(Pro)étale cohomology
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Lecture 11: Homological algebra II

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In this lecture and the next we discuss two kinds of completion.

1. A complex $C^\bullet$ is *Postnikov complete* if it is the derived limit of its truncations; namely,
$$C^\bullet = R\lim_{n\in\mathbb{N}} \tau^{\geq -n} C^\bullet$$
(we explain this notation below).
2. For l a prime, a complex $C^\bullet$ of $\mathbb{Z}_l$-modules is *derived l-complete* if it is the derived limit of its reductions mod l^n; namely,
$$C^\bullet = R\lim(C^\bullet/l^n)$$
(we explain this notation next week).

The goal of this lecture is to show that every complex in $D(X_{\mathrm{proet}})$ is Postnikov complete. This allows us to reduce arguments to the case of bounded below complexes. (Which we can usually reduce to complexes concentrated in degree zero).

Next week we will discuss derived l-completeness and hopefully prove the claim that l-adic cohomology is actual sheaf cohomology on X_{proet}.

1 The weakly contractible site

In this section we show an equivalence of categories $\mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}}) \cong \mathrm{Shv}(X_{\mathrm{proet}})$ and deduce some consequences. The most important consequence is that isomorphisms, resp. epimorphisms of abelian group objects, in $\mathrm{Shv}(X_{\mathrm{proet}})$ are detected on objects of $X^{\mathrm{wc}}_{\mathrm{proet}}$.

Definition 1. Define $X^{\mathrm{wc}}_{\mathrm{proet}} \subseteq X_{\mathrm{proet}}$ to be the full subcategory of weakly contractible objects. Equip it with the topology whose coverings are families of the form
$$\{W_i \subseteq W\}_{i=1}^n$$
where $W = \sqcup_{i=1}^n W_i$.

Exercise 2. Show that the above definition does define a Grothendieck topology.

Exercise 3. Show that the inclusion $\mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}}) \subseteq \mathrm{PSh}(X^{\mathrm{wc}}_{\mathrm{proet}})$ admits a left adjoint $a : \mathrm{PSh}(X^{\mathrm{wc}}_{\mathrm{proet}}) \to \mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}})$ which satisfies:

$$aF(W) = \operatorname*{colim}_{W = W_1 \sqcup \cdots \sqcup W_n} F(W_1) \times \cdots \times F(W_n) \tag{1}$$

where the colimit is over the filtered poset of decompositions of W into finitely many clopens.

Lemma 4. *Let X be a scheme and $W \in X^{\mathrm{wc}}_{\mathrm{proet}}$. Then the functor*

$$ev_W : \mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}}, \mathcal{A}\mathrm{b}) \to \mathcal{A}\mathrm{b}; \qquad F \mapsto F(W)$$

commutes with colimits.

Proof. The analogous functor $ev_W : \mathrm{PSh}(X^{\mathrm{wc}}_{\mathrm{proet}}, \mathcal{A}\mathrm{b}) \to \mathcal{A}\mathrm{b}$ always commutes with colimits (for any site). The obstacle for sheaves is usually that colimits in Shv are calculated by taking the colimit in PSh and then sheafifying. However, sheafification on $X^{\mathrm{wc}}_{\mathrm{proet}}$ takes the particularly nice form

$$aF(W) = \operatorname*{colim}_{W = W_1 \sqcup \cdots \sqcup W_n} F(W_1) \times \cdots \times F(W_n). \tag{2}$$

Since $A_1 \sqcup A_2 = A_1 \times A_2$ in $\mathcal{A}\mathrm{b}$, there are only colimits in Eq.(2). Then since colimits commute with colimits, for any diagram of sheaves $(F_\lambda)_{\lambda \in \Lambda}$ in $\mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}}, \mathcal{A}\mathrm{b})$ we have

$$(a \operatorname{colim}_\lambda F_\lambda)(W) = \operatorname{colim}_\lambda (aF_\lambda)(W).$$

□

Remark 5. In Lemma 4 we can replace $\mathcal{A}\mathrm{b}$ with any category admitting small colimits in which finite products are isomorphic to finite coproducts.

