Derived Algebraic Geometry
Shane Kelly, UTokyo
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If you find mistakes in these notes, please email me: shanekelly64[at]gmail[dot]com.

1 Motivation

References:

`http://math.stanford.edu/~vakil/245/245class1.pdf`

Theorem 1 (Bezout's Theorem, Version I). *Suppose that $f(x, y), g(x, y) \in \mathbb{R}[x, y]$ are two polynomials of degree d and e respectively, and*

$$C = \{(a, b) \in \mathbb{R}^2 \mid f(a, b) = 0\}$$
$$C' = \{(a, b) \in \mathbb{R}^2 \mid g(a, b) = 0\}$$

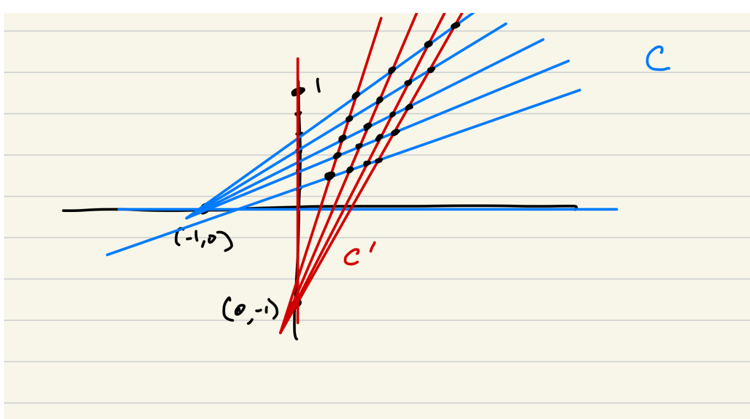
the corresponding curves in $\mathbb{R}^2$. Then, if $C \cap C'$ is finite, we have

$$|C \cap C'| \leq d \cdot e.$$

Example 2. Take

$$f(x, y) = \prod_{i=0}^{d-1} (dy - i(1 + x)),$$
$$g(x, y) = \prod_{j=0}^{e-1} (ex - j(1 + y)).$$

So C is the union of the lines through $(-1, 0)$ and $(0, \frac{i}{d})$ for $0 \leq i < d$. Similarly, C' is the union of the lines through $(0, -1)$ and $(\frac{j}{e}, 0)$ for $0 \leq j < e$. Each of the former lines intersects each of the latter lines exactly once. Hence, there are $d \cdot e$ points in common.

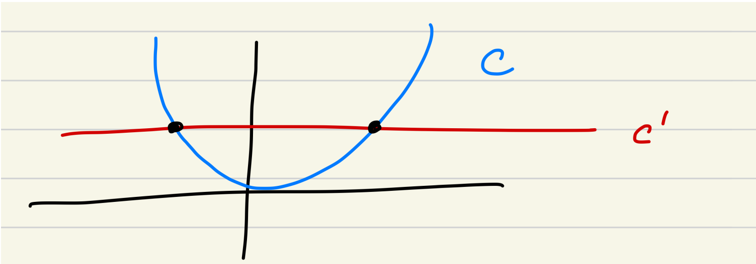


Now let's look at why we have "$\leq d \cdot e$" and not "$= d \cdot e$".

Example 3. Take

$$\begin{aligned} f(x,y) &= y - x^2, \\ g(x,y) &= y - 1. \end{aligned}$$

We can parametrise C' as $\{(t,1) : t \in \mathbb{R}\}$. Restricting $f(x,y)$ along this map $\gamma : \mathbb{R} \to \mathbb{R}^2; t \mapsto (t,1)$ we see that $\gamma(t) = (t,1)$ is in C if and only if t is a solution of $f(\gamma(t)) = 1 - t^2 = (1-t)(1+t)$. So we get $2 = 1 \cdot 2$ solutions.



On the other hand if we had chosen

$$g(x,y) = y + 1,$$

then we would end up with $1 + t^2$ which has no real solutions. However, if we use $\mathbb{C}$ instead of $\mathbb{R}$, this problem goes away.

