(Pro)étale cohomology
Shane Kelly, UTokyo
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Comments welcome!: shanekelly64[at]gmail[dot]com.

Lecture 7: Curves

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At the beginning of the course we gave the Weil conjectures as motivation for the development of étale cohomology. The four properties that would allow étale cohomology to imply the Weil conjectures were:

1. Compatibility with singular cohomology.
2. Poincaré Duality.
3. Lefschetz Trace Formula.
4. Riemann Hypothesis.

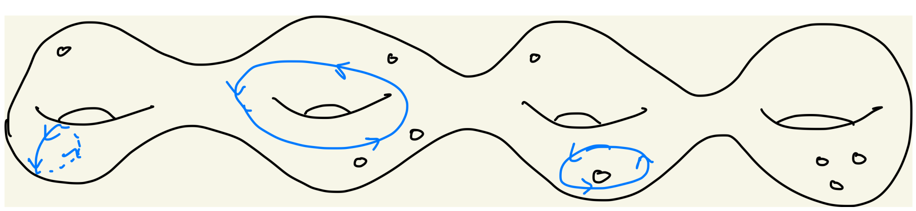
The last two are beyond the scope of this course. In this lecture we calculate the cohomology of curves over algebraically closed fields, and state Poincaré Duality for curves.

Definition 0.1. *In this lecture "curve" means smooth connected dimension one variety over an algebraically closed field* $k = \overline{k}$.

1 Some topology

Suppose that $k = \mathbb{C}$, and U is a curve. Then the associated topological space $U(\mathbb{C})$ is homeomorphic to a sphere with g-handles attached M_g and some points removed

$$U(\mathbb{C}) \cong M_g \setminus \{x_1, \dots, x_m\}$$



Consequently, we have the following.[1]

$$H^r_{\text{sing}}(U(\mathbb{C}),\mathbb{Q}) = \begin{cases} \begin{array}{c|ccc} r\backslash m & 0 & 1 & >1 \\ \hline 0 & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 1 & \mathbb{Q}^{2g} & \mathbb{Q}^{2g} & \mathbb{Q}^{2g+m-1} \\ 2 & \mathbb{Q} & 0 & 0 \\ >2 & 0 & 0 & 0 \end{array} \end{cases}$$

$$H^r_{\text{sing},c}(U(\mathbb{C}),\mathbb{Q}) = \begin{cases} \begin{array}{c|ccc} r\backslash m & 0 & 1 & >1 \\ \hline 0 & \mathbb{Q} & 0 & 0 \\ 1 & \mathbb{Q}^{2g} & \mathbb{Q}^{2g} & \mathbb{Q}^{2g+m-1} \\ 2 & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ >2 & 0 & 0 & 0 \end{array} \end{cases}$$

Remark 1.1.

1. The symmetry here comes from a canonical pairing, known as Poincaré Duality [Hatcher, "Algebraic Topology", Theorems 3.2 and 3.35].

2. If $U(\mathbb{C})$ was a non-orientable manifold such as $\mathbb{RP}^2 = S^2/\{z \sim -z\}$, then we can still get a duality if instead of the constant sheaf $\mathbb{Q}$ we use an appropriate locally constant sheaf (cf. Hatcher, "Algebraic Topology", Theorem 3H.6). In the étale theory, the sheaf

$$\mu_n(V) = \{n\text{th roots of unity of } \Gamma(V, \mathcal{O}_V)\}$$

plays this rôle. Since we are using an algebraically closed field, μ_n is (non-canonically) isomorphic to the constant sheaf $\mathbb{Z}/n$, however, the automorphisms of k will act differently on $\mathbb{Z}/n$ and μ_n.

3. Let us also point out that calculating the above tables is straightforward using very basic properties of singular cohomology (see the footnote). On the other hand, the calculation of étale cohomology of curves, even over an algebraically closed field, uses some serious algebraic results.

   (a) *Commutative algebra.* Proposition 2.3 describes divisors on smooth algebras.

   (b) *Field theory.* Theorem 2.5 (Hilbert's Theorem 90 and Tsen's theorem) describes vanishing of Galois cohomology of certain fields.

   (c) *Algebraic geometry.* Theorem 3.4 describes torsion and cotorsion in abelian varieties, and hence in the Picard group.