Proposition 6. *Suppose that $\iota : D \subseteq C$ is a full subcategory of a small category C, and C and D are equipped with topologies τ_D, τ_C. Consider the following conditions:*

(C1) Every τ_C-covering $\{Y_i \to W\}_{i \in I}$ of an object of D is refinable by a τ_D-covering $\{W_j \to W\}_{j \in J}$ in the sense that there exist factorisations

$$W_j \to Y_{i_j} \to W.$$

(C2) Every object Y of C admits a τ_C-covering family

$$\{W_i \to Y\}_{i \in I}$$

such that the W_i are in D.

1. *If (C1) is satisfied then the canonical functor*

$$\iota^* : \mathrm{Shv}(D) \to \mathrm{Shv}(C)$$

is fully faithful.

2. *If (C2) is satisfied then a morphism $F \to G$ in $\mathrm{Shv}(C)$ is an isomorphism if and only if*
$$F(W) \to G(W)$$
is an isomorphism for every $W \in D$.
3. *If (C1) and (C2) are both satisfied then the adjoints*
$$\iota^* : \mathrm{Shv}(D) \rightleftarrows \mathrm{Shv}(C) : \iota_*$$
are inverse equivalences of categories.

Remark 7. We don't need to assume C small for part (2).

Proof.
1. Since $\iota : D \subseteq C$ is fully faithful the left Kan extension $\iota^p : \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ is fully faithful. The condition (C1) implies that restriction $\iota_p : \mathrm{PSh}(C) \to \mathrm{PSh}(D)$ sends τ_C-sheaves to τ_D-sheaves, giving a functor $\iota_* = \iota_p|_{\mathrm{Shv}(C)} : \mathrm{Shv}(C) \to \mathrm{Shv}(D)$. Recall that ι_* admits a left adjoint $\iota^* : \mathrm{Shv}(D) \to \mathrm{Shv}(C)$ given by $\iota^* = a \circ \iota^p \circ \mathrm{inc}$ where inc is the inclusion $\mathrm{Shv} \subseteq \mathrm{PSh}$ and $a : \mathrm{PSh} \to \mathrm{Shv}$ is sheafification. The condition (C1) also implies that if F is a sheaf, then the sheafification of its left Kan extension doesn't affect values on D. That is, $(a \circ \iota^p \circ \mathrm{inc}\, F)(W) = F(W)$ for $W \in D$. Consequently, the unit is a natural isomorphism $\mathrm{id} \xrightarrow{\sim} \iota_*\iota^*$. In other words, ι^* is fully faithful.
2. Suppose $F \to G$ is a morphism in $\mathrm{Shv}(C)$. We want to show that $F(Y) \to G(Y)$ is an isomorphism for each $Y \in C$. Using (C2) we find a τ_C-covering $\{W_i \to Y\}_{i \in I}$ with $W_i \in D$. The products $W_i \times_Y W_j$ probably won't be in D so choose coverings of these $\{W_{ijk} \to W_i \times_Y W_j\}_{k \in K_{ij}}$. Since F is a sheaf each $F(W_i \times_Y W_j) \to \prod F(W_{ijk})$ is injective. So the τ_C-sheaf condition
$$F(Y) = \mathrm{eq}\left(\prod_{i \in I} F(W_i) \rightrightarrows \prod_{i,j} F(W_i \times_Y W_j)\right)$$
becomes
$$F(Y) = \mathrm{eq}\left(\prod_{i \in I} F(W_i) \rightrightarrows \prod_{i,j,k} F(W_{ijk})\right).$$
The same is true for G, and since the $F(W_i) \to G(W_i)$ and $F(W_{ijk}) \to G(W_{ijk})$ are isomorphisms, we deduce that $F(Y) \to G(Y)$ is an isomorphism.
3. By the first part it remains only to show that the counit $\iota^*\iota_* \to \mathrm{id}$ is a natural isomorphism. By the second part this is an isomorphism if and only if $\iota^*\iota_*F(W) \to F(W)$ is an isomorphism for each $W \in D$. Clearly $\iota_*F(W) = F(W)$ and we saw in the first part that ι^* doesn't change the values of a sheaf on D, so $\iota^*\iota_*F(W) \cong F(W)$.

