(Pro)étale cohomology
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Spring Semester 2024

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Lecture 6: Functoriality

May 16th, 2024

In this lecture we show how the category $\mathrm{Shv}_{\mathrm{et}}(X)$ of étale sheaves on a scheme X can be reconstructed from $\mathrm{Shv}_{\mathrm{et}}(Z)$ sheaves on a closed subscheme $Z \subseteq X$ and $\mathrm{Shv}_{\mathrm{et}}(U)$ sheaves on its open complement $U = X - Z$ via a functor $i^*j_* : \mathrm{Shv}_{\mathrm{et}}(U) \to \mathrm{Shv}_{\mathrm{et}}(Z)$, see Theorem 4.1.

The material in this lecture works for quasi-compact quasi-separated schemes (e.g., Noetherian and separated), where we recall that étale morphisms are by definition locally of finite presentation (this means finite type if working only with Noetherian schemes).

However, the main application in the next lecture will be to curves, and in the second half of the course the main example will be things of dimension zero.

So feel free to assume all schemes are dimension ≤ 1.

References for this section include:
[Maclane, Categories for the working mathematician]
[Milne, Etale cohomology]
[Stacks Project]

1 Presheaf adjunctions

Definition 1.1. *Suppose that $\pi : C \to D$ is a functor. We denote the functor induced by composition as*

$$\begin{aligned} \pi_p : \mathrm{PSh}(D) &\to \mathrm{PSh}(C); \\ F &\mapsto F \circ \pi \end{aligned}$$

Remark 1.2. If $\pi : C \to D$ and $\theta : D \to E$ are two functors, we have

$$(\theta \circ \pi)_p = \pi_p \circ \theta_p.$$

Recall that given a functor $C \to D$ and an object $Y \in D$, the *comma category* $(Y \downarrow \pi)$ is the category whose objects are pairs $(X, \begin{smallmatrix} Y \\ \downarrow \\ \pi(X) \end{smallmatrix})$ consisting of an object X of C and a morphism $X \to \pi(Y)$ of D. Morphisms are those morphisms of C making a commutative triangle. That is

$$\hom((X, \begin{smallmatrix} Y \\ \downarrow \\ \pi(X) \end{smallmatrix}), (X', \begin{smallmatrix} Y' \\ \downarrow \\ \pi(X') \end{smallmatrix})) = \left\{ X \to X' \mid \begin{smallmatrix} & Y & \\ \swarrow & & \searrow \\ \pi(X) & \to & \pi(X') \end{smallmatrix} \text{ commutes} \right\}.$$

See [MacLane, Categories for the working mathematician, pp.45-47].

Definition 1.3. *Let $\pi : C \to D$ be a functor and suppose C is small. Give a presheaf $F \in \mathrm{PSh}(C)$ and $Y \in D$ define*

$$(\pi^p F)(Y) = \varinjlim_{Y \to \pi(X)} F(X)$$

where the colimit is over the comma category $(Y \downarrow \pi)$.

Remark 1.4. There is also a right adjoint to π_p defined in an analogous way, but we will not use it.

Exercise 1.5. Using the universal property of the colimit, show that a morphism $Y \to Y'$ in D induces a morphism $(\pi^p F)(Y') \to (\pi^p F)(Y)$, and that this makes $\pi^p F$ into a presheaf on C. Hint.[1]

Show that a morphism of presheaves $F \to G$ induces a morphism $\pi^p F \to \pi^p G$, and that this makes π^p into a functor

$$\pi^p : \mathrm{PSh}(C) \to \mathrm{PSh}(D).$$

Exercise 1.6. Show that π^p is the unique colimit preserving functor making following square commutative

$$\begin{tikzcd} C \arrow[r, "\pi"] \arrow[d, "h_-"'] & D \arrow[d, "h_-"] \\ \mathrm{PSh}(C) \arrow[r, "\pi^p"'] & \mathrm{PSh}(D) \end{tikzcd}$$

Here h is the Yoneda functor $X \mapsto \hom(-, X)$. So in particular, show that

$$\pi^p h_X = h_{\pi X}$$

for any $X \in C$.

Exercise 1.7 (Harder)**.** Let $G \in \mathrm{PSh}(D)$ be an arbitrary presheaf.

