \to K$ to map $e_\varepsilon : \Lambda^2_\varepsilon \to K$ in two canonical ways ($\varepsilon = 0, 2$), using the degeneracies $\Delta^1 \to \Delta^0 \stackrel{\sigma_{\varepsilon/2}}{\to} \Delta^1 \stackrel{e}{\to} K$. Then for any extension $\Lambda^2_1 \to \Delta^2 \dashrightarrow K$, the edge $\Delta^1 \stackrel{\delta_{\varepsilon/2}}{\to} \Delta^2 \dashrightarrow K$ can be thought of as a choice of left, resp. right, inverse.
3. The higher extensions can be thought of as higher compositions / inverses.
4. If $\pi : X \to Y$ is a Kan fibration, then for every $y : \Delta^0 \to Y$, the fibre $\Delta^0 \times_Y X$ is a Kan complex.
5. Suppose we have $x : \Delta^0 \to X$ and an edge $e : \Delta^1 \to Y$ such that $\pi x = \sigma_0 e$. The lifting condition applied to Λ^1_1 assures a lift $\Delta^1 \dashrightarrow X \to Y$, from which we get $x' : \Delta^0 \stackrel{\sigma_1}{\to} \Delta^1 \dashrightarrow X$. Choosing a x' for every x in this way, we get a map $(\pi^{-1}y)_0 \to (\pi^{-1}y')_0$ (where y, y' are the endpoints of e). We can think of this as part of an ∞-functor $\pi^{-1}y \to \pi^{-1}y'$ associated to the edge $e \in Y_1$.

Just as for topological spaces, we can define the notion of homotopy equivalence of simplicial sets.

Definition 31. If K, L are two simplicial sets, we get a new simplicial set $K \times L$ by setting

$$(K \times L)_n = K_n \times L_n.$$

Exercise 32.

1. Given $[m] \to [n]$ in Δ, describe the associated morphisms of sets $(K \times L)_n \to (K \times L)_m$.
2. Let X, Y be topological spaces and show that $\operatorname{Sing}(X \times Y) = (\operatorname{Sing} X) \times (\operatorname{Sing} Y)$.
3. Draw the topological spaces $\Delta^1_{\text{top}} \times \Delta^1_{\text{top}}$ and $\Delta^1_{\text{top}} \times \Delta^2_{\text{top}}$. Describe all non-degenerate simplices[4] in Δ^1, Δ^2, $\Delta^1 \times \Delta^1$ and $\Delta^1 \times \Delta^2$.

Definition 33. Let $f, g : K \rightrightarrows L$ be two morphisms of simplicial sets. A *homotopy* from f to g is a morphism

$$h : K \times \Delta^1 \to L$$

such that $h(-, 0) = f(-)$ and $h(-, 1) = g(-)$. Here, $h(-, 0)$ (resp. $h(-, 1)$) means the the composition $K \cong K \times \Delta^0 \to K \times \Delta^1 \to L$ where $\Delta^0 \to \Delta^1$ corresponds to $[0] \to [1]; 0 \mapsto 0$ (resp. $0 \mapsto 1$).

Exercise 34. Suppose that $f, g : X \rightrightarrows Y$ are two continuous morphisms of topological spaces which are homotopic. Show that $\operatorname{Sing} f, \operatorname{Sing} g : \operatorname{Sing} X \rightrightarrows \operatorname{Sing} Y$ are homotopic.

[4] A simplex $\sigma \in K_n$ is called *non-degenerate* if it is not of the form $p^*\sigma$ for some surjection $p : [n] \to [n-1]$.

Remark 35. Unlike Top, being homotopic is not an equivalence relation on $\hom(K, L)$ for general $K, L \in \mathcal{S}\mathrm{et}_\Delta$ since there is no appropriate morphism of simplicial sets $\Delta^1 \to \Lambda^2_1$. However, we will see below that K is a Kan complex, then simplicial homotopy does become an equivalence relation on $\hom(L, K)$ for any L.

Definition 36. A morphism of simplicial sets $K \to L$ is a *(Quillen) weak equivalence* if $|K| \to |L|$ is a weak equivalence of topological spaces.

4 Geometric realisation

In this section we consider the left adjoint $|-|$ to Sing. To begin with the just assume it exists, but this existence will follow from the concrete descriptions we obtain.

We will heavily use colimits of topological spaces so to be clear, we recall a construction.

Definition 37 (Colimits of topological spaces)**.** Let I be a category and $X : I \to \mathrm{Top}$ a functor. The colimit of this diagram can be constructed explicitly as follows. The underlying set of $\varinjlim_{i\in I} X_i$ is the colimit taken in the category of sets. That is, it is the quotient of the disjoint union $\sqcup_{i\in I} X_i$ by the equivalence relation *generated by* $x_i \in X_i$ is equivalent to $x_j \in X_j$ if there exists $u : i \to j$ in I such that $X_u(x_i) = x_j$.