$\square$

Corollary 8. *A morphism $f : F \to G$ in $\mathrm{Shv}(X_{\mathrm{proet}})$ is an isomorphism if and only if $F(W) \to G(W)$ is an isomorphism for each $W \in X^{\mathrm{wc}}_{\mathrm{proet}}$.*

Definition 9. Let C be a site. A morphism $f : F \to G$ in $\mathrm{Shv}(C)$ is an *epimorphism* if for every $X \in C$ and $s \in G(X)$ there exists a covering family $\{U_i \to X\}_{i \in I}$ and sections $t_i \in F(U_i)$ such that $s|_{U_i} = f(t_i)$ in $G(U_i)$.

$$\begin{tikzcd} \prod F(U_i) \arrow[r] & \prod G(U_i) \\ & G(X). \arrow[u] \end{tikzcd}$$

Remark 10. It can be shown that since $\mathrm{Shv}(C)$ is a topos, the above definition is equivalent to the categorical one. Namely, $f : F \to G$ is an epimorphism in the sense of Definition 9 if and only if $\hom(G, H) \to \hom(F, H)$ is injective for every H,

Example 11. Recall that we write $j : C \to \mathrm{Shv}(C)$ for the Yoneda functor. Suppose that $W_0, W_1 \in X^{\mathrm{wc}}_{\mathrm{proet}}$ are non-empty. Show that $jW_0 \sqcup jW_1 \to j(W_0 \sqcup W_1)$ is an epimorphism of sheaves in $\mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}})$ but that $(jW_0 \sqcup jW_1)(W_0 \sqcup W_1) \to j(W_0 \sqcup W_1)(W_0 \sqcup W_1)$ is not a surjection of sets. Deduce that Lemma 4 is false if we use $\mathcal{S}\mathrm{et}$ instead of $\mathcal{A}\mathrm{b}$.

Corollary 12. *A morphism $f : F \to G$ in $\mathrm{Shv}(X_{\mathrm{proet}}, \mathcal{A}\mathrm{b})$ is an epimorphism if and only if $F(W) \to G(W)$ is an epimorphism for each $W \in X^{\mathrm{wc}}_{\mathrm{proet}}$.*

Proof.

$$\begin{aligned} F \to G \text{ is an epi.} &\iff \mathrm{coker}(F \to G) \to 0 \text{ is an iso.} \\ &\overset{\text{Cor.12}}{\iff} \mathrm{coker}(F \to G)(W) \to 0 \text{ is an iso. } \forall\, W \in X^{\mathrm{wc}}_{\mathrm{proet}} \\ &\overset{\text{Cor.4}}{\iff} \mathrm{coker}(F(W) \to G(W)) \to 0 \text{ is an iso. } \forall\, W \in X^{\mathrm{wc}}_{\mathrm{proet}} \\ &\iff F(W) \to G(W) \text{ is an epi. } \forall\, W \in X^{\mathrm{wc}}_{\mathrm{proet}}. \end{aligned}$$

□

Corollary 13. *For every scheme X restriction induces an equivalence of categories*

$$\mathrm{Shv}(X_{\mathrm{proet}}) \xrightarrow{\sim} \mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}}).$$

Remark 14. We cannot apply Proposition 6 directly because X_{proet} is not small. For a way around this see [Bhatt, Scholze, The pro-étale topology for schemes, Remark 4.1.2].

2 The derived category

We recall that the derived category of a Grothendieck abelian category $\mathcal{A}$ is constructed as follows.

