(Pro)étale cohomology
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Lecture 8: Proétale cohomology

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1 Weil conjectures

We began the course with the question:

Question 1. Given a smooth projective variety $X/\mathbb{F}_q$, how many $\mathbb{F}_{q^n}$-points does X have for each n? That is, calculate

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}\right).$$

This lead to the Weil conjectures:

Theorem 2 (Weil conjectures)**.** *If X is a smooth projective variety of dimension d over $\mathbb{F}_q$.*

1. *(Rationality) $Z(X,t)$ is a rational function of t. In other words we have $Z(X,t) \in \mathbb{Q}(t) \cap \mathbb{Q}[[t]] \subseteq \mathbb{Q}((t))$.*
2. *(Functional equation) There is an integer e such that*
$$Z(X, q^{-d}t^{-1}) = \pm q^{ed/2} t^e Z(X,t).$$
3. *(Riemann Hypothesis) We can write*
$$Z(X,t) = \frac{P_1(t)P_3(t)\dots P_{2d-1}(t)}{P_0(t)P_2(t)\dots P_{2d}(t)}$$
with $P_i(t) \in \mathbb{Z}[t]$, and such that the roots of $P_i(t)$ have absolute value $q^{-i/2}$. Moreover, $P_0(t) = 1-t$ and $P_{2d}(t) = 1 - q^d t$.
4. *(Betti numbers) If X comes from a smooth projective variety over $\mathcal{O}_\mathfrak{p} \subseteq \mathbb{C}$ for some number ring $\mathcal{O}$ and prime $\mathfrak{p}$ (e.g., $\mathcal{O}_\mathfrak{p} = \mathbb{Z}_{(p)}$),*
$$\deg P_i(t) = \dim_\mathbb{Q} H^i(X(\mathbb{C}), \mathbb{Q}).$$

The strategy was to develop a cohomology theory

$$H^\bullet : (\text{Varieties}/k)^{op} \to \text{graded } \mathbb{Q}\text{-vector spaces}$$

for arbitrary varieties over any field k, which satisfied the following properties for smooth projective varieties X.

1. (Finiteness) $\dim H^\bullet(X)$ is finite, and $H^i(X) = 0$ for $i \notin \{0, 1, \ldots, 2\dim X\}$.
2. (Poincaré Duality) There is a canonical isomorphism $H^{2\dim X}(X) \cong \mathbb{Q}$ and a natural perfect pairing
$$H^i(X) \times H^{2d-i}(X) \to \mathbb{Q}$$
3. (Lefschetz Trace Formula)
$$|X(\mathbb{F}_{q^m})| = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}(\phi_i^m)$$
where $X_{\overline{\mathbb{F}}_q} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $\phi : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ is the Frobenius morphism, and $\phi_i : H^i(X_{\overline{\mathbb{F}}_q}) \to H^i(X_{\overline{\mathbb{F}}_q})$ is the induced morphism.
4. (Compatibility) If $k = \mathbb{C}$ then $H^\bullet(X)$ is isomorphic to singular cohomology.

Then,

$$\begin{array}{rcl} \text{(Lefschetz Trace Formula)} & \Rightarrow & \text{(Rationality)} \\ \text{(Poincaré Duality)} & \Rightarrow & \text{(Functional equation)} \\ \text{(Compatibility)} & \Rightarrow & \text{(Betti numbers)} \\ \text{Eigenvalues } \alpha_{i,j} \text{ of } \phi_i | H^i(X_{\overline{\mathbb{F}}_q}) & & \\ \text{satisfy } |\alpha_{i,j}| = q^{-i/2} & \Rightarrow & \text{(Riemann Hypothesis)} \end{array}$$

Very early Serre showed that due to the existence of supersingular elliptic curves, there cannot be any cohomology theory with the above properties taking values in $\mathbb{Q}$-vector spaces.

However, we saw in the last lecture that for curves, étale cohomology with $\mathbb{Z}/l^n$-coefficients has Poincaré Duality and

$$\operatorname{rank}_{\mathbb{Z}/l^n} H^i_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/l^n) = \dim_{\mathbb{Q}} H^i_{\text{sing}}(X(\mathbb{C}), \mathbb{Q}).$$

2 l-adic cohomology

This leads us to define:

Definition 3.
$$H^i_{\text{et}}(X, \mathbb{Q}_l) := \left(\varprojlim_{n \geq 1} H^i_{\text{et}}(X, \mathbb{Z}/l^n) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \tag{1}$$

Then we obtain the following.