We equip $\sqcup_{i\in I} X_i/\sim$ with the finest topology such that the canonical morphisms $\iota_i : X_i \to \sqcup_{i\in I} X_i/\sim$ are continuous. Explicitly, a subset $U \subseteq \sqcup_{i\in I} X_i/\sim$ is open if and only if $\iota_i^{-1}(U)$ is open for all i.

Exercise 38. Suppose that $Z_1, Z_2 \subseteq X$ are two subspaces of a topological space X. Show that if Z_1, Z_2 are both closed, then $Z_1 \cup Z_2$ is homeomorphic to $Z_1 \sqcup_{Z_1 \cup Z_2} Z_2$. Give an example of subspaces $Z_1, Z_2 \subseteq X$ (not closed) such that $Z_1 \cup Z_2$ is not homeomorphic to $Z_1 \sqcup_{Z_1\cap Z_2} Z_2$.

Geometric realisation as a colimit of representables. Suppose that the left adjoint $|\cdot| : \mathcal{S}\mathrm{et}_\Delta \to \mathrm{Top}$ exists. Being left adjoint to Sing forces the following properties:

1. Since

$$\hom_{\mathrm{Top}}(|\Delta^n|, -) = \hom_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, \mathrm{Sing}\,-) = (\mathrm{Sing}\,-)_n = \hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, -)$$

by coYoneda we must have

$$|\Delta^n| = \Delta^n_{\mathrm{top}}.$$

2. Since $|-|$ is a left adjoint it has to preserve colimits;

$$|\varinjlim(-)| = \varinjlim |-|$$

Since every presheaf can canonically be written as a colimit of representables, this completely determines $|-|$. Explicitly, given a simplicial set $K \in \mathcal{S}\mathrm{et}_\Delta$ let $\Delta_{/K}$ be the category whose objects are pairs $([n], k)$ where $n \in \mathbb{N}$ and $k \in K_n$; i.e., elements of $\amalg_{\mathbb{N}} K_n$. A morphism $([n], k) \to ([n'], k')$ is a morphism $\sigma : [n] \to [n']$ such that

$K_{n'} \to K_n$ sends k' to k. Then $K = \varinjlim_{([n],k)\in\Delta_{/K}} \Delta^n$ (see Exercise 39) so we must have

$$|K| = \varinjlim_{([n],k)\in\Delta_{/K}} \Delta^n_{\text{top}}.$$

Exercise 39. Let $F : C^{op} \to \mathcal{S}\text{et}$ be a presheaf of sets. Show that

$$F = \varinjlim_{\substack{c\in Ob\ C,\\ \hom(-,c)\xrightarrow{s}F}} \hom_C(-,c)$$

where the colimit is indexed by the category $C_{/F}$ described above (recall that $F(c) \cong \hom(\hom(-,c),F)$ by Yoneda's Lemma).

Exercise 40. Let I be the category associated to the partially ordered set of the sub-linearly ordered sets of $[n]$ of size n and $n-1$. Let J be the category obtained from I by removing the object $\{0,\ldots,j{-}1,j{+}1,\ldots,n\}$. Show that $\partial\Delta^n_{\text{top}} = \varinjlim_{L\in I} \Delta^L_{\text{top}}$. Using the fact that $|-|$ preserves colimits deduce that $|\partial\Delta^n| = \partial\Delta^n_{\text{top}}$.

Geometric realisation as a homotopy colimit. In general, any colimit can be written as a coequalisers of coproducts , $\varinjlim_{i\in I} X_i = \text{coeq}(\sqcup_{i\xrightarrow{u}j\in I} X_i \rightrightarrows \sqcup_{i\in I} X_i)$. If we do this for the above colimit, we get $K = \text{coeq}\left(\coprod_{[n]\xrightarrow{\sigma}[m]} \coprod_{k\in K_m} \Delta^n \rightrightarrows \coprod_{n\in\Delta} \coprod_{k\in K_n} \Delta^n\right)$ where one morphism sends the kth copy of Δ^n to the $\sigma^* k$th copy of Δ^n, and the other morphism is the canonical $\sigma : \Delta^n \to \Delta^m$ from the kth copy to the kth copy. So we must have $|K| = \text{coeq}\left(\coprod_{[n]\xrightarrow{\sigma}[m]} \coprod_{k\in K_m} \Delta^n_{\text{top}} \rightrightarrows \coprod_{n\in\Delta} \coprod_{k\in K_n} \Delta^n_{\text{top}}\right)$ which can be written as

$$|K| = \text{coeq}\left(\coprod_{[n]\xrightarrow{\sigma}[m]} K^\delta_m \times \Delta^n_{\text{top}} \rightrightarrows \coprod_{n\in\Delta} K^\delta_n \times \Delta^n_{\text{top}}\right)$$

where X^δ means the set X equipped with the discrete topology. We will see in a few weeks that this is a model for the homotopy colimit of discrete spaces

$$\text{hocolim}_{[n]\in\Delta} K^\delta_n$$

in Top.