The category of chain complexes Ch$(\mathcal{A})$. We begin with the category of chain complexes Ch$(\mathcal{A})$. Examples are $\Delta^1 = \mathrm{Cone}(\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z})$ equipped with the two canonical morphisms $\iota_0, \iota_1 : \mathbb{Z} \to \Delta^1$. There is a unique functor preserving colimits in both variables $\otimes : \mathcal{A} \times \mathcal{A}\mathrm{b} \to \mathcal{A}$ such that $- \otimes \mathbb{Z} : \mathcal{A} \to \mathcal{A}$ is the identity functor, and this extends to a functor $\mathrm{Ch}(\mathcal{A}) \times \mathrm{Ch}(\mathcal{A}\mathrm{b}) \to \mathrm{Ch}(\mathcal{A})$.

The homotopy category $K(\mathcal{A})$. A morphism $f : C^\bullet \to D^\bullet$ in Ch$(\mathcal{A})$ is *null homotopic* if there is a commutative diagram of the form

$$\begin{tikzcd} C^\bullet \arrow[d, "\iota_0"'] \arrow[dr, "0"] & \\ C^\bullet \otimes \Delta^1 \arrow[r] & D^\bullet \\ C^\bullet \arrow[u, "\iota_1"] \arrow[ur, "f"'] & \end{tikzcd}$$

Null homotopic morphisms form subgroups of the $\hom_{\mathrm{Ch}(\mathcal{A})}(C^\bullet, D^\bullet)$, preserved under composition, and one defines $K(\mathcal{A})$ to have the same objects as Ch$(\mathcal{A})$ and $\hom_{K(\mathcal{A})} = \hom_{\mathrm{Ch}(\mathcal{A})} / \{\text{null homotopic morphisms}\}$.

The derived category $D(\mathcal{A})$. Finally, one says that an object $Q^\bullet$ of $K(\mathcal{A})$ is *q.i.-local* if for every quasi-isomorphism $f : A^\bullet \to B^\bullet$ the induced morphism $\hom_{K(\mathcal{A})}(f, Q^\bullet)$ is an isomorphism. The *derived category* is the full subcategory

$$\mathrm{inc} : D(\mathcal{A}) \subseteq K(\mathcal{A})$$

of $q.i.$-local objects. It is a theorem that the inclusion admits a left adjoint

$$\mathrm{loc} : K(\mathcal{A}) \to D(\mathcal{A}).$$

The functor loc is the universal functor which inverts quasi-isomorphisms.

Derived functors. Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ between Grothendieck abelian categories, the *(right) derived functor* RF can be defined as the composition

$$RF : D(\mathcal{A}) \stackrel{\mathrm{inc}}{\to} K(\mathcal{A}) \stackrel{F}{\to} K(\mathcal{B}) \stackrel{\mathrm{loc}}{\to} D(\mathcal{B}).$$

The adjunction unit $\mathrm{id} \to \mathrm{inc} \circ \mathrm{loc}$ induces a natural transformation

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\mathrm{loc}"'] \arrow[dr, phantom, "\swarrow"] & K(\mathcal{B}) \arrow[d, "\mathrm{loc}"] \\ D(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B}) \end{tikzcd}$$

The functor RF is the initial functor equipped with such a natural transformation.

Definition 15. If $F : K(\mathcal{A}) \to K(\mathcal{B})$ is a functor and $C^\bullet \in K(\mathcal{A})$ an object, we say that F *is already derived at* $C^\bullet$ if the morphism $F(C^\bullet) \to F(\mathrm{inc}\,\mathrm{loc}(C^\bullet))$ is a quasi-isomorphism. In this case

$$\mathrm{loc}(F(C^\bullet)) \to RF(\mathrm{loc}(C^\bullet))$$

an isomorphism in $D(\mathcal{B})$.

Exercise 16. Suppose that $F : \mathcal{A} \to \mathcal{B}$ is *exact*. Show that $F : K(\mathcal{A}) \to K(\mathcal{B})$ is already derived at every object.

3 ∞-categories

An ∞-categorical way of defining the derived category, at least for $\mathcal{A} = \mathrm{Shv}(C)$ is as follows. See [Lurie, Derived Algebraic Geometry I: Stable ∞-Categories] (and [Higher topos theory] and [Higher algebra]) for more details.