1. Show that there is canonical isomorphism $\hom(\pi^p h_Y, G) \cong \hom(h_Y, \pi_p G)$. Here $h_Y = \hom(-, Y)$. Hint.[2]

2. Show that any presheaf $F \in \mathrm{PSh}(C)$ can be written as a colimit of representable presheaves. Hint.[3]

3. Deduce that for any $F \in \mathrm{PSh}(C)$ (not necessarily representable) there is a canonical isomorphism

$$\hom_{\mathrm{PSh}(D)}(\pi^p F, G) \cong \hom_{\mathrm{PSh}(C)}(F, \pi_p G).$$

[1]Note first that there is a canonical functor $(Y' \downarrow \pi) \to (Y \downarrow \pi)$.

[2]Use Exercise 1.6 and note that the right side of the isomorphism is $(\pi_p G)(Y)$.

[3]Consider the comma category $(h \downarrow F)$ where $h : C \to \mathrm{PSh}(C); X \mapsto \hom(-, X)$ is Yoneda.

Corollary 1.8. *The pair* (π^p, π_p) *is an adjunction* $\mathrm{PSh}(D) \rightleftarrows \mathrm{PSh}(C)$.

Exercise 1.9. Show that for any two composable functors $C \xrightarrow{\pi} D \xrightarrow{\theta} E$ we have

$$\theta^p \circ \pi^p = (\theta \circ \pi)^p.$$

Hint.[4]

Example 1.10. Suppose that $f : Y \to X$ is a morphism of topological spaces, and let $\pi : \mathrm{Op}(X) \to \mathrm{Op}(Y); U \mapsto f^{-1}U$ be the induced functor between the categories of open subsets of X, Y. Then π_p is the usual push-forward $\mathrm{PSh}(Y) \to \mathrm{PSh}(X)$ and π^p is the usual inverse image of presheaves functor $\mathrm{PSh}(X) \to \mathrm{PSh}(Y)$.

Example 1.11. If C is any small category and $\pi : C \to *$ the unique morphism towards the category $*$ with a unique object, and a unique morphism (i.e., id), then via the identifications $\mathrm{PSh}(*) \cong \mathrm{Set}$ and $\mathrm{PSh}(C) \cong \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set})$, the functor π^p is identified with the functor

$$\begin{aligned} \varinjlim : \mathrm{Fun}(C^{\mathrm{op}}, \mathrm{Set}) &\to \mathrm{Set} \\ F &\mapsto \varinjlim_{c \in C} F(c) \end{aligned}$$

Exercise 1.12. Suppose that C is a category with fibre products, e.g., $C = \mathrm{Et}_{/X}$ for some scheme X.

1. Show that for every object $V \in C$ there is an adjunction

$$\gamma : C_{/V} \rightleftarrows C : \pi$$

2. Show that

$$\gamma_p \cong \pi^p$$

   Hint.[5] Deduce that in this case we have

$$\pi^p h_Y = h_{W \times Y}.$$

Remark 1.13. In the situation of Exercise 1.12 we have three functors, each one left adjoint to the one on its right

$$\gamma^p \vdash \gamma_p \cong \pi^p \vdash \pi_p.$$

Exercise 1.14 (Base change)**.** Suppose that $V \to X$ is an étale morphism of schemes, $Y \to X$ any other morphism of schemes, and consider the cartesian square

$$\begin{tikzcd} Y \times_X V \arrow[r, "b"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ Y \arrow[r, "a"'] & X \end{tikzcd}$$

[4] Use Exercise 1.2, Corollary 1.8, and uniqueness of adjoints.
[5] Use uniqueness of adjoints.

1. Show that the square

$$\begin{tikzcd} \mathrm{Et}_{/Y\times_X V} \arrow[d, "\gamma"'] & \mathrm{Et}_{/V} \arrow[l, "\pi"'] \arrow[d, "\gamma"] \\ \mathrm{Et}_{/Y} & \mathrm{Et}_{/X} \arrow[l, "\pi"] \end{tikzcd}$$

   is commutative (up to natural isomorphism).

2. Deduce that the two squares

$$\begin{tikzcd} \mathrm{PSh}(\mathrm{Et}_{/Y\times_X V}) \arrow[r, "\pi_p"] & \mathrm{PSh}(\mathrm{Et}_{/V}) \\ \mathrm{PSh}(\mathrm{Et}_{/Y}) \arrow[u, "\gamma_p"] \arrow[r, "\pi_p"'] & \mathrm{PSh}(\mathrm{Et}_{/X}) \arrow[u, "\gamma_p"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{PSh}(\mathrm{Et}_{/Y\times_X V}) \arrow[d, "\gamma^p"] & \mathrm{PSh}(\mathrm{Et}_{/V}) \arrow[l, "\pi^p"'] \arrow[d, "\gamma^p"] \\ \mathrm{PSh}(\mathrm{Et}_{/Y}) & \mathrm{PSh}(\mathrm{Et}_{/X}) \arrow[l, "\pi^p"] \end{tikzcd}$$

   are commutative (up to natural isomorphism).