$\mathbb{R}$	$\rightsquigarrow$	$\mathbb{C}$

The case

$$g(x,y) = y$$

produces $f(\gamma(t)) = t^2$ which has only one solution. We can correct for this by taking into account the square. The modern way of doing this is to move from geometry to algebra.

geometry	$\rightsquigarrow$	algebra
affine varieties	$\rightsquigarrow$	rings
$\mathbb{C}^2$	$\rightsquigarrow$	$\mathbb{C}[x,y]$
curves in $\mathbb{C}^2$	$\rightsquigarrow$	quotients of $\mathbb{C}[x,y]$
intersection	$\rightsquigarrow$	tensor product

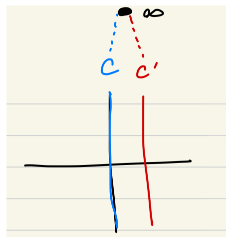
Then we have

$$\begin{array}{c}\#\text{ points in the} \\ \text{intersection}\end{array} = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[x,y]}{y - x^2} \otimes_{\mathbb{C}[x,y]} \frac{\mathbb{C}[x,y]}{y} \right) = \dim_{\mathbb{C}} \frac{\mathbb{C}[t]}{t^2} = 2.$$

Example 4. Take

$$f(x, y) = x,$$
$$g(x, y) = x - 1.$$

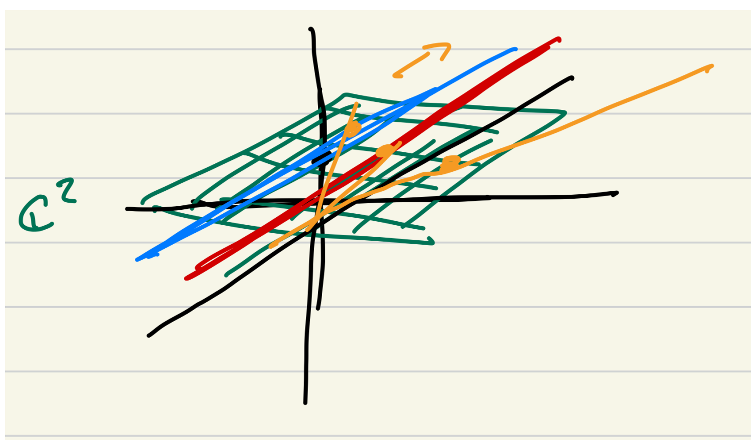
Then C and C' are parallel lines, so there are no points in common.



We fix this problem by adjoining the points at infinity where those two parallel lines should meet. We do this by embedding $\mathbb{C}^2$ into $\mathbb{C}^3$; $(x, y) \mapsto (x, y, 1)$. Now we can identify points of $\mathbb{C}^2$ with the non-horizontal lines through the origin.

$$\mathbb{C}^2 \cong \{\{(at, bt, ct) \in \mathbb{C}^3 : t \in \mathbb{C}\} : c \neq 0\}$$
$$(x, y) \mapsto \{(xt, yt, t) \in \mathbb{C}^3 : t \in \mathbb{C}\} =: (X : Y : 1)$$

Our two lines $f(x, y) = x$ and $g(x, y) = x - 1$ now correspond to planes $f(X, Y, Z) = X$ and $g(X, Y, Z) = X - 1$ which meet in the line $(0 : 1 : 0) = \{(0, t, 0) \in \mathbb{C}^3 : t \in \mathbb{C}\}$.



With a little more thought, one can see that the horizontal lines $\mathbb{C}^3$ are in bijection with parallel classes of lines in $\mathbb{C}^2$. We define

$$\mathbb{CP}^2 := \{\text{ lines in } \mathbb{C}^3 \text{ through the origin}\}.$$

Polynomial functions $\sum a_{ij} x^i y^j \in \mathbb{C}[x, y]$ of degree d on $\mathbb{C}^2 \subseteq \mathbb{CP}^2$ now correspond to homogeneous polynomials $\sum a_{ij} X^i Y^j Z^{d-i-j} \in \mathbb{C}[X, Y, Z]$ of degree d.

affine plane	$\rightsquigarrow$	projective plane
points in $\mathbb{C}^2$	$\rightsquigarrow$	lines through the origin in $\mathbb{C}^3$
polynomials in $\mathbb{C}[x, y]$	$\rightsquigarrow$	homogeneous polynomials in $\mathbb{C}[X, Y, Z]$

Theorem 5 (Bezout's Theorem, Version II). *Suppose k is an algebraically closed field, and $f(X,Y,Z), g(X,Y,Z) \in k[X,Y,Z]$ are two homogeneous polynomials of degree d and e respectively, with corresponding curves $C, C' \subseteq \mathbb{P}^2$. Then, if $C \cap C'$ is finite, we have*

$$|C \cap C'| = d \cdot e$$

as long as points are counted wth multiplicity.