We assume that we have some good theory of ∞-categories with good notions of equivalence, functor categories, Yoneda, slice categories, (co)limits, etc. such that every classical category is an ∞-category. The theory of quasi-categories described in [Higher topos theory] is one such theory. A model agnostic, axiomatic approach is being developed, for example, by Denis-Charles Cisinski (see his teaching page for more details).

The category of *simplicial abelian groups* $\mathcal{A}\mathrm{b}_\Delta$ (or *animated abelian groups*) can defined as the category of those functors

$$A : (\mathcal{A}\mathrm{b}^{\text{finite free}})^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}_\infty$$

which preserve products $A(M \oplus N) = A(M) \times A(N)$. Here $\mathcal{A}\mathrm{b}^{\text{finite free}}$ is the 1-category of finite free abelian groups and $\mathcal{G}\mathrm{pd}_\infty$ is the category of ∞-groupoids, namely, the category of those ∞-categories in which every 1-morphism is an equivalence. Sometimes objects of $\mathcal{G}\mathrm{pd}_\infty$ are called *anima*. One defines $\Omega : \mathcal{A}\mathrm{b}_\Delta \to \mathcal{A}\mathrm{b}_\Delta$ as $A \mapsto * \times_A *$ and $\mathcal{D}(\mathcal{A}\mathrm{b})$ as the inverse limit

$$\mathcal{D}(\mathcal{A}\mathrm{b}) = \lim(\cdots \xrightarrow{\Omega} \mathcal{A}\mathrm{b}_\Delta \xrightarrow{\Omega} \mathcal{A}\mathrm{b}_\Delta \xrightarrow{\Omega} \mathcal{A}\mathrm{b}_\Delta)$$

taken in the ∞-category of ∞-categories. This is ∞-categorical derived category of abelian groups. The category of chain complexes $K(\mathrm{PSh}(C), \mathcal{A}\mathrm{b})$ above is replaced by the category of functors

$$\mathrm{PSh}(C, \mathcal{D}(\mathcal{A}\mathrm{b})) = \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{D}(\mathcal{A}\mathrm{b})).$$

A functor $C^{\mathrm{op}} \to \mathcal{D}(\mathcal{A}\mathrm{b})$ is said to be a *hypersheaf* if for every hypercovering $Y_\bullet \to Y$ the morphism

$$F(Y) \to \lim_{n \in \Delta} F(Y_n)$$

in $D(\mathcal{A}\mathrm{b})$ is an equivalence. The derived category $D(\mathrm{Shv}(C, \mathcal{A}\mathrm{b}))$ above is then replaced with the full subcategory

$$\mathcal{D}(C) \subseteq \mathrm{PSh}(C, \mathcal{D}(\mathcal{A}\mathrm{b}))$$

of hypersheaves. The inclusion of 1-categories into ∞-categories admits a left adjoint

$$\tau_{\leq 1} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}$$

and we have:

$$\begin{array}{ccc}
 & \tau_{\leq 1} & \\
\mathcal{A}\mathrm{b}_\Delta & \mapsto & D^{\leq 0}(\mathcal{A}\mathrm{b}) \\
\Omega(\bullet) & \mapsto & \tau^{\leq 0}(\bullet[-1]) \\
\mathcal{D}(\mathcal{A}\mathrm{b}_\Delta) & \mapsto & D(\mathcal{A}\mathrm{b}) \\
\mathrm{PSh}(C, D(\mathcal{A}\mathrm{b})) & \mapsto & K(\mathrm{PSh}(C, \mathcal{A}\mathrm{b})) \\
\mathcal{D}(C) & \mapsto & D(C)
\end{array}$$

One advantage of the ∞-category setup is one cannot talk about functors which are not derived. Every functor is automatically derived. A disadvantage is that calculations are usually impossible. Sometimes this can be an advantage as it prevents one from performing unnecessarily complicated calculations. The reverse can also happen; there are situations where things can be more technically complicated in the ∞-categorical world.

