(Pro)étale cohomology
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Lecture 9: Commutative algebra II

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1 Overview

Last week we saw the definition of the proétale site X_{proet} of a scheme X. We also saw an example[1] of a scheme X for which every Zariski covering $\{U_i \to X\}$ is refinable by a Zariski covering $\{X_j \to X\}_{j=1}^n$ for which $X = \sqcup_{j=1}^n X_j$. Schemes which satisfy the proétale version of this are called *weakly contractible*.

Definition 1. A scheme Y is called *weakly contractible* if for every proétale covering $\{V_i \to Y\}_{i \in I}$ there exists a decomposition $Y = \sqcup_{k \in K} Y_k$ and factorisations

$$Y_j \to V_{i_j} \to Y.$$

The existence of enough weakly contractible proétale coverings means that proétale sheaves are determined by these weakly contractible objects, and on these the sheaf condition is particularly simple. We prove the following proposition in a later lecture.

Proposition 2. *Let X be a scheme and let $X_{\text{proet}}^{\text{wc}} \subseteq X_{\text{proet}}$ denote the fully subcategory of weakly contractible objects. Equip it with the induced topology. Then*

$$\operatorname{Shv}(X_{\text{proet}}) \cong \operatorname{Shv}(X_{\text{proet}}^{\text{wc}}).$$

Moreover, a presheaf $F \in \operatorname{PSh}(X_{\text{proet}}^{\text{wc}})$ is a sheaf if and only if it sends finite coproducts to products:

$$F(\sqcup_{i=1}^n Y_i) = \sqcap_{i=1}^n F(Y_i).$$

The goal of this lecture is the following, from which Proposition 2 follows relatively easily. (This week we will only achieve the Zariski and topological versions of this goal. We will finish the proétale version next week).

Goal 3. Every scheme X admits a proétale covering $\{Y_i \to X\}_{i \in I}$ such that each Y_i is affine and weakly contractible.

[1] Namely, the spectrum of $\operatorname{colim}_{n \in \mathbb{N}} \left(\mathbb{Z}[\frac{1}{p_1 \dots p_n}] \times \mathbb{Z}_{(p_1)} \times \cdots \times \mathbb{Z}_{(p_n)} \right)$ where p_i is the ith prime.

Since every scheme admits an open affine covering, it suffices to treat the case when X is affine. In this case, there are roughly four steps:

1. (Zariski version) Build a surjective proZariski morphism $X^Z \to X$ with X^Z weakly contractible for the Zariski topology.
   Set theoretically, X^Z is the disjoint union $\sqcup_{x\in X}\operatorname{Spec}(\mathcal{O}_{X,x})$ of the localisations of X at every point, but X^Z is equipped with a coarser topology than $\sqcup_{x\in X}\operatorname{Spec}(\mathcal{O}_{X,x})$. The closed points of X^Z are in bijection with the points of X.
2. (Topological version) Build a surjective morphism $T \to (X^Z)^{cl}$ to the set $(X^Z)^{cl}$ of closed points of X^Z from a profinite set[2] T which is weakly contractible as a profinite set.
3. (Dimension zero case) Give T a structure of affine scheme X_0 such that all residue fields are separably closed.
4. (General case) Henselise along $X_0 \to X$ to produce the desired $f : Y \to X$.
   The closed points $y \in Y^{\mathrm{cl}}$ will be in bijection with the points of T, and local rings $\mathcal{O}_{Y,y}$ at these points will be strict henselisations of their images $\mathcal{O}_{Y,y} \cong \mathcal{O}^{sh}_{X,f(y)}$.

$$\begin{tikzcd}
X_0 \arrow[r] & X^Z \arrow[r] & X \\
T \arrow[u] \arrow[r] & (X^Z)^{cl} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Example 4. If X has finitely many points (e.g., $X = \operatorname{Spec}(R)$ with R a discrete valuation ring, or more generally, a localisation of a Dedekind domain at finitely many primes) then X^Z in Step 1 *is* just the disjoint union $\amalg_{x\in X}\operatorname{Spec}(\mathcal{O}_{X,x})$ of the localisations at each point of X. Step 2 is unnecessary because $(X^Z)^{cl}$ is finite. That is, $T = (X^Z)^{cl}$. Step 3 just chooses separable closures $k(x)^{sep}$ for each $k(x)$, and Step 4 produces the disjoint union of the strict henselisations $\sqcup_{x\in X}\operatorname{Spec}(\mathcal{O}^{sh}_{X,x}) \to X$.