2 Sheaf adjunctions

Definition 2.1. *Suppose that C, D are sites, i.e., categories equipped with Grothendieck topologies. A functor $\pi : C \to D$ is called* continuous *if for every sheaf F on D, the presheaf $\pi_p F$ is a sheaf on C.*

Exercise 2.2. Suppose $\pi : C \to D$ sends fibre products to fibre products. Show that π is continuous if it sends covers to covers.

Example 2.3. If $Y \to X$ is a morphism topological spaces then the induced morphism of sites $\mathrm{Op}(X) \to \mathrm{Op}(Y)$ is continuous.

Example 2.4. If $f : Y \to X$ is a morphism of schemes, then

$$\pi : \mathrm{Et}_X \to \mathrm{Et}_Y; \qquad W \mapsto Y \times_X W$$

is continuous. If f is an étale morphism of schemes then

$$\gamma : \mathrm{Et}_Y \to \mathrm{Et}_X; \qquad (W \to Y) \mapsto (W \to Y \to X)$$

is also continuous.

Definition 2.5. *Suppose $\pi : C \to D$ is a continuous functor between sites. The induced functor on sheaves is denoted*

$$\pi_* : \mathrm{Shv}(D) \to \mathrm{Shv}(C).$$

The composition of π^p with sheafification $L : \mathrm{PSh}(D) \to \mathrm{Shv}(D)$ and the inclusion $\iota : \mathrm{Shv}(C) \to \mathrm{PSh}(C)$ is denoted

$$\pi^* = L \circ \pi^p \circ \iota : \mathrm{Shv}(C) \to \mathrm{Shv}(D).$$

Exercise 2.6. Suppose we are in the situation of Definition 2.5. Show that

$$\pi^* : \mathrm{Shv}(C) \rightleftarrows \mathrm{Shv}(D) : \pi_*$$

is an adjunction. Hint.[6]

Exercise 2.7. Show that if $C \xrightarrow{\pi} D \xrightarrow{\theta} E$ are continuous functors between sites then $(\theta \circ \pi)_* = \pi_* \circ \theta_*$ and $\theta^* \circ \pi^* = (\theta \circ \pi)^*$.

Definition 2.8. *If $f : Y \to X$ is a morphism of schemes, we write*

$$f^* := \pi^*, \qquad f_* := \pi_*$$

where π is $\mathrm{Et}_{/X} \to \mathrm{Et}_{/Y}; W \mapsto Y \times_X W$. If f is étale, so π has a left adjoint $\gamma : (W \to Y) \mapsto (W \to Y \to X)$ then we write

$$f_! := \gamma^*$$

Note that since $\gamma_* = \pi^*$, we get three functors, each left adjoint to the one on its right

$$f_! \vdash f^* \vdash f_*.$$

Lemma 2.9. *Let $f : Y \to X$ be a morphism of schemes. Then f^* preserves small colimits and finite limits of diagrams of sheaves. In particular, it preserves exact sequences of sheaves of abelian groups.*

Proof. The functor f^* automatically preserves colimits because it is a left adjoint. To show that it preserves finite limits it is sufficient to show that ι, a, and π^p preserve finite limits.

ι. The functor ι is a right adjoint, so preserves all small limits.

L. The sheafification functor L preserves finite limits by its construction.

π^p. The functor π^p doesn't necessarily preserve finite limits in general, cf.Example 1.11, however when π is $W \mapsto Y \times_X W$, the under categories $(V \downarrow \pi)$ admit finite limits. That is, they are filtered. Consequently, the colimits we used to define π^p are filtered colimits in this case, and filtered colimits commute with finite limits. □

Exercise 2.10. Let $f : Y \to X$ and $X' \to X$ be morphisms of schemes. Show that we have

$$f^* h_{X'} = h_{Y \times_X X'}.$$

Hint.[7]

If f is étale, and $V \to Y$ an étale morphism, show that we have

$$f_! h_V = h_V.$$

[6] Note that $\mathrm{Shv} \to \mathrm{PSh}$ is fully faithful.

[7] Use adjunction and Yoneda.