As a tower of relative cells complexes. Finally, recall that one defines a simplex $\sigma \in K_n$ to be *degenerate* if $\sigma \in \bigcup_{\hom([n],[n-1])} \text{im}(K_{n-1} \to K_n)$ and *non-degenerate* if it is not degenerate. Write $NK_n \subseteq K_n$ for the set of non-degenerate simplicies of dimension n, and for $n \geq -1$ define $\text{sk}_{-1} K = \varnothing$ and let

$$\text{sk}_n K = \bigcup_{\substack{0\leq j\leq n\\ \sigma\in NK_j}} \text{im}(\Delta^j \xrightarrow{\sigma} K) \subseteq K$$

be the smallest subsimplicial set containing all $\sigma \in NK_j; j \leq n$. Note that sk_n is functorial, and $\partial\Delta^n = \operatorname{sk}_{n-1}\Delta^n$. In particular, given any $\sigma \in NK_n$, we have a corresponding morphism $\partial\Delta^n \to \operatorname{sk}_{n-1} K$. In fact, one sees that there exist cocartesian squares

$$\begin{array}{ccc} \coprod_{NK_n} \partial\Delta^n & \longrightarrow & \operatorname{sk}_{n-1} K \\ \downarrow & & \downarrow \\ \coprod_{NK_n} \Delta^n & \longrightarrow & \operatorname{sk}_n K. \end{array}$$

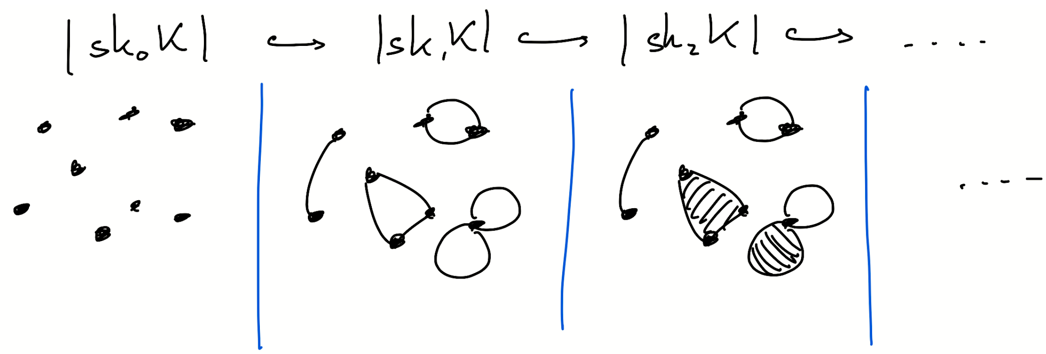
We have seen that $|\Delta^n| = \Delta^n_{\text{top}}$ and by Exercise 40 we have $|\partial\Delta^n| = \partial\Delta^n_{\text{top}}$. Consequently, since $|-|$ preserves all colimits there are cocartesian squares

$$\begin{array}{ccc} \coprod_{NK_n} \partial\Delta^n_{\text{top}} & \longrightarrow & |\operatorname{sk}_{n-1} K| \\ \downarrow & & \downarrow \\ \coprod_{NK_n} \Delta^n_{\text{top}} & \longrightarrow & |\operatorname{sk}_n K|. \end{array}$$

Morevoer, $K = \bigcup_{j\geq 0} \operatorname{sk}_j K = \varinjlim(\operatorname{sk}_0 K \to \operatorname{sk}_1 K \to \dots)$ so

$$|K| = \varinjlim\Big(|\operatorname{sk}_0 K| \to |\operatorname{sk}_1 K| \to |\operatorname{sk}_2 K| \to \dots\Big).$$

In other words, we obtain $|K|$ be sequentially glueing cells Δ^n_{top} along their boundaries $\partial\Delta^n_{\text{top}} \to |\operatorname{sk}_{n-1} K|$.