[1] The first table can be calculated easily using Mayer-Vietoris sequences, and cohomology groups of spheres, and homotopy invariance, the second table is calculated easily using the closed / open complement long exact sequence for cohomology with compact support.

2 $\mathbb{G}_m$-coefficients

Recall that $\mathbb{G}_m$ is the sheaf

$$\mathbb{G}_m : V \mapsto \Gamma(V, \mathcal{O}_V)^* \cong \hom(V, \operatorname{Spec} \mathbb{Z}[t, t^{-1}]).$$

To calculate the cohomology of $\mathbb{G}_m$ we also use the étale sheaf which sends $V \in \mathrm{Et}_{/X}$ to

$$\mathrm{Div}(V) = \oplus_{v \in V^{(1)}} \mathbb{Z}.$$

Here $V^{(1)}$ is the set of points of V of codimension one. Equivalently, one can define

$$\mathrm{Div} = \left(\oplus_{x \in X^{(1)}} i_{x*}\mathbb{Z}\right)(V)$$

where $i_x : x \to X$ is the inclusion associated to $x \in X$ and $\mathbb{Z}$ is the constant sheaf associated to $\mathbb{Z}$ on $\mathrm{Shv}_{\mathrm{et}}(k(x), \mathrm{Ab})$.

Exercise 2.1. Prove the isomorphism $\Gamma(V, \mathcal{O}_V)^* \cong \hom_{\mathsf{Sch}}(V, \operatorname{Spec} \mathbb{Z}[t, t^{-1}])$ in the case V is an affine scheme.

Exercise 2.2. Using the fact that étale morphisms preserve codimension of points, prove the isomorphism $\oplus_{V^{(1)}} \mathbb{Z} \cong (\oplus_{X^{(1)}} i_{x*}\mathbb{Z})(V)$

We will use the following commutative algebra results to calculate the cohomology of $\mathbb{G}_m$.

Proposition 2.3 ([Stacks project, 034P, 0AFT, 0AG0, 00TT)**.** *Suppose that A is a smooth local k-algebra. Then A is a unique factorisation domain. In particular:*

1. *all height 1 prime ideals are principal, and*
2. *every element $a \in A$ factors into the product*

   $$a = u p_1^{e_1} \dots p_n^{e_n}$$

   where u is a unit, the principal ideals $\langle p_i \rangle$ are distinct prime ideals of height one, and $e_1, \dots, e_n > 0$.

The decomposition in 2. is unique in the sense that if $a = v q_1^{f_1} \dots q_m^{f_m}$ is a second factorisation, then there is an isomorphism $\sigma : \{1, \dots, n\} \xrightarrow{\sim} \{1, \dots, m\}$ such that $f_{\sigma(i)} = e_i$ and $\langle q_{\sigma(i)} \rangle = \langle p_i \rangle$.

Proposition 2.4 (Milne, Exam.II.3.9)**.** *For any curve X, there is an exact sequence of sheaves on X_{et},*

$$0 \to \mathbb{G}_m \to g_*\mathbb{G}_{m,K} \to \mathrm{Div} \to 0,$$

where $g : \eta \to X$ is the inclusion of the generic point. Here we write $\mathbb{G}_{m,K} := \mathbb{G}_m \in \mathrm{Shv}_{\mathrm{et}}(K)$ for $\mathbb{G}_m$ in $\mathrm{Shv}_{\mathrm{et}}(K)$ to distinguish it from the $\mathbb{G}_m \in \mathrm{Shv}_{\mathrm{et}}(X)$.

Proof. To show the sequence is exact, consider a local ring $A = \mathcal{O}_{Y,y}$ of some $Y \in \mathrm{Et}_{/X}$. By Proposition 2.3 there is a canonical isomorphism of abelian groups

$$\mathrm{Frac}(A)^*/A^* \cong \bigoplus_{\substack{\mathfrak{p}\in\mathrm{Spec}(A)\\ \mathrm{height}(\mathfrak{p})=1}} \mathbb{Z}.$$

Note that the right hand side is exactly $\mathrm{Div}(A)$. So the sequence of abelian groups

$$0 \to A^* \to \mathrm{Frac}(A)^* \to \mathrm{Div}(A) \to 0$$

is exact. This means the sequence in the statement is exact as a sequence of Zariski sheaves, and therefore as a sequence of étale sheaves since sheafification $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Et}_{/X}) \to \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/X})$ is exact. □

We also use the following two field theory results.