Exercise 2.11 (Base change)**.** Suppose that $V \to X$ is an étale morphism of schemes, $Y \to X$ any other morphism of schemes, and consider the cartesian square

$$\begin{tikzcd} Y \times_X V \arrow[r, "b"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ Y \arrow[r, "a"'] & X \end{tikzcd}$$

Show that the two squares

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/Y\times_X V}) \arrow[r, "b_*"] & \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/V}) \\ \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/Y}) \arrow[r, "a_*"'] \arrow[u, "g^*"] & \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/X}) \arrow[u, "f^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/Y\times_X V}) \arrow[d, "g_!"] & \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/V}) \arrow[l, "b^*"'] \arrow[d, "f_!"] \\ \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/Y}) & \mathrm{Shv}_{\mathrm{et}}(\mathrm{Et}_{/X}) \arrow[l, "a^*"] \end{tikzcd}$$

are commutative (up to natural isomorphism). That is, we have

$$g_! b^* \cong a^* f_!, \qquad f^* a_* \cong b_* g^*.$$

Hint.[8]

3 Immersions

For ease of exposition, from this point on we work with sheaves of abelian groups. Everything below also has a version for sheaves of sets.

Proposition 3.1. *Suppose that $i : Z \to X$ is a closed immersion of schemes. Then the adjunction counit*

$$i^* i_* \to \mathrm{id}$$

is an isomorphism.

Proof. If suffices to prove that for every sheaf F and geometric point $\overline{x} \to Z$ we have $(i^* i_* F)_{\overline{x}} \cong F_{\overline{x}}$. Since[9] $(i^* -)_{\overline{x}} \to (-)_{\overline{x}}$ is an isomorphism, we need to show that $(i_* F)_{\overline{x}} \to F_{\overline{x}}$ is an isomorphism. Explicitly, this is the morphism

$$\varinjlim_{\overline{x}\to W\to X} F(Z\times_X W) \to \varinjlim_{\overline{x}\to V\to Z} F(V)$$

induced by the functor[10] $\Phi : (\overline{x} \downarrow \mathrm{Et}_{/X}) \to (\overline{x} \downarrow \mathrm{Et}_{/Z})$, $W \mapsto Z \times_X W$. More explicitly, it is the morphism

$$\varinjlim_{W\in(\overline{x}\downarrow \mathrm{Et}_{/X})} F(\Phi(W)) \to \varinjlim_{V\in(\overline{x}\downarrow \mathrm{Et}_{/Z})} F(V).$$

[8] Use Exercise 1.14.

[9] This is because $(-)_{\overline{x}} = \iota^*$ where $\iota : \overline{x} \to Z$ is the structural morphism.

[10] Here, $(\overline{x} \downarrow \mathrm{Et}_{/X})$ is the category of factorisations $\overline{x} \to V \to X$ with $V \in \mathrm{Et}_{/X}$ and similar for $(\overline{x} \downarrow \mathrm{Et}_{/Z})$.

So it suffices to show that Φ is cofinal. This follows from the following two geometric facts which we take as a black box.

1. For every factorisation $\overline{x} \to V \to Z$ with $V \in \mathrm{Et}_{/Z}$, there exists a factorisation $\overline{x} \to W \to X$ with $W \in \mathrm{Et}_{/X}$ such that $Z \times_X W \cong V$. [Stacks Project, 04FW].

2. For every $W, W' \in \mathrm{Et}_{/X}$ and morphism $f : Z \times_X W \to Z \times_X W'$ in $\mathrm{Et}_{/Z}$, there exist morphisms $W \overset{g}{\leftarrow} W'' \overset{h}{\to} W'$ in $Et_{/X}$ such that $Z \times_X g$ is an isomorphism and f is the morphism $(Z \times_X h) \circ (Z \times_X g)^{-1}$. [Stacks Project, 04FV].

$\square$

Exercise 3.2. Suppose that $i : Z \to X$ is a closed immersion and $j : U \to X$ the open complement. Show that a sheaf of abelian groups F is zero if and only if

$$j^*F = 0 \qquad \text{and} \qquad i^*F = 0.$$

Hint.[11]

Deduce that a morphism of sheaves $\phi : F \to G$ is an isomorphism, resp. monomorphism, resp. epimorphism,[12] if and only if $j^*\phi$ and $i^*\phi$ are isomorphisms, resp. monomorphisms, resp. epimorphisms. More over, a sequence

$$0 \to F \to G \to H \to 0$$

of sheaves of abelian groups is exact if and only if it is exact after applying j^* and i^*.