$$\begin{tikzcd}
\sqcup_{x\in X}\operatorname{Spec}(\mathcal{O}^{sh}_{X,x}) \arrow[r] & \sqcup_{x\in X}\operatorname{Spec}(\mathcal{O}_{X,x}) \arrow[r] & X \\
\sqcup_{x\in X}\operatorname{Spec}(k(x)^{\mathrm{sep}}) \arrow[u] \arrow[r] & \sqcup_{x\in X}\operatorname{Spec}(k(x)) \arrow[u] \arrow[ur] &
\end{tikzcd}$$

2 ProZariski covers of affine schemes

Definition 5. Let A be a ring and consider the *set* $\mathfrak{P}(A)$ of prime ideals. A subset $S \subseteq \mathfrak{P}(A)$ is *constructible* if it is a <u>finite</u> union of sets of the form

$$D(a_1, \dots, a_n; b) = \{\mathfrak{p} \mid a_1, \dots, a_n \in \mathfrak{p}, b \notin \mathfrak{p}\}.$$

[2]A *profinite set* is a topological space which is a filtered inverse limit $\lim_{\lambda\in\Lambda} F_\lambda$ of some system $\{F_\lambda\}_{\lambda\in\Lambda}$ of finite discrete spaces F_λ.

where $a_1, \dots, a_n, b \in A$.

The *constructible topology* on $\mathfrak{P}(A)$ is the topology[3] whose open sets are arbitrary unions of constructible subsets.

Example 6.

1. If $\mathrm{Spec}(A)$ has dimension zero, that is, all primes are maximal, then the Zariski topology is already the constructible topology.[4] We will see below that in this case, the topological space is a profinite set.

– – – – – picture: Cantor set – – – – – –

2. If A is Noetherian, integral, and Krull dimension one, then with constructible topology, $\mathfrak{P}(A)$ is the one point compactification of the set $\mathrm{Spec}(A)^{\mathrm{cl}}$ of Zariski closed points of $\mathrm{Spec}(A)$. That is, the opens in the constructible topology are:
   (a) Any set of Zariski closed points.
   (b) Any complement of finitely many Zariski closed points.
   In particular, the generic point is closed in the constructible topology.

– – – – – picture: convergent sequence – – – – – –

Exercise 7. Show that the family of constructible sets is closed under finite union, finite intersection, and complement.

Exercise 8. Consider $\mathfrak{P}(A)$ with the constructible topology. Show that:

1. Every element of $\mathfrak{P}(A)$ is closed in the constructible topology.
2. If $\{W_i\}_{i \in I}$ is a set of constructible subsets such that $\mathfrak{P}(A) = \cup_{i \in I} W_i$, then there exists a finite set of disjoint constructible subsets $X_1, \dots, X_n$ such that $\mathfrak{P}(A) = \sqcup_{j=1}^n X_j$ and for each j we have $X_j \subseteq W_{i_j}$ for some i_j. Hint.[5] Hint.[6]

Our first proposition is that the constructible topology is the Zariski topology for some appropriate A-algebra.

Proposition 9. *There exists a ring homomorphism $A \to A_{\mathrm{cons}}$ such that:*

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[3] Here we mean topology in the classical sense of a set equipped with a collection of *open* subsets. We are not talking about Grothendieck topologies.

[4] It is enough to show that for any $f \in A$ the subscheme $\mathrm{Spec}(A/f)$ is open in the Zariski topology. Since we only care about the topological space, we can assume A is reduced. An A-module is flat if and only if it's localisations are flat. Since A is dimension zero and reduced, all local rings are fields. So all localisations $(A/f)_\mathfrak{p}$ are free $A_\mathfrak{p}$-modules. Since A/f is a finitely presented A-module, this means it is projective. So there is a section $A/f \to A$ to the canonical projection $A \to A/f$. Let $e \in A$ be the image of the unit $1 \in A/f$ under this projection. Then one checks that $A/f \cong A[e^{-1}]$. Indeed, $A \to A/f \to A$ is multiplication by e, so $ef = 0$. So for every prime $\mathfrak{p}$ with $e \notin \mathfrak{p}$ we have $f \in \mathfrak{p}$. That is, $\mathrm{Spec}(A[e^{-1}]) \subseteq \mathrm{Spec}(A/f)$. On the other hand by construction, e is sent to 1 in A/f. That is, $1 = e + fg$ for some g. Since no prime contains 1 this means that $f \in \mathfrak{p} \Rightarrow e \notin \mathfrak{p}$, or in other words, $\mathrm{Spec}(A/f) \subseteq \mathrm{Spec}(A[e^{-1}])$.