Proposition 2.5.

1. Hilber's Theorem 90. *For any scheme X, we have $H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*) \cong H^1_{\mathrm{et}}(X, \mathcal{O}_X^*)$. In particular, if K is any field then*

$$H^1_{\mathrm{et}}(K, \mathbb{G}_m) = 0.$$

2. Tsen's theorem *[Stacks project, 03RD, 03R8], Let X be a smooth curve over an algebraically closed field k with fraction field $k(X)$. Then*

$$H^r_{et}(k(X), \mathbb{G}_m) = 0, \qquad r > 1.$$

Theorem 2.6 (Milne, III.2.22(d), III.4.9). *Let U be a curve. Then $R\Gamma(U, \mathbb{G}_m)$ is quasi-isomorphic to the complex*

$$[\cdots \to 0 \to k(U)^* \stackrel{val}{\to} \bigoplus_{u\in U^{(1)}} \mathbb{Z} \to 0 \to \dots]$$

concentrated in degrees 0 and 1. In particular,

$$H^r_{\mathrm{et}}(U, \mathbb{G}_m) = \begin{cases} \Gamma(U, \mathcal{O}_U)^*, & r = 0 \\ \mathrm{Pic}(U), & r = 1 \\ 0, & r > 1. \end{cases}$$

Here, the Picard group $\mathrm{Pic}(U)$ can be defined by the exact sequence

$$k(U)^* \to \oplus_{x\in U}\mathbb{Z} \to \mathrm{Pic}(U) \to 0. \tag{1}$$

Proof. Consider the long exact sequence associated to the divisor sequence of Proposition 2.4

$$\begin{aligned} 0 &\to H^0_{\mathrm{et}}(U, \mathbb{G}_m) \to H^0_{\mathrm{et}}(U, g_*\mathbb{G}_{m,K}) \to H^0_{\mathrm{et}}(U, \mathrm{Div}) \to \\ &\to H^1_{\mathrm{et}}(U, \mathbb{G}_m) \to H^1_{\mathrm{et}}(U, g_*\mathbb{G}_{m,K}) \to H^1_{\mathrm{et}}(U, \mathrm{Div}) \to \\ &\to H^2_{\mathrm{et}}(U, \mathbb{G}_m) \to H^2_{\mathrm{et}}(U, g_*\mathbb{G}_{m,K}) \to H^2_{\mathrm{et}}(U, \mathrm{Div}) \to \dots. \end{aligned}$$

We always have $R^0 F = F$, so the top row is

$$0 \to \Gamma(U, \mathcal{O}_U)^* \to k(U)^* \to \underset{x \in U^{(1)}}{\oplus} \mathbb{Z} \to$$

So to finish the proof, it suffices to show that

$$H^r_{\text{et}}(U, g_* \mathbb{G}_{m,K}) = 0, \qquad \text{and} \qquad H^r_{\text{et}}(U, \text{Div}) = 0, \qquad \text{for all } r \geq 1.$$

The latter is easy. Since U is a curve, all codimension one points are closed. Moreover, since k is algebraically closed, the are all isomorphic to $\text{Spec}(k)$. By $\text{Div} = \oplus i_{u*}\mathbb{Z}$, for $r > 0$ we have

$$H^r_{\text{et}}(U, \text{Div}) = H^r_{\text{et}}(U, \oplus i_{u*}\mathbb{Z}) \overset{Ex.2.7}{\cong} \oplus H^r_{\text{et}}(U, i_{u*}\mathbb{Z}) \overset{Ex.2.8}{\cong} \oplus H^r_{\text{et}}(\text{Spec}(k), \mathbb{Z}) \overset{Ex.2.9}{\cong} 0.$$

Showing $H^r_{\text{et}}(U, g_* \mathbb{G}_{m,K}) = 0$ is harder. This is the content of Proposition 2.5. □

Exercise 2.7. Show $H^n_{\text{et}}(X, \oplus_{i \in I} F_i) \cong \oplus_{i \in I} H^n_{\text{et}}(X, F_i)$. Hint.[2] Hint.[3]

Exercise 2.8 (Harder)**.** Show that if $i : Z \to X$ is a closed immersion then $H^n_{\text{et}}(X, i_* F) \cong H^n_{\text{et}}(Z, F)$. Hint.[4]

Exercise 2.9. Show that since k is algebraically closed, $H^n_{\text{et}}(\text{Spec}(k), F) = 0$ for any $F \in \text{Shv}_{\text{et}}(\text{Spec}(k), \text{Ab})$, and all $n > 0$.