Theorem 4. *The $\mathbb{Q}_l$-vector spaces $H^i_{\text{et}}(X, \mathbb{Q}_l)$ satisfy (Finiteness), (Poincaré Duality), (Lefschetz Trace Formula), and (Riemann Hypothesis).*

The Poincaré Duality that we saw in the last lecture was a special case of a much more general form of duality which is encoded in a six functor formalism.

Theorem 5. *For "nice" morphisms between "nice" $\mathbb{Z}[1/l]$-schemes $f : Y \to X$, and "nice" objects $E \in D(X_{\text{et}}, \mathbb{Z}/l^n)$ there are adjunctions*

$$(f^*, f_*) : D(Y_{\text{et}}, \mathbb{Z}/l^n) \rightleftarrows D(X_{\text{et}}, \mathbb{Z}/l^n)$$
$$(f_!, f^!) : D(X_{\text{et}}, \mathbb{Z}/l^n) \rightleftarrows D(Y_{\text{et}}, \mathbb{Z}/l^n)$$
$$(- \otimes E, \mathscr{H}om(E, -)) : D(X_{\text{et}}, \mathbb{Z}/l^n) \rightleftarrows D(X_{\text{et}}, \mathbb{Z}/l^n)$$

satisfying a number of properties such as a Proper Base Change, Künneth, and Projection formulas.

Remark 6. In this framework, (Duality) becomes an isomorphism

$$f_* \mathscr{H}\text{om}(F, f^! G) \xrightarrow{\sim} \mathscr{H}\text{om}(f_! F, G). \tag{2}$$

In order to have these functors for sheaves of $\mathbb{Z}_l$-modules, some work is needed.

Definition 7 ([BS, Def.3.5.3])**.** For a scheme X, define $\text{Shv}_{\text{et}}(X, \mathbb{Z}/l^\bullet)$ to be the category of $\mathbb{N}$-indexed systems

$$\cdots \to F_2 \to F_1 \to F_0$$

in $\text{Shv}_{\text{et}}(X, \mathcal{A}\text{b})$ such that $F_n \in \text{Shv}_{\text{et}}(X, \mathbb{Z}/l^n)$ for each n. The derived category of this abelian category is denoted by $D(X_{\text{et}}, \mathbb{Z}/l^\bullet)$. We consider its full subcategory

$$D_{Ek}(X, \mathbb{Z}_l) \subseteq D(X_{\text{et}}, \mathbb{Z}/l^\bullet)$$

consisting of those systems of complexes $(\cdots \to K_2 \to K_1 \to K_0)$ such that each

$$K_{n+1} \otimes^L_{\mathbb{Z}/l^{n+1}} \mathbb{Z}/l^n \to K_n$$

is an isomorphisms in $D(X_{\text{et}}, \mathbb{Z}/l^n)$.

Theorem 8 (Ekedahl)**.** *Under reasonable hypotheses on X, the functors $f^*, f_*, f_!, f^!, \otimes, \mathscr{H}om$ can be extended to the categories $D_{Ek}(X, \mathbb{Z}_l)$ in a sensible way.*

We also have a very nice Galois theory.

Definition 9. Let X be a connected scheme, $\overline{x} \to X$ a geometric point, FEt_X the category of finite étale X-schemes, and consider the functor

$$\Phi : \text{FEt}_X \to \mathcal{S}\text{et}; \qquad Y \mapsto Y_{\overline{x}} = \hom_X(\overline{x}, Y).$$

The *étale fundamental group* of X is the profinite group

$$\pi_1^{\text{et}}(X, \overline{x}) = \text{Aut}(\Phi).$$

That is, an element of $\pi_1^{\text{et}}(X, \overline{x})$ is a system $(\sigma_Y)_{Y \in \text{FEt}_X}$ of automorphisms $\sigma_Y : \Phi(Y) \xrightarrow{\sim} \Phi(Y)$ indexed by objects $Y \in \text{FEt}_X$ subject to the naturality condition that for every morphism $Y' \to Y$ in FEt_X the corresponding square is commutative.

Example 10.