[5] Show that there is a finite subset $I' \subseteq I$ such that we still have $\mathfrak{P}(A) = \cup_{i \in I'} W_i$.

[6] Using Exercise 7, note that if W, W' are constructible then $W \cap W'$ and $W \setminus (W \cap W')$ are also constructible.

1. $\mathfrak{P}(A_{\mathrm{cons}}) \to \mathfrak{P}(A)$ *is a bijection.*
2. $\mathfrak{P}(A_{\mathrm{cons}}) \to \mathfrak{P}(A)$ *induces isomorphisms on all residue fields.*
3. *The Zariski topology on* $\mathfrak{P}(A_{\mathrm{cons}})$ *is the constructible topology on* $\mathfrak{P}(A)$.

The proof of this proposition takes some setting up.

Construction 10 (Stacks Project, 096U)**.** Let A be a ring and $E \subseteq A$ a finite set of elements. For each decomposition $E = E_+ \sqcup E_-$ consider the A-algebra

$$A_{E_+,E_-} = \tfrac{A}{\langle a \in E_+ \rangle}[\tfrac{1}{\prod_{b\in E_-} b}]. \tag{1}$$

Remark 11. For a ring A, ideal I, and multiplicatively closed subset $S \subseteq A$, we will identify $\mathfrak{P}((A/I)[S^{-1}])$ with the corresponding subset of $\mathfrak{P}(A)$.

Exercise 12. Suppose $E = E_+ \sqcup E_- = \{a_1, \dots, a_n\} \sqcup \{b_1, \dots, b_m\}$. Show that $\mathfrak{P}(A_{E_+,E_-})$ is precisely $D(a_1, \dots, a_n; \Pi b_j) \subseteq \mathfrak{P}(A)$.

Exercise 13. Show that for a fixed $E \subseteq A$, the *set* $\mathfrak{P}(A)$ is the disjoint union of the *sets* $\mathfrak{P}(A_{E_+,E_-})$

$$\mathfrak{P}(A) = \coprod_{E=E_+\sqcup E_-} \mathfrak{P}(A_{E_+,E_-})$$

over all decompositions of E. That is, for each prime $\mathfrak{p} \subseteq A$ there exists a unique decomposition $E = E_+ \sqcup E_-$ such that $\mathfrak{p} \in \mathfrak{P}(A_{E_+,E_-})$.

Note that given any inclusion $E \subseteq F$ of finite subsets of a ring A, and a decomposition $F = F_+ \sqcup F_-$ of F, there is an induced decomposition $E = (E\cap F_+) \sqcup (E \cap F_-)$ of E. Moreover, there is a canonical ring homomorphism

$$A_{(E\cap F_+),(E\cap F_-)} \to A_{F_+,F_-}$$

−−−−−−−picture: stratification of the plane by finitely many curves and their intersections−−−−

Exercise 14. Show that we have:
1. $\mathrm{Spec}(R_0 \times R_1) = \mathrm{Spec}(R_0) \sqcup \mathrm{Spec}(R_1)$ for rings R_0, R_1, and
2. $\mathrm{Spec}(\mathrm{colim}\, R_\lambda) = \lim \mathrm{Spec}(R_\lambda)$ for any filtered diagram of rings $\{R_\lambda\}_{\lambda\in\Lambda}$.

Here Spec are considered as topological spaces. So the limit and disjoint union are in the category of topological spaces.

Proof of Proposition 9. Set

$$A_{\mathrm{cons}} = \varinjlim_{\substack{E\subseteq A \\ E \text{ finite}}} \prod_{E=E_+\sqcup E_-} A_{E_+,E_-}. \tag{2}$$

By Exercise 13 each $\sqcup_{E=E_+\sqcup E_-}\mathfrak{P}(A_{E_+,E_-})$ is bijective to $\mathfrak{P}(A)$, so the colimit also has this property. The claim about residue fields is also clear, since localisation and quotient of rings don't change residue fields.