Now what about higher dimension? The above adjustments (algebraically closing the field, moving to projective space, counting with multiplicity) are quite robust.

Theorem 6 (Bezout's Theorem, Version III). *Suppose k is algebraically closed, $f_1, \ldots, f_n \in k[X_0, \ldots, X_n]$ are n homogeneous polynomials of degrees $d_1, \ldots, d_n$ respectively, with corresponding hypersurfaces $V_1, \ldots, V_n$. Then, if $\cap_i V_i$ is finite, we have*

$$|\bigcap_i V_i| = \prod_i d_i$$

as long as points are counted wth multiplicity.

Let's try and do better. Let's consider varieties of higher codimension.

Example 7 (Ravi Vakil). Let $V = P_0$ be a plane in $\mathbb{P}^4$, and let $V' = P_1 \cup P_2$ be the union of two different planes such that $P_1 \cap P_2$ is a single point. We want to know what

$$V \cap V'$$

looks like as V varies. Recall that planes in $\mathbb{P}^4$ correspond to 3-dimensional subspaces of $\mathbb{C}^5$. So for P, Q any two planes in $\mathbb{P}^4$, the intersection $P \cap Q$ is either a point, a line, or a plane ($P \cap Q$ cannot be empty because $3 + 3 > 5$). We want $V \cap V'$ to be finite, so we only care about the case that $P_0 \cap P_1$ and $P_0 \cap P_2$ both consist of a single point. So ignoring multiplicity, we have

$$|V \cap V'| = 1 \text{ or } 2 \qquad \text{(counting without multiplicity)}$$

according to whether $V \cap V' = P_1 \cap P_2$ or not. One can check that in the latter case, there is no multiplicity. So in the former case we want the unique point to have multiplicity 2.

Choosing coordinates appropriately,[1] we can assume that the point is the origin

[1] Let $e_0 \in \mathbb{C}^5$ be a generator for $P_1 \cap P_2 = V \cap V' \cong \mathbb{C} \subseteq \mathbb{C}^5$, let e_1, e_2 be any two linearly independent vectors in $P_1 \setminus (P_1 \cap P_2)$ and e_3, e_4 any two linearly independent vectors in $P_2 \setminus (P_1 \cap P_2)$. Let a, b, c be generators for P_0. Since $V \cap V' = P_1 \cap P_2$ we can assume $a = e_0$, and $b = \sum_{i=1}^4 b_i e_i$ and $c = \sum_{i=1}^4 c_i e_i$. If $(b_1, b_2) = \lambda(c_1, c_2)$ for some λ, then $b - \lambda c \in P_2$, contradicting the fact that $P_0 \cap P_2$ is a one dimensional vector space in $\mathbb{C}^5$. Similarly, we can't have $(b_3, b_4) = \lambda(c_3, c_4)$. In other words, the matricies $\left[\begin{smallmatrix} b_1 & c_1 \\ b_2 & c_2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} b_3 & c_3 \\ b_4 & c_4 \end{smallmatrix}\right]$ are invertible. Using the inverses of these matricies, we can replace e_1, e_2, e_3, e_4 with f_1, f_2, f_3, f_4 such that $b = f_1 + f_3$ and $c = f_2 + f_4$. Then e_0, f_1, f_2, f_3, f_4 is the desired basis.

in $\mathbb{A}^4 \subseteq \mathbb{P}^4$, and our three planes (intersected with $\mathbb{A}^4$ are:

$$\begin{aligned} \mathbb{A}^4 \cap P_0: &\qquad w = y; x = z \\ \mathbb{A}^4 \cap P_1: &\qquad w = x = 0 \\ \mathbb{A}^4 \cap P_2: &\qquad y = z = 0 \end{aligned}$$