1. If X is the spectrum of a field k then
$$\pi_1^{\mathrm{et}}(X, \overline{x}) \cong \mathrm{Gal}(k^{sep}/k).$$
2. If X is a smooth $\mathbb{C}$-variety then
$$\pi_1^{\mathrm{et}}(X, \overline{x}) \cong \pi_1(X(\mathbb{C}))^{\wedge};$$
the profinite completion[1] of the usual fundamental group. [Szamuely, Galois groups and fundamental groups, Thm 5.7.4], [SGA1, Exposé XII, Cor 5.2]

Theorem 11 (Stacks Project, Tags 0BNB, 0BMY,0BN4)**.** *Let X be a connected scheme and $\overline{x} \to X$ a geometric point. Then Φ induces an equivalence of categories*
$$\mathrm{FEt}_X \cong \pi_1^{\mathrm{et}}(X, \overline{x})\text{-FinSet}$$
with the category of finite sets equipped with a continuous $\pi_1^{\mathrm{et}}(X, \overline{x})$-action.

There is also a linear version of this. Recall that $\mathrm{Loc}_X(R)$ is the category of *local systems* with R-coefficients. That is, sheaves F of R-modules such that for some covering $\{f_i : U_i \to X\}$, each $f_i^* F$ is isomorphic to the constant sheaf R^n for some n. Similar to the case of topological spaces, π_1 determines the category of local systems.

Proposition 12. *If X is a connected locally noetherian $\mathbb{Z}_{(l)}$-scheme, then there is an equivalence of categories*
$$\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \varprojlim \mathrm{Loc}_X(\mathbb{Z}/l^n) \cong \left\{ \begin{array}{c} \textit{continuous finite dimensional} \\ \mathbb{Q}_l\textit{-linear representations of } \pi_1^{et}(X) \end{array} \right\}.$$

All of this is not quite as nice as it could be though.

Problem 13.

1. *The definition $H^i_{\mathrm{et}}(X, \mathbb{Q}_l) := \left(\varprojlim_{n \geq 1} H^i_{\mathrm{et}}(X, \mathbb{Z}/l^n) \right) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ is a ad hoc, and not very pleasant to work with.*
2. *The categories $D(X_{\mathrm{et}}, \mathbb{Z}/l^\bullet)$ are horrible to work with.*
3. *The equivalence between local systems and π_1-representations is no longer true in general if one uses, honest $\mathbb{Q}_l$-local systems instead of the ad hoc $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} \mathrm{Loc}_X(\mathbb{Z}/l^n)$ (cf. [Bhatt-Scholze, Pro-étale topology, Example 7.4.9] for an example due to Deligne).*

Question 14. So why can't we just use sheaves of $\mathbb{Z}_l$-coefficients?

$$\boxed{\text{Representability!}}$$

[1] The profinite completion of a group G is the inverse limit over all surjective maps to finite groups $\lim_{G \twoheadrightarrow F} F$.

Finite coefficients work so well due to the equivalence of categories.

Theorem 15. *There is equivalence of categories*

$$\mathrm{FEt}(X) \cong \mathrm{Loc}_X(\mathrm{FinSet})$$

between the category of finite étale X-schemes and the category of locally constant étale sheaves.

This suggests that we should enlarge the category Et_X to include filtered limits.

3 The proétale topology

Definition 16. A morphism of schemes $Y \to X$ is *weakly étale* if both $Y \to X$ and $Y \to Y \times_X Y$ are flat.

Example 17.

1. Étale morphisms are weakly étale.
2. If $\ldots \to Y_2 \to Y_1 \to Y_0$ is a sequence of étale X-schemes, then $\lim Y_n \to X$ is weakly étale.
3. In particular, $\lim_{n\in\mathbb{N}}(\sqcup_{\mathbb{Z}/l^n})X \to X$ is weakly étale.

Exercise 18. Show that for any $Y \in \mathrm{Et}_{/X}$ we have

$$\hom_X(Y, \lim_{n\in\mathbb{N}}(\sqcup_{\mathbb{Z}/l^n})X) = \mathbb{Z}_l(Y)$$

where $\mathbb{Z}_l$ is the constant étale sheaf associated to $\mathbb{Z}_l$. That is, $\mathbb{Z}_l$ is representable by the scheme $\lim_{n\in\mathbb{N}}(\sqcup_{\mathbb{Z}/l^n})X$.