By Exercise 12, every constructible subset is open in $\mathrm{Spec}(A_{\mathrm{cons}})$. Conversely, by definition of the limit topology,[7] every open of $\mathrm{Spec}(A_{\mathrm{cons}})$ is a union of opens in the $\mathrm{Spec}(A_{E_+,E_-})$. Since each $\mathrm{Spec}(A_{E_+,E_-})$ is affine, every such open is a finite union of "standard" opens in the $\mathrm{Spec}(A_{E_+,E_-})$, that is, one of the form $\mathrm{Spec}(A_{E_+,E_-}[f^{-1}])$. Clearly, we can assume $f \in A$, since multiplying f by a unit of A_{E_+,E_-} doesn't change the localisation $A_{E_+,E_-}[f^{-1}]$. In this case we have $\mathrm{Spec}(A_{E_+,E_-}[f^{-1}]) = \mathrm{Spec}(A_{E_+,E_-\cup\{f\}})$. So every Zariski open in $\mathrm{Spec}(A_{\mathrm{cons}})$ is a union of constructibles of $\mathfrak{P}(A)$. $\square$

We can use the construction above to make a more interesting affine scheme.

Proposition 15. *The ring homomorphism $A \to A_{\mathrm{cons}}$ factors through a surjection*

$$A \to A_w \overset{surj.}{\twoheadrightarrow} A_{\mathrm{cons}}$$

such that:

1. *The canonical morphism $\coprod_{\mathfrak{p}\in\mathrm{Spec}(A)} \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$ factors through a bijection*
$$\coprod_{\mathfrak{p}\in\mathfrak{P}(A)} \mathfrak{P}(A_{\mathfrak{p}}) \xrightarrow{\sim} \mathfrak{P}(A_w).$$
2. $\mathrm{Spec}(A_{\mathrm{cons}})$ *is the space* $\mathrm{Spec}(A_w)^{\mathrm{cl}}$ *of closed points of* $\mathrm{Spec}(A_w)$.
3. *Every prime of A_w is contained in a* unique *maximal ideal of A_w.*
4. *A_w is* ind-Zariski *in the sense that it is a filtered colimit of finite products of localisations of A:*
$$A_w = \mathrm{colim}_{\lambda\in\Lambda} \prod_{i\in I_\lambda} A[S_{\lambda,i}^{-1}].$$

Sketch of proof. Given a finite set $E \subseteq A$ and a decomposition $E = E_+ \sqcup E_-$ define

$$A^{\sim}_{E_+,E_-} := A[S^{-1}_{E_+,E_-}]$$

where $S_{E_+,E_-} \subseteq A$ is the set of elements which are sent to units in A_{E_+,E_-}. That is, $S_{E_+,E_-} = A \times_{A_{E_+,E_-}} (A_{E_+,E_-})^*$. Note that this comes equipped with a canonical surjection

$$A^{\sim}_{E_+,E_-} \twoheadrightarrow A_{E_+,E_-} \tag{3}$$

Since every element of A_{E_+,E_-} can be written as a/s with $a \in A$ and $s \in S_{E_+,E_-}$, cf.Eq.(1). Next, define

$$A_w = \varinjlim_{\substack{E\subseteq A\\ E \text{ finite}}} \prod_{E=E_+\sqcup E_-} A^{\sim}_{E_+,E_-}$$

so we get (4) by definition. It comes with canonical maps

$$A \to A_w \overset{(*)}{\to} A_{\mathrm{cons}} = \varinjlim_{\substack{E\subseteq A\\ E \text{ finite}}} \prod_{E=E_+\sqcup E_-} A_{E_+,E_-}.$$

[7]For any diagram of topological spaces X_λ, the opens of $\lim X_\lambda$ are unions of pullbacks of opens in X_λ. That is, unions of sets of the form $U \times_{X_\lambda} \lim X_\lambda$ for $U \subseteq X_\lambda$ open.

The map (*) is surjective because the Eq.(3) are surjective. So we get the surjectivity claim.

We leave it to the reader to check (1), (2), and (3). ☐

Proposition 16. *For every open covering* $\{U_i\}_{i\in I}$ *of* $\mathrm{Spec}(A_w)$ *there exists a finite decomposition* $\mathrm{Spec}(A_w) = X_1 \sqcup \cdots \sqcup X_n$ *and factorisations*

$$X_j \subseteq U_{i_j} \subseteq X$$

Proof. In Exercise 8 we saw that the analogous claim holds for $\mathrm{Spec}(A_{\mathrm{cons}})$. That is, there is a decomposition $\mathrm{Spec}(A_{\mathrm{cons}}) = Z_1 \sqcup \cdots \sqcup Z_n$ such that each Z_j is contained in some U_{i_j}. Looking more carefully at $\mathrm{Spec}(A_{\mathrm{cons}})$ one sees that the decomposition $Z_1 \sqcup \cdots \sqcup Z_n$ is achieved by some $E \subseteq A$. So we get an analogous decomposition $\mathrm{Spec}(A_w) = X_1 \sqcup \cdots \sqcup X_n$.