3 μ_n-coefficients

Recall that μ_n is the sheaf

$$\mu_n : V \mapsto \{a \in \Gamma(V, \mathcal{O}_V)^* : a^n = 1\} \cong \hom(V, \text{Spec}\, \mathbb{Z}[t]/(t^n - 1)).$$

Exercise 3.1. Prove the isomorphism above in the case that V is affine.

Exercise 3.2 (Milne, Pg.125)**.** Using the fact that $n : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]; t \mapsto t^n$ is an étale morphism, prove that the sequence of étale sheaves

$$1 \to \mu_n \to \mathbb{G}_m \overset{n}{\to} \mathbb{G}_m \to 1 \tag{2}$$

is exact (now n is the morphism $\Gamma(V, \mathcal{O}_V)^* \to \Gamma(V, \mathcal{O}_V)^*; a \mapsto a^n$, cf. Exercise 2.1).

[2] Show that $\Gamma(X, \oplus_{i \in I} F_i) = \oplus_{i \in I} \Gamma(X, F_i)$.

[3] Show that for a Grothendieck abelian category $\mathcal{A}$ the functors $D(\mathcal{A}) \to K(\mathcal{A})$ and $K(\mathcal{A}) \to D(\mathcal{A})$ preserves small sums.

[4] Using the fact that i_* is exact and has an exact left adjoint i^*, show that there are commutative squares

$$\begin{array}{ccc} K(\text{Shv}_{\text{et}}(Z, \text{Ab})) & \longrightarrow & K(\text{Shv}_{\text{et}}(X, \text{Ab})) \\ \downarrow & & \downarrow \\ D(\text{Shv}_{\text{et}}(Z, \text{Ab})) & \longrightarrow & D(\text{Shv}_{\text{et}}(X, \text{Ab})) \end{array} \qquad \begin{array}{ccc} K(\text{Shv}_{\text{et}}(Z, \text{Ab})) & \longrightarrow & K(\text{Shv}_{\text{et}}(X, \text{Ab})) \\ \uparrow & & \uparrow \\ D(\text{Shv}_{\text{et}}(Z, \text{Ab})) & \longrightarrow & K(\text{Shv}_{\text{et}}(X, \text{Ab})) \end{array}$$

Definition 3.3. *The exact sequence Eq.*(2) *is called the* Kummer sequence*.*

For Proposition 3.5 we need some input from the theory of abelian varieties. An *abelian variety* over our algebraically closed field k is a smooth projective k-variety A equipped with morphisms $\mu : A \times A \to A$, $\epsilon : \mathrm{Spec}(k) \to A$, and $\iota : A \to A$, satisfying the axioms of an abelian group. Sometimes $\mu(a, b)$ is written $a + b$.

For a smooth projective curve X, the map $\mathrm{Div} \to \mathbb{Z}$; $\sum n_x x \mapsto \sum n_x$ induces a group homomorphism

$$\mathrm{Pic}(X) \overset{\deg}{\to} \mathbb{Z}$$

We define $\mathrm{Pic}^0(X) = \ker(\deg)$. It turns out that $\mathrm{Pic}^0(X)$ has the structure of an abelian variety.

Theorem 3.4. *Suppose X is a smooth projective curve.*

1. *There exists an abelian variety A and an isomorphism of abelian groups*

$$\hom(\mathrm{Spec}(k), A) \cong \mathrm{Pic}^0(X).$$

2. *For any abelian variety A and n coprime to char(k), the multiplication by n map $A \to A$; $a \mapsto a + \cdots + a$ is surjective with kernel isomorphic to $(\mathbb{Z}/n)^{2 \dim A}$.*

Proposition 3.5. *Let X be a projective curve of genus g and $n \neq$ char.(k). Then*

$$H^r_{\mathrm{et}}(X, \mu_n) = \begin{cases} \mu_n(k), & r = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g}, & r = 1 \\ \mathbb{Z}/n\mathbb{Z}, & r = 2 \\ 0, & r > 2 \end{cases}$$

Any automorphism of X acts trivially on H^2 (but not necessarily trivially on H^0 or H^1).