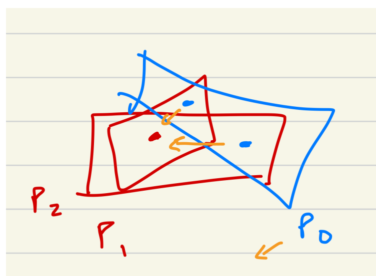
So $\mathbb{A}^4 \cap V'$ has coordinate ring $\mathbb{C}[w,x,y,z]/(w,x)(y,z)$ and $\mathbb{A}^4 \cap P_0$ has coordinate ring $\mathbb{C}[w,x,y,z]/(w-y, x-z)$. Now,

$$\frac{\mathbb{C}[w,x,y,z]}{(w,x)(y,z)} \otimes_{\mathbb{C}[w,x,y,z]} \frac{\mathbb{C}[w,x,y,z]}{(w-y,x-z)} \cong \frac{\mathbb{C}[w,x]}{(x^2,xy,w^2)}$$

which has dimension 3. So, in fact, we get

$$|V \cap V'| = 3 \text{ or } 2 \qquad \text{(counting with multiplicity)}$$

As we slide P_0 around, our two distinct points have joined to become three points. Where did the extra point come from?



The solution to the above problem came from Serre. The idea is that we should be using "homotopy types" not just sets.

sets	$\rightsquigarrow$	homotopy types
abelian groups	$\rightsquigarrow$	chain complexes of abelian groups
R-modules	$\rightsquigarrow$	chain complexes of R-modules

For use in algebra, often chain complexes are a good enough model for homotopy types, so we will use these here. A (connective) chain complex is a sequence of R-modules

$$M_\bullet = (\cdots \to M_2 \overset{d(2)}{\to} M_1 \overset{d(1)}{\to} M_0)$$

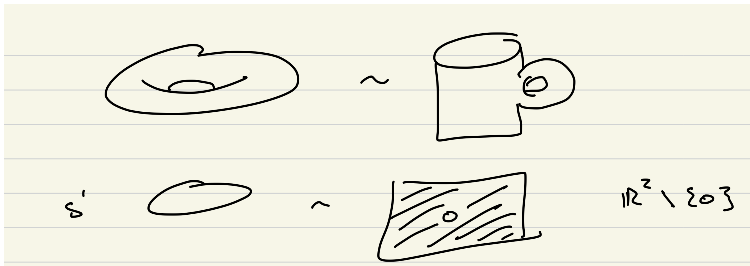
such that $d(n-1) \circ d(n) = 0$ for every n. A morphism of chain complexes $M_\bullet \to N_\bullet$ is a sequence of morphisms $M_n \to N_n$ forming commutative squares. The *homology* of a chain complexes is

$$H_n(M) = \frac{\ker(M_n \to M_{n-1})}{\operatorname{im}(M_{n+1} \to M_n)}$$

A morphism $f : M_\bullet \to N_\bullet$ inducing an isomorphism on homology is called a *quasi-isomorphism*.

Principle. *Quasi-isomorphism type* is what we actually want to work with, not specific representatives of a given quasi-isomorphism class.

This is a version of the principle that topological spaces should be studied up to *homotopy type*. That is, if we can bend or stretch a space X into a space Y, then X and Y should be considered as the same. We will make this formal next week.



There is a canonical extension of $\otimes$ to Ch_R, namely

$$(M_\bullet \otimes N_\bullet)_n = \bigoplus_{i+j=n} M_i \otimes M_j, \qquad d(m \otimes n) = (dm) \otimes n + (-1)^{deg\ m} m \otimes dn$$

but it does not preserve quasi-isomorphisms. Indeed, $M = [\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}]$ is quasi-isomorphic to $N = [\cdots \to 0 \to 0 \to \mathbb{Z}/2]$ but $M \otimes N$ is not quasi-isomorphic to $N \otimes N$. However, there is a sub-category $\mathrm{Ch}_R^{free} \subseteq \mathrm{Ch}_R$ on which $\otimes$ *does* preserves quasi-isomorphisms, the category of complexes of free modules, and every complex is quasi-isomorphic to one in Ch_R^{free}. In fact, there is even a functor