4 Deriving products and limits

We begin with a lemma which explains how to get limits from products.

Lemma 17. *Suppose that* $(\cdots \to F_2 \xrightarrow{t_1} F_1 \xrightarrow{t_0} F_0)$ *is a sequence of epimorphisms in* $\mathrm{Shv}(X_{\mathrm{proet}}, \mathcal{A}\mathrm{b})$*. Then there is a short exact sequence*

$$0 \to \lim F_n \to \prod F_n \overset{\mathrm{id}-t}{\to} \prod_n F_n \to 0$$

where $\mathrm{id}-t$ *is* $(\ldots, s_1, s_0) \mapsto (\ldots, s_1 - t_1 s_2, s_0 - t_0 s_1)$ *appropriately interpreted.*

Proof. By Corollary 12 we can detect short exact sequences by evaluating on the $W \in X^{\mathrm{wc}}_{\mathrm{proet}}$. Evaluation preserves limits. Again by Corollary 12 it preserves epimorphisms. So we have reduced the problem to the analogous question about a sequence of epimorphisms $(\ldots A_2 \to A_1 \to A_0)$ in $\mathcal{A}\mathrm{b}$. There the problem is a standard exercise. $\square$

Exercise 18. Show that if $(\ldots A_2 \to A_1 \to A_0)$ is a sequence of epimorphisms in $\mathcal{A}\mathrm{b}$ then

$$0 \to \lim A_n \to \prod A_n \overset{\mathrm{id}-t}{\to} \prod_n A_n \to 0$$

is a short exact sequence.

In general if C is a site then the category of $\mathbb{N}$-indexed products (resp. $\mathbb{N}$-indexed sequences) in $\mathrm{Shv}(C)$ is also a category of sheaves. Namely, the category of sheaves on $\sqcup_{n\in\mathbb{N}} C$, resp., $C \times \mathbb{N}$, for an appropriate topology where $\mathbb{N} = \{0 \to 1 \to 2 \to \ldots\}$ is considered as a category.

Exercise 19. Suppose that C is a site. Define topologies on $\sqcup_{n\in\mathbb{N}} C$ and $C \times \mathbb{N}$ such that there are equivalences

$$\prod_{n\in\mathbb{N}} \mathrm{Shv}(C) \cong \mathrm{Shv}(\sqcup_{n\in\mathbb{N}} C), \qquad \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Shv}(C)) \cong \mathrm{Shv}(C \times \mathbb{N}).$$

More generally, given any small category I, define a topology on $C \times I$ such that there is an equivalence

$$\mathrm{Fun}(I, \mathrm{Shv}(C)) \cong \mathrm{Shv}(C \times I).$$

Proposition 20. *Let $\mathcal{A} = \mathrm{Shv}(X_{\mathrm{proet}}, \mathbb{Z})$. The functor*

$$\begin{aligned} K(\sqcup_{n\in\mathbb{N}}\mathcal{A}) &\to K(\mathcal{A}) \\ \{C^\bullet(n)\}_{n\in\mathbb{N}} &\mapsto \prod_{n\in\mathbb{N}} C^\bullet(n) \end{aligned}$$

is already derived at every object.

Proof. It suffices to show that $\prod_{n\in\mathbb{N}} : \sqcap_{n\in\mathbb{N}}\mathrm{Shv}(X_{\mathrm{proet}}, \mathcal{A}\mathrm{b}) \to \mathrm{Shv}(X_{\mathrm{proet}}, \mathcal{A}\mathrm{b})$ is an exact functor. It is left exact because it is a right adjoint. Suppose that $\{F(n)\}_{n\in\mathbb{N}} \to \{G(n)\}_{n\in\mathbb{N}}$ is a sequence of epimorphisms. By Corollary 12 for $W \in X^{\mathrm{wc}}_{\mathrm{proet}}$ each $F(n)(W) \to G(n)(W)$ is a surjection of abelian groups, so $\prod_{n\in\mathbb{N}} F(n)(W) \to \prod_{n\in\mathbb{N}} G(n)(W)$ is a surjection of abelian groups. Since $W \in X^{\mathrm{wc}}_{\mathrm{proet}}$ was arbitrary, by Corollary 12 $\prod_{n\in\mathbb{N}} F(n) \to \prod_{n\in\mathbb{N}} G(n)$ is an epimorphism. $\square$