Proof. Consider the long exact sequence associated to the Kummer sequence

$$\begin{aligned} 0 &\to H^0_{\mathrm{et}}(X, \mu_n) \to H^0_{\mathrm{et}}(X, \mathbb{G}_m) \to H^0_{\mathrm{et}}(X, \mathbb{G}_m) \to \\ &\to H^1_{\mathrm{et}}(X, \mu_n) \to H^1_{\mathrm{et}}(X, \mathbb{G}_m) \to H^1_{\mathrm{et}}(X, \mathbb{G}_m) \to \\ &\to H^2_{\mathrm{et}}(X, \mu_n) \to H^2_{\mathrm{et}}(X, \mathbb{G}_m) \to H^2_{\mathrm{et}}(X, \mathbb{G}_m) \to \dots . \end{aligned}$$

Since X is *projective* we have $\mathbb{G}_m(X) = k^*$. So then by Theorem 2.6, this long exact sequence becomes

$$\begin{aligned} 0 &\to H^0_{\mathrm{et}}(X, \mu_n) \to k^* \overset{(-)^n}{\to} k^* \to \\ &\to H^1_{\mathrm{et}}(X, \mu_n) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X) \to \\ &\to H^2_{\mathrm{et}}(X, \mu_n) \to 0 \to 0 \to \dots . \end{aligned}$$

We automatically have $H^0_{\text{et}}(X,\mu_n) = \mu_n(k)$. Since k is *algebraically closed*, the map $k^* \overset{(-)^n}{\to} k^*$ is surjective. So it remains only to show that

$$\ker(\operatorname{Pic}(X) \to \operatorname{Pic}(X)) \cong (\mathbb{Z}/n\mathbb{Z})^{2g} \tag{3}$$

$$\operatorname{coker}(\operatorname{Pic}(X) \to \operatorname{Pic}(X)) \cong (\mathbb{Z}/n\mathbb{Z}) \tag{4}$$

These follow from the theory of abelian varieties. As mentioned above, the group $\operatorname{Pic}(X)$ sits in a short exact sequence $0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \overset{\deg}{\to} \mathbb{Z} \to 0$, where deg is induced by the degree map $\operatorname{Div} \to \mathbb{Z}; \sum n_i x_i \mapsto \sum n_i$, and $\operatorname{Pic}^0(X)$ has the structure of an abelian variety of dimension g. Then we get Eq.(3) and Eq.(4) from Theorem 3.4. □

Remark 3.6. If $k = \mathbb{C}$, then $\operatorname{Pic}^0(X)$ can be identified with $\mathbb{C}^g/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^{2g}$ by integrating holomorphic differential forms around curves inside the Riemann surface $X(\mathbb{C})$.

4 Compact support (for curves)

Definition 4.1 (Milne, page 91, 93)**.** *Let U be a curve, and $j : U \to X$ its smooth compactification. That is, j is the unique dense open embedding into a smooth projective curve X.* Cohomology with compact support *of a sheaf $F \in \operatorname{Shv}_{\text{et}}(U)$ are the cohomology groups of $j_!F \in \operatorname{Shv}_{\text{et}}(X)$*

$$H^r_c(U,F) := H^r_{\text{et}}(X, j_!F).$$

Remark 4.2. Equivalently, we can define

$$R\Gamma_c(U,F) := \operatorname{Cone}(R\Gamma(X,F) \to R\Gamma(Z,i^*F))[-1]$$

and

$$H^r_c(U,F) = H^r(R\Gamma_c(U,F)).$$

where $X \setminus Z = U$.

Remark 4.3. In general, $H^r_{\text{et}}(X, j_!F) \neq R^r\Gamma(X, j_!-)$.