Definition 19. The category X_{proet} of weakly étale X-schemes is equipped with the coarsest topology[2] such that:

1. Zariski coverings are coverings, and
2. $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is a covering for every surjective $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ in X_{proet}.

Theorem 20. *Let X be a connected noetherian scheme.*

1. *We have*
$$H^i(X_{\mathrm{proet}}, \mathbb{Q}_l) \cong H^i(X_{\mathrm{et}}, \mathbb{Q}_l)$$
where the right hand side is the limit Eq.(1), and the left hand side is honest sheaf cohomology of $\mathbb{Q}_l$.
2. *The six functors of Theorem 5 work for the honest derived categories $D(X_{\mathrm{proet}}, \mathbb{Z}_l)$.*

[2] In other words, $\{Y_i \to Y\}_{i\in I}$ is a covering if there exists a Zariski covering $\{U_j \to Y\}_{j\in J}$, surjections $V_j \to U_j$, a map $\sigma : J \to I$ and factorisations $V_j \to Y_{\sigma(j)} \to Y$.

3. *If* $X = \mathrm{Spec}(k)$ *is the spectrum of a field, then the subcategory of quasicompact quasiseparated objects* $X^{qcqs}_{\mathrm{proet}}$ *is canonically isomorphic to the category of profinite continuous (not necessarily finite)* $\mathrm{Gal}(k^{sep}/k)$*-sets*

$$\mathrm{Spec}(k)^{qcqs}_{\mathrm{proet}} \cong \left\{ \begin{array}{c} \textit{Profinite sets equipped with a} \\ \textit{continuous } \mathrm{Gal}(k^{sep}/k) \textit{ action.} \end{array} \right\}$$

4. *Honest* $\mathbb{Q}_l$*-local systems on* X *are equivalent to continuous representations of* $\pi_1^{\mathrm{proet}}(X)$ *on finite dimensional* $\mathbb{Q}_l$*-vector spaces.*

4 Proétale schemes

Definition 21. A morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes is *proétale* if there exists a cofiltered[3] system $(B_\lambda)_{\lambda\in\Lambda}$ of étale finite presentation A-algebras such that $B = \varinjlim B_\lambda$. The system (B_λ) is called a *presentation* for B.

Exercise 22. Let $(B_\lambda)_{\lambda\in\Lambda}$ be a cofiltered system of rings. Let $\mathfrak{P}(C)$ denote the set of prime ideals of a ring C, and $\mathrm{Spc}(C)$ the underlying topological space of $\mathrm{Spec}(C)$, i.e., $\mathrm{Spc}(C)$ is $\mathfrak{P}(C)$ equipped with its Zariski topology.

1. Show that
$$\mathfrak{P}(\varinjlim B) = \varprojlim \mathfrak{P}(B_\lambda).$$
2. Show that for any $f \in B_\lambda$ with image $\overline{f} \in \varinjlim B_\lambda$, the set $D(\overline{f}) \subseteq \mathfrak{P}(\varinjlim B_\lambda)$ of primes not containing $\overline{f}$ is the preimage of the set $D(f) \subseteq \mathfrak{P}(B_\lambda)$ of primes not containing f, under the canonical map $\pi : \mathfrak{P}(\varinjlim B_\lambda) \to \mathfrak{P}(B_\lambda)$. That is, show $D(\overline{f}) = \pi^{-1}(D(f))$.
3. Deduce that
$$\mathrm{Spc}(\varinjlim B_\lambda) = \varprojlim \mathrm{Spc}(B_\lambda).$$

Exercise 23. Let k be an algebraically closed field. Using Exercise 22, show that for every profinite set S, there exists a proétale k-scheme $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ with $S \cong \mathrm{Spc}(B)$.

Exercise 24. Let k be a field and $k \subseteq k^{sep}$ a separable closure. Show that the $\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)$ is proétale.

Exercise 25. Suppose that $\mathrm{Spec}(B) \to \mathrm{Spec}(A), \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ are proétale. Show that $\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is proétale. Hint.[4]

[3] A system is cofiltered if (i) it is nonempty, (ii) for every pair of objects $B_\lambda, B_{\lambda'}$ there is a third object $B_{\lambda''}$ and morphisms in the system $B_\lambda \to B_{\lambda''}, B_{\lambda'} \to B_{\lambda''}$, and (iii) for any pair of parallel morphisms in the system $B_\lambda \rightrightarrows B_{\lambda'}$ there exists a morphism in the system $B_{\lambda'} \to B_{\lambda''}$ such that the two compositions are equal.