Corollary 41. *For $K \in \mathcal{S}\mathrm{et}_\Delta$ there is a bijection of* sets

$$|K| \cong K_0 \amalg \left(\coprod_{n>0} \coprod_{NK_n} (\Delta^n_{top})^\circ \right)$$

where $(\Delta^n_{top})^\circ$ means the interior of Δ^n_{top}. In particular, a simplex $k \in K_n$ is non-degenerate if and only if the induced continuous morphism $(\Delta^n_{top})^\circ \to |K|$ is injective, and degenerate if and only if it factors via a linear projection $(\Delta^n_{top})^\circ \to (\Delta^m_{top})^\circ \to |K|$ for some $m < n$ and some non-degenerate $\Delta^m \to K$.

5 (Co)fibrant replacement

To finish we state the following without proof.

Theorem 42.

1. *For every $K \in \mathcal{S}\mathrm{et}_\Delta$, the morphism $K \to \operatorname{Sing}|K|$ is a weak equivalence.*
2. *For every $X \in \mathrm{Top}$, the morphism $|\operatorname{Sing} X| \to X$ is a weak equivalence.*
3. *If K, L are Kan complexes then a morphism $K \to L$ is a weak equivalence if and only if it is a homotopy equivalence.*
4. *For any simplicial sets K, L a morphism $|K| \to |L|$ is a weak equivalence if and only if it is a homotopy equivalence.*

Exercise 43. Show that part 1 of the theorem follows from part 2.

Exercise 44. Kan fibrations don't just lift through $\Lambda^n_i \to \Delta^n$, but have the stronger property that they also lift through $\Lambda^n_i \times L \to \Delta^n \times L$ for any simplicial set L, [Goerss, Jardine, Cor.I.4.3 Cor.I.4.6]. That is, if $K \to M$ is a Kan fibration then any commutative square

$$\begin{array}{ccc} \Lambda^n_i \times L & \longrightarrow & K \\ \downarrow & \nearrow & \downarrow \\ \Delta^n \times L & \longrightarrow & M \end{array}$$

admits a dashed morphism making commutative triangles. Using this, show that if K is a Kan complex, then simplicial homotopy is an equivalence relation on $\hom_{\mathcal{S}\mathrm{et}_\Delta}(L, K)$.

Exercise 45. Let $\mathcal{K}\mathrm{an} \subseteq \mathcal{S}\mathrm{et}_\Delta$ denote the full subcategory of Kan complexes, and write $h\mathcal{K}\mathrm{an}$ for the category with the same objects as $\mathcal{K}\mathrm{an}$, and hom sets homotopy equivalences classes of morphisms $\hom_{h\mathcal{K}\mathrm{an}}(K, L) = \hom_{\mathcal{S}\mathrm{et}_\Delta}(K, L)/\sim$. Consider the functor

$$\mathcal{S}\mathrm{et}_\Delta \to h\mathcal{K}\mathrm{an}; \qquad K \mapsto \operatorname{Sing}|K|.$$

Using the theorem above show that for any category C, the induced functor

$$\operatorname{Fun}(h\mathcal{K}\mathrm{an}, C) \to \operatorname{Fun}(\mathcal{S}\mathrm{et}_\Delta, C)$$

is fully faithful and it's essential image is the category of functors $\mathcal{S}\mathrm{et}_\Delta \to C$ which send all weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$ to isomorphisms in C. In other words, $\mathcal{S}\mathrm{et}_\Delta \to h\mathcal{K}\mathrm{an}$ is the localisation along the class $\mathcal{W}$ of weak equivalences.

$$\mathcal{S}\mathrm{et}_\Delta[\mathcal{W}^{-1}] \cong h\mathcal{K}\mathrm{an}.$$

Here Fun means the category whose objects are functors, and morphisms are natural transformations.

Definition 46. A *CW-complex* is a topological space that can be written as $X = \varinjlim(X(-1) \to X(0) \to X(1) \to X(2) \to \dots)$ such that $X(-1) = \varnothing$ and for $n \geq 0$ the nth transition morphism is a pushout of the form

$$\begin{array}{ccc} \coprod_{I_n} \partial\Delta^n_{\text{top}} & \xrightarrow{g_n} & X(n-1) \\ \downarrow & & \downarrow \\ \coprod_{I_n} \Delta^n_{\text{top}} & \dashrightarrow & X(n). \end{array}$$

for some maps g_n and sets I_n. Write $\text{CW} \subseteq \text{Top}$ for the full subcategory of CW-complexes. Note that the geometric realisation of a simplicial set is a CW-complex.

Exercise 47. The same exercise as above but with $\text{CW} \subseteq \text{Top}$ instead of $\mathcal{K}\text{an} \subseteq \mathcal{S}\text{et}_\Delta$. So

$$\text{Top}[\mathcal{W}^{-1}] \cong h\text{CW}.$$

Exercise 48. Using the theorem above show that the adjunction $(|-|, \text{Sing})$ induces an equivalence of categories

$$h\mathcal{K}\text{an} \cong h\text{CW}.$$