Exercise 4.4. Let $i : Z \to X$ be the closed complement to $j : U \to X$ in the definition of cohomology with compact support. Using the short exact sequence $0 \to j_!j^* \to \text{id} \to i_*i^* \to 0$ show that for any sheaf $F \in \operatorname{Shv}_{\text{et}}(X)$, there is a long exact sequence

$$\cdots \to H^r_c(U,j^*F) \to H^r_{\text{et}}(X,F) \to H^r_{\text{et}}(Z,i^*F) \to H^{r+1}_c(U,j^*F) \to \dots$$

Corollary 4.5. *Let U be a curve, $U \to X$ the smooth compactification, and $m = \#(X \setminus U)$. Choose an isomorphism $\mu_n \cong \mathbb{Z}/n$ (that is, choose a primitive nth root of unity in k^*). Then*

$$H^r_c(U,\mathbb{Z}/n) \cong \left\{ \begin{array}{c|ccc} r\backslash m & 0 & 1 & >1 \\ \hline 0 & \mathbb{Z}/n & 0 & 0 \\ 1 & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g+m-1} \\ 2 & \mathbb{Z}/n & \mathbb{Z}/n & \mathbb{Z}/n \\ >2 & 0 & 0 & 0 \end{array} \right.$$

Here g is the genus of the compactification, and these identifications depend on the isomorphism $\mathbb{Z}/n \cong \mu_n$.

Exercise 4.6. Prove Corollary 4.5 using Proposition 3.5, Exercise 4.4 and $Z \cong \amalg_{i=1}^N \operatorname{Spec}(k)$ (and that k is algebraically closed). Cf. Exercises 2.7, 2.8, 2.9.

Note: the groups $H^r_c(U, \mathbb{Z}/n)$ are all $\mathbb{Z}/n$-modules (since we can work in the category of sheaves of $\mathbb{Z}/n$-modules). Moreover, every free module is projective. Hence, any short exact sequence of the form $0 \to (\mathbb{Z}/n)^a \to H^r_c(U, \mathbb{Z}/n) \to (\mathbb{Z}/n)^b \to 0$ is split.

5 Poincaré duality for curves

Definition 5.1. *An étale sheaf $F \in \mathrm{Shv}_{\mathrm{et}}(X)$ is* locally constant *if there is some étale covering $\{U_i \to X\}_{i \in I}$ such that each $F|_{U_i}$ is a constant sheaf.*

Remark 5.2. Any finite étale morphism $Y \to X$ induces a locally constant sheaf $\hom_X(-, Y)$ with finite fibres. In fact, there is an equivalence of categories between finite étale morphisms to X, and locally constant sheaves with finite fibres, cf. Milne Prop.V.1.1

Theorem 5.3 (Poincaré Duality. Milne Thm.V.2.1)**.** *Let F be a locally constant sheaf of $\mathbb{Z}/n$-modules with finite fibres on a curve U. There is a canonical perfect pairing of finite groups*

$$H^r_c(U, F) \times H^{2-r}_{\mathrm{et}}(U, \check{F}(1)) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Here $\check{F}(1)$ is the sheaf $V \mapsto \hom_{\mathrm{Shv}_{\mathrm{et}}(V)}(F|_V, \mu_n|_V)$.

Remark 5.4. This pairing is canonically isomorphic to the pairing induced by composition in $D(\mathrm{Shv}_{\mathrm{et}}(X, \mathbb{Z}/n))$

$$\hom(X, j_! F[r]) \times \hom(j_! F[r], \mu_n[2]) \to \hom(X, \mu_n[2])$$

Unfortunately, we do not have time for the proof.

Corollary 5.5. *Let U be a curve, $U \to X$ the smooth compactification, and $m = \#(X \setminus U)$. Choose an isomorphism $\mu_n \cong \mathbb{Z}/n$ (that is, choose a primitive root of unity in k^*). Then*

$$H^r_{\mathrm{et}}(U, \mathbb{Z}/n) \cong \left\{ \begin{array}{c|ccc} r \backslash m & 0 & 1 & > 1 \\ \hline 0 & \mathbb{Z}/n & \mathbb{Z}/n & \mathbb{Z}/n \\ 1 & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g+m-1} \\ 2 & \mathbb{Z}/n & 0 & 0 \\ > 2 & 0 & 0 & 0 \end{array} \right.$$

Here g is the genus of the compactification g, and these identifications depend on the isomorphism $\mathbb{Z}/n \cong \mu_n$.

Exercise 5.6. Prove the corollary using Poincaré duality and Corollary 4.5.