[4] Let $B = \varinjlim_{\lambda\in\Lambda} B_\lambda$ and $C = \varinjlim_{\mu\in M} C_\mu$ be presentations and consider the system $(B_\lambda \otimes_A C_\mu)_{(\lambda,\mu)\in\Lambda\times M}$.

Exercise 26. Show that

$$\operatorname{Spc}(k^{sep} \otimes_k k^{sep}) \cong Gal(k^{sep}/k)$$

as *topological spaces.* Hint.[5] Hint.[6]

Exercise 27. Let A be a ring and $\mathfrak{p} \in \operatorname{Spec}(A)$ a point. Show that the canonical morphism $\operatorname{Spec}(A_{\mathfrak{p}}) \to \operatorname{Spec}(A)$ is proétale.

Example 28. Let p_n be the nth prime number (so $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, \dots$). For any $n \in \mathbb{N}$, the map

$$X_n := \operatorname{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}]) \amalg (\sqcup_{i=1}^n \operatorname{Spec}(\mathbb{Z}_{(p_i)})) \to \operatorname{Spec}(\mathbb{Z})$$

is proétale. Moreover, there are canonical morphisms $X_{n+1} \to X_n$ induced by the canonical proétale morphisms

$$\operatorname{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}, \tfrac{1}{p_{n+1}}]) \amalg \operatorname{Spec}(\mathbb{Z}_{p_{n+1}}) \to \operatorname{Spec}(\mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}]).$$

Consequently, $X := \varprojlim X_n$ is a proétale $\operatorname{Spec}(\mathbb{Z})$ scheme. As a set, we have

$$X = \{\eta\} \amalg (\sqcup_{n \geq 1} \{\eta_i, \mathfrak{p}_i\})$$

where $\{\eta_i, \mathfrak{p}_i\}$ correspond to the points of $\operatorname{Spec}(\mathbb{Z}_{(p_i)})$, and η corresponds to the generic points of the $\operatorname{Spec}(\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_n}])$'s. The open sets of X are disjoint unions of sets of the form

$$\{\eta_i\}, \qquad \{\eta_i, \mathfrak{p}_i\}, \qquad X \setminus (\sqcup_{i=1}^N \{\eta_i, \mathfrak{p}_i\}).$$

In particular, every open covering of X can be refined by one which is a finite family of sets of the above form. These sets' corresponding rings of functions are

$$\mathbb{Q}, \qquad \mathbb{Z}_{(p_i)}, \qquad \varinjlim_{n \to \infty} \mathbb{Z}[\tfrac{1}{p_1}, \dots, \tfrac{1}{p_n}] \times (\mathbb{Z}_{(p_N)} \times \mathbb{Z}_{(p_{N+1})} \times \cdots \times \mathbb{Z}_{(p_n)}).$$

The latter is a subring of $\prod_{i>N} \mathbb{Z}_{(p_i)}$ with $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_n}]$ embedded diagonally into $\prod_{i>n} \mathbb{Z}_{(p_i)}$. Here is a picture.

Exercise 29. Consider the X from Example 28. Show that for every open covering $\{U_i \to X\}_{i \in I}$ the associated morphism $\amalg U_i \to X$ admits a section. Deduce that for every open covering $\{U_i \to X\}_{i \in I}$ there exists a finite decomposition $X = \sqcup_{i=1}^n V_i$ into clopens,[7] a map $\sigma : \{1, \dots, n\} \to I$ and factorisations $V_i \subseteq U_{\sigma i} \subseteq X$.

[5] Recall that if L/k is a (finite) Galois extension, then $\operatorname{Spec}(L \otimes_k L) \cong \amalg_{Gal(L/k)} \operatorname{Spec}(L)$.

[6] Recall also that a separable closure k^{sep}/k is the union of the finite Galois subextensions $k \subseteq L \subseteq k^{sep}$ and $Gal(k^{sep}/k) \cong \varprojlim_{k \subseteq L \subseteq k^{sep}} Gal(L/k)$.

[7] I.e., each $V_i \subseteq X$ is both closed and open.