Since every point of $\mathrm{Spec}(A_w)$ specialises to a unique point of $\mathrm{Spec}(A_{\mathrm{cons}})$, the inclusion $Z_j \subseteq U_{i_j}$ implies an inclusion $X_j \subseteq U_{i_j}$. ☐

3 Compacta

Definition 17. A *compactum* is a topological space X which is compact and Hausdorff. That is, such that,

1. every open covering of X is refinable by a finite one, and
2. for every two distinct points $x_0, x_1 \in X$ there exist disjoint opens U_0, U_1 with $x_\varepsilon \in U_\varepsilon$.

Example 18.

1. Every finite discrete space is a compactum.
2. Closed subspaces of compacta are compacta.
3. Products of compacta are compacta (Tychonoff's theorem).
4. Limits of compacta are compacta.
5. Profinite sets are compacta.

Exercise 19. For any ring A, show that the topological space of $\mathrm{Spec}(A_{\mathrm{cons}})$ is a compactum.

Definition 20 (Stacks Project, 08YN)**.** A compactum X is called *extremally disconnected* if for every continuous surjection $Y \to X$ from a compactum, there exists a continuous section $X \to Y$.

Example 21. Any finite discrete space is extremally disconnected.

Proposition 22. *For any topological space* X *the category* $\mathrm{Compacta}_{X/}$ *of maps* $X \to Y$ *towards compacta has an initial object* $X \to \beta X$.

Definition 23. The space βX is called the *Stone-Čech compactification* of X.

Proof. The main observation is that if $X \to Y$ is a continuous map with dense image and Y is Hausdorff, then the cardinality of Y is bounded: $|Y| < 2^{2^{|X|}}$. This is [Stacks Project, 0909]. It is not hard to show, but we will not do it.[8]

[8] The idea is that each point Y is determined by the set of it's "nice" neighbourhoods in $f(X)$.

Choose a set I of objects $X \to Y$ of $\mathrm{Compacta}_{X/}$ such that every morphism towards a compactum with dense image is represented in I. This is possible by the previous observation. Let βX be the closure of the image of X in

$$\prod_{X \to Y \in I} Y.$$

Then this satisfies the required property basically by construction.

Every map $f : X \to K$ towards a compactum factors through the closure of the image $X \to \overline{f(X)} \to K$. The former morphism is isomorphic to one in I, so it factors through the product, and hense through βX,

$$X \to \beta X \subseteq \prod_{X \to Y \in I} Y \overset{proj.}{\to} \overline{f(X)} \subseteq K.$$

Since $X \to \beta X$ has dense image, any two factorisations $X \to \beta X \rightrightarrows K$ must agree. Indeed, given any two such maps we can consider the factorisation $X \to \beta X \times_{K \times K} K \subseteq \beta X$. Since $X \subseteq \beta X$ is dense and the inclusion $\beta X \times_{K \times K} K \subseteq \beta X$ is closed, we must have $\beta X \times_{K \times K} K = \beta X$. This means the two maps $\beta X \rightrightarrows K$ are the same. $\square$

Corollary 24.

1. *If X is discrete, then βX is extremally disconnected.*
2. *Every compactum X admits a continuous surjection $Y \to X$ from an extremally disconnected compactum Y.*

Proof.

1. If X is discrete, and $Y \to \beta X$ any surjection from a compactum, we can choose a lift $X \to Y \to \beta X$. By the universal property of βX this comes with a factorisation $X \to \beta X \to Y \to \beta X$. Since $X \to \beta X$ is initial in $\mathrm{Compacta}_{X/}$, there is a unique commutative triangle $\begin{tikzcd}[row sep=small] & \beta X \arrow[dd] \\ X \arrow[ur] \arrow[dr] & \\ & \beta X \end{tikzcd}$. So the composition $\beta X \to Y \to \beta X$ must be the identity $\mathrm{id}_{\beta X}$.
2. Let X^δ be X with the discrete topology. Then $X^\delta \to X$ factors as $X^\delta \to \beta(X^\delta) \to X$. Since $X^\delta \to X$ is surjective, so is $\beta(X^\delta) \to X$.

$\square$