Proposition 21. *Let $\mathcal{A} = \mathrm{Shv}(X^{\mathrm{wc}}_{\mathrm{proet}}, \mathbb{Z})$. Suppose that*

$$(\cdots \to C^\bullet(2) \to C^\bullet(1) \to C^\bullet(0)) \in \mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ch}(\mathcal{A}))$$

is a sequence of chain complexes such that each $C^i(n+1) \to C^i(n)$ is an epimorphism of sheaves. Then lim *is already derived at $\{C^\bullet(n)\}_{n\in\mathbb{N}} \in K(\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{A}))$.*

Proof. Let $\{C^\bullet(n)\}_{n\in\mathbb{N}} \to \{Q^\bullet(n)\}_{n\in\mathbb{N}}$ be a fibrant replacement. So each $C^\bullet(n) \to Q^\bullet(n)$ is a quasi-isomorphism and $\{Q^\bullet(n)\}_{n\in\mathbb{N}}$ is a model for $\mathrm{inc}\, L\{C^\bullet(n)\}_{n\in\mathbb{N}}$ in $K(\mathcal{A})$. Now it suffices to show that $\lim_n C^\bullet(n) \to \lim_n Q^\bullet(n)$ is a quasi-isomorphism. We will show in Proposition 24 that each $Q^i(n+1) \to Q^i(n)$ is a split epimorphism. Consequently, by Lemma 17 we have a morphism of short exact sequences of chain complexes.

$$\begin{tikzcd} 0 \arrow[r] & \lim_n C^\bullet(n) \arrow[r] \arrow[d] & \prod_n C^\bullet(n) \arrow[r, "\mathrm{id}-t"] \arrow[d] & \prod_n C^\bullet(n) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \lim_n Q^\bullet(n) \arrow[r] & \prod_n Q^\bullet(n) \arrow[r, "\mathrm{id}-t"] & \prod_n Q^\bullet(n) \arrow[r] & 0 \end{tikzcd}$$

Each $C^\bullet(n) \to Q^\bullet(n)$ is a quasi-isomorphism, so since countable products are exact, the $\prod_n C^\bullet(n) \to Q^\bullet(n)$ are quasi-isomorphisms. Considering the long exact sequence of cohomology objects, it follows that $\lim_n C^\bullet(n) \to \lim_n Q^\bullet(n)$ is a quasi-isomorphism. $\square$

Corollary 22. *For each $C^\bullet \in D(X_{\mathrm{proet}})$ there is an isomorphism*

$$C^\bullet \xrightarrow{\sim} R\lim \tau^{\geq n} C^\bullet$$

in $D(X_{\mathrm{proet}})$.

Proof. The sequences of sheaves $\cdots \to (\tau^{\geq n-1}C^\bullet)^i \to (\tau^{\geq n}C^\bullet)^i \to \cdots \to (\tau^{\geq 0}C^\bullet)^i$ look like

$$\cdots = C^i = \cdots = C^i \twoheadrightarrow C^i/\operatorname{im}(C^{i-1}) \to 0 = \cdots = 0.$$

In particular, degreewise they are sequences of epimorphisms of sheaves. Consequently by Proposition 21 the functor lim is already derived at $\{\tau^{\geq -n}C^\bullet\}_{n\in\mathbb{N}}$. It is clear that $C^\bullet = \lim \tau^{\geq -n}C^\bullet$ so the result follows. ☐