$$\mathcal{F} : \mathrm{Ch}_R \to \mathrm{Ch}_R^{free}$$

equipped with a natural transformation $\mathcal{F}(-) \to \mathrm{id}$ such that $\mathcal{F}M_\bullet \to M_\bullet$ is a quasi-isomorphism for every $M_\bullet$. So

$$\otimes^L : (M_\bullet, N_\bullet) \mapsto \mathcal{F}M_\bullet \otimes \mathcal{F}N_\bullet$$

does preserves quasi-isomorphisms in both variables.

Exercise 8.

1. Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, and $F = \oplus_{i \in I} R$ is a free R-module, then $0 \to A \otimes F \to B \otimes F \to C \otimes F \to 0$ is a short exact sequence.
2. Show that if $M_\bullet \to N_\bullet$ is a quasi-isomorphism and $F = \oplus_{i \in I} R$ is a free module, then $M_\bullet \otimes F \to N_\bullet \otimes F$ is a quasi-isomorphism.
3. Show that $\otimes$ commutes with cone[2], shift[3], and colimit[4] in each variable.

[2] $Cone(M_\bullet \xrightarrow{f} N_\bullet)_n = M_{n-1} \oplus N_n$ with differential $d(m, n) = (dm, dn + (-1)^{deg\ m} fm)$.
[3] $(M_\bullet[1])_n = M_{n-1}$ with the same differentials.
[4] Colimits of chain complexes are computed degere-wise. That is $(\varinjlim_\lambda M_{\lambda,\bullet})_n = \varinjlim_\lambda M_{\lambda,n}$.

4. Let $F_\bullet$ be a complex of free modules. Let $\tau_{\le n}F_\bullet$ be the chain complex such that $(\tau_{\le n}F)_m = F_m$ if $m \le n$ and 0 otherwise. Show that:
   (a) $\tau_{\le n}F = Cone(F_n[n-1] \to \tau_{\le n-1}F)$ and $F_\bullet = \varinjlim(\tau_{\le 0}F \to \tau_{\le 1}F \to \dots)$.
5. Let $F_\bullet$ be a complex of free modules and $M_\bullet \to N_\bullet$ a quasi-isomorphism. Show that $M_\bullet \otimes F_\bullet \to N_\bullet \otimes F_\bullet$ is a quasi-isomorphism.

If we have a specific $M_\bullet, N_\bullet$, since we only care about quasi-isomorphism class, not the actual chain complexes, we can choose any convenient quasi-isomorphisms $F_\bullet \to M_\bullet$, $G_\bullet \to N_\bullet$ with $F_\bullet, G_\bullet \in \mathrm{Ch}_R^{free}$ to calculate $\otimes^L$. We don't have to use the $\mathcal{F}M_\bullet$ and $\mathcal{F}N_\bullet$ coming from choice of $\mathcal{F}(-) \to \mathrm{id}$. In fact, for a fixed $F_\bullet \in \mathrm{Ch}_R^{free}$ the functor

$$- \otimes F_\bullet : \mathrm{Ch}_R \to \mathrm{Ch}_R$$

preserves quasi-isomorphisms so we only need to choose a nice model $F_\bullet$ of $N_\bullet$.

Indeed, if $F_\bullet \to N_\bullet$ is a quasi-isomorphism, then $\mathcal{F}F_\bullet \to \mathcal{F}N_\bullet$ will be a quasi-isomorphism,[5] so

$$M_\bullet \otimes F_\bullet \leftarrow (\mathcal{F}M_\bullet) \otimes F_\bullet \leftarrow (\mathcal{F}M_\bullet) \otimes (\mathcal{F}F_\bullet) \to (\mathcal{F}M_\bullet) \otimes (\mathcal{F}N_\bullet)$$

are quasi-isomorphisms.