Remark 23. As we see in Proposition 24, every object of $D(X_{\mathrm{proet}})$ is isomorphic (in $D(X_{\mathrm{proet}})$) to a sequence $\{Q^\bullet(n)\}_{n\in\mathbb{N}}$ such that each $Q^i(n+1) \to Q^i(n)$ is a split epimorphism. In particular, by Lemma 17 and Proposition 20, for any $\{C^\bullet(n)\}_{n\in\mathbb{N}}$ in $K(\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Shv}(X_{\mathrm{proet}}, \mathbb{Z})))$ we have

$$R\lim(\operatorname{loc} C^\bullet(n)) \cong \operatorname{loc}(\operatorname{Cone}\left(\prod_n C^\bullet(n) \to \prod_n C^\bullet(n)\right))$$

in $D(X_{\mathrm{proet}})$. We will use this in the sequel.

A Fibrancy of sequences

Recall that an object $Q^\bullet \in \mathrm{Ch}(\mathrm{Shv}(C, \mathcal{A}\mathrm{b}))$ is called *fibrant* if for every morphism $A^\bullet \to B^\bullet$ in $\mathrm{Ch}(\mathrm{Shv}(C, \mathcal{A}\mathrm{b}))$ which is a quasi-isomorphism and a monomorphism, every morphism $A^\bullet \to Q^\bullet$ admits a factorisation

$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[rr, bend left] & B^\bullet \arrow[r, dashed] & Q^\bullet. \end{tikzcd}$$

Proposition 24. *Let $\mathcal{A}$ be a Grothendieck abelian category and $\{Q^\bullet(n)\}_{n\in\mathbb{N}} \in \mathrm{Ch}(\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{A}))$ a fibrant object. Then each $Q^i(n+1) \to Q^i(n)$ is a split epimorphism.*

Proof. The functor $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ sending a chain complex $C^\bullet$ to C^i admits a left adjoint δ. Namely, the functor sending $A \in \mathcal{A}$ to the complex

$$\delta A = [\cdots \to 0 \to A = A \to 0 \to \dots]$$

where the complex is concentrated in degrees i and $i+1$. Similarly, the functor

$$\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathrm{Ch}(\mathcal{A})) \to \mathrm{Ch}(\mathcal{A})$$

send a sequence $\{C^\bullet(n)\}_{n\in\mathbb{N}}$ to the nth term $C^\bullet(n)$ admits a left adjoint ρ_n. Namely, the functor sending $C^\bullet$ to the sequence

$$\rho_n(C^\bullet) = \{\cdots \to 0 \to \underbrace{C^\bullet}_{n} \to \underbrace{C^\bullet}_{n-1} \to \cdots \to \underbrace{C^\bullet}_{0}\}$$

with $n+1$ non-zero terms.

Now define $A = \rho_n\delta(Q^i(n))$ and $B = \rho_{n+1}\delta(Q^i(n))$ (notice that A uses ρ_n and B uses ρ_{n+1}). Since complexes in the image of δ are acyclic, the canonical inclusion

$A \subseteq B$ is a quasi-isomorphism. We also have the canonical morphism $A \to Q^\bullet$ adjoint to $\mathrm{id} : Q^i(n) = Q^i(n)$. Since $\{Q^\bullet(n)\}_{n\in\mathbb{N}}$ is fibrant, we get a factorisation

$$A \to B \dashrightarrow Q^\bullet. \tag{3}$$

In positions $n+1$ and n, Eq.(3) is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \lambda(Q^i(n)) \arrow[r] \arrow[d, "="'] & Q^\bullet(n+1) \arrow[d] \\
\lambda(Q^i(n)) \arrow[r, "="'] & \lambda(Q^i(n)) \arrow[r] & Q^\bullet(n)
\end{tikzcd}$$

in $\mathrm{Ch}(\mathcal{A})$. The right hand square is adjoint to a commutative square

$$\begin{tikzcd}
Q^i(n) \arrow[r] \arrow[d, "="'] & Q^i(n+1) \arrow[d] \\
Q^i(n) \arrow[r, "="'] & Q^i(n)
\end{tikzcd}$$

in $\mathcal{A}$. Hence, $Q^i(n+1) \to Q^i(n)$ is a split epimorphism. $\square$