The extension of the function $\dim : \mathbb{C}\text{-Mod} \to \mathbb{Z}$ to $\mathrm{Ch}_\mathbb{C}$ is

$$\dim M_\bullet = \sum_{i \in \mathbb{Z}} (-1)^i \dim H_i M.$$

Now let's come back to the above example. Instead of

$$\frac{\mathbb{C}[w,x,y,z]}{(w,x)(y,z)} \otimes_{\mathbb{C}[w,x,y,z]} \frac{\mathbb{C}[w,x,y,z]}{(w-y,x-z)} \tag{1}$$

we should have been considering

$$\frac{\mathbb{C}[w,x,y,z]}{(w,x)(y,z)} \otimes^L_{\mathbb{C}[w,x,y,z]} \frac{\mathbb{C}[w,x,y,z]}{(w-y,x-z)}. \tag{2}$$

To do this calculation, we choose a term-wise free complex representing $\frac{\mathbb{C}[w,x,y,z]}{(w-y,x-z)}$.

$$F_\bullet := \left[R \xrightarrow{(w-y,x-z)} R \oplus R \xrightarrow{(x-z,y-w)} R \right] \tag{3}$$

where $R := \mathbb{C}[x,w,y,z]$.

Exercise 9.

1. Suppose that R is a ring and $f \in R$ a non-zero divisors. Show that $[R \xrightarrow{f} R]$ is quasi-isomorphic to R/f.

[5] Consider the square $\begin{smallmatrix} \mathcal{F}F_\bullet & \to & \mathcal{F}N_\bullet \\ \downarrow & & \downarrow \\ F_\bullet & \to & N_\bullet \end{smallmatrix}$.

2. Suppose that R is a ring, $I \subset R$ an ideal, $f \in R$ an element which is a non-zero divisor in R/I (and R), and $F_\bullet$ a chain complex of free modules quasi-isomorphic to R/I. Show that $Cone(F_\bullet \xrightarrow{f} F_\bullet)$ is quasi-isomorphic to $R/I + (f)$.
3. Suppose that R is a ring and $f_1, \ldots, f_n \in R$ are elements such that f_i is not a zero divisor in $R/(f_1, \ldots, f_{i-1})$. By induction, show that

$$[R \xrightarrow{f_1} R] \otimes \cdots \otimes [R \xrightarrow{f_n} R]$$

is quasi-isomorphic to $R/(f_1, \ldots, f_n)$.

On the other hand, we have the short exact sequence of R-modules

$$0 \to \frac{\mathbb{C}[w,x,y,z]}{(w,x)(y,z)} \to \mathbb{C}[w,x] \oplus \mathbb{C}[y,z] \to \mathbb{C} \to 0$$

(morphisms send the various coordinates to zero) which is the algebraic manifestation of the fact that V is two planes glued at a point. Indeed, a function on $V = P_1 \cup P_2$ should be the same as a function on P_1 and a function on P_2 which agree on $P_1 \cap P_2$.

Applying $- \otimes_R F_\bullet$ to this, we get a short exact sequence of chain complexes,

$$0 \to \frac{\mathbb{C}[w,x,y,z]}{(w,x)(y,z)} \otimes_R F_\bullet \to \Big(\mathbb{C}[w,x] \otimes_R F_\bullet\Big) \oplus \Big(\mathbb{C}[y,z] \otimes_R F_\bullet\Big) \to \mathbb{C} \otimes_R F_\bullet \to 0.$$

By the Snake Lemma, this leads to a long exact sequence of homology groups

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners] \\
H_3 \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, rounded corners] \\
H_2 \arrow[r] & 0 \arrow[r] & \mathbb{C} \arrow[dll, rounded corners] \\
H_1 \arrow[r] & 0 \arrow[r] & \mathbb{C} \oplus \mathbb{C} \arrow[dll, rounded corners] \\
H_0 \arrow[r] & \mathbb{C} \oplus \mathbb{C} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

From this we can read directly that

$$\begin{aligned}
\dim_{\mathbb{C}} H_n &= 0 \text{ for } n > 1 \\
\dim_{\mathbb{C}} H_1 &= 1 \\
\dim_{\mathbb{C}} H_0 &= 3
\end{aligned}$$

So the multiplicity according to Serre is

$$2 = 3 - 1 + 0 - 0 + 0 - ...$$

as expected.

Question 10. Find a geometric interpretation of H_1.