Exercise 3.3. Suppose that $i : Z \to X$ is a closed immersion and $j : U \to X$ the open complement. Show the following identities. Hint.[13] Hint.[14]

$$\begin{aligned} j^*j_* &= \mathrm{id}, & j^*j_! &= \mathrm{id}, \\ j^*i_* &= 0, & i^*j_! &= 0 \end{aligned}$$

Using this, show that for any sheaf $F \in \mathrm{Shv}_{\mathrm{et}}(X)$ the sequence

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0$$

is exact.

Exercise 3.4. Suppose that $i : Z \to X$ is a closed immersion and $j : U \to X$ is the open complement. Let $F \in \mathrm{Shv}_{\mathrm{et}}(X)$ be a sheaf.

1. Show that $j_!, j_*$, and i_* are fully faithful.

[11] Recall that a sheaf of abelian groups $F \in \mathrm{Shv}_{\mathrm{et}}(X)$ is zero if and only if the stalk $F_{\overline{x}}$ is zero for every geometric point $\overline{x} \to X$.

[12] By monomorphism, resp. epimorphism, we mean that $\ker \phi \cong 0$ resp. $\operatorname{coker} \phi \cong 0$.

[13] Use Exercise 2.11.

[14] Consider the cartesian squares for $U \times_X U \cong U$ and $Z \times_X U \cong \varnothing$.

2. Show that F is in the image of $j_!$ if and only if

$$F(V) = 0 \qquad \text{for all } V \notin \mathrm{Et}_U. \tag{1}$$

3. Show that F is in the image of j_* if and only if

$$F(V) \xrightarrow{\sim} F(V \times_X U) \qquad \text{for all } V \in \mathrm{Et}_X. \tag{2}$$

4. Show that F is in the image of i_* if and only if

$$F(V) = 0 \qquad \text{for all } V \in \mathrm{Et}_U. \tag{3}$$

4 The localistion sequences

To recap, associated to a closed immersion $i : Z \to X$ and its open complement $j : U \to X$ we now have five functors

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{et}}(Z) \arrow[r, "i_*"] & \mathrm{Shv}_{\mathrm{et}}(X) \arrow[l, bend right, "i^*"'] \arrow[r, "j^*"] & \mathrm{Shv}_{\mathrm{et}}(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

satisfying the identities

$$\begin{array}{ccc} j^*j_* = \mathrm{id}, & i_*i^* = \mathrm{id}, & j^*j_! = \mathrm{id}, \\ j^*i_* = 0, & & i^*j_! = 0 \end{array}$$

where the outer four identities follow directly from base change, i.e., they're formal, and the central one uses geometric facts about étale morphisms.

Using these five functors, we will describe $\mathrm{Shv}_{\mathrm{et}}(X)$ as the comma category

$$T(X) := \Big(\mathrm{Shv}_{\mathrm{et}}(Z) \downarrow (i^*j_* : \mathrm{Shv}_{\mathrm{et}}(U) \to \mathrm{Shv}_{\mathrm{et}}(Z))\Big).$$

Explicitly, $T(X)$ is the category whose objects are triples (F_1, F_2, ϕ) consisting of two objects $F_1 \in \mathrm{Shv}_{\mathrm{et}}(Z), F_2 \in \mathrm{Shv}_{\mathrm{et}}(U)$, and a morphism $\phi : F_1 \to i^*j_*F_2$. Morphisms $(F_1, F_2, \phi) \to (F_1', F_2', \phi')$ are pairs of morphisms $(F_1 \xrightarrow{\psi_1} F_1', F_2 \xrightarrow{\psi_2} F_2')$ such that the square commutes.

$$\begin{tikzcd} F_1 \arrow[r, "\phi"] \arrow[d, "\psi_1"'] & i^*j_*F_2 \arrow[d, "i^*j_*\psi_2"] \\ F_1' \arrow[r, "\phi'"'] & i^*j_*F_2' \end{tikzcd}$$

Theorem 4.1 (Milne Thm.II.3.10)**.** *The functor $t : \mathrm{Shv}_{\mathrm{et}}(X) \to T(X)$*

$$t : F \mapsto \Big(i^*F, \qquad j^*F, \qquad i^*(F \xrightarrow{\eta} j_*j^*F)\Big)$$

is an equivalence of categories. Here $\eta : \mathrm{id} \to j_*j^*$ is the adjunction unit.

Proof. Given a triple (F_1, F_2, ϕ) in $T(X)$ define

$$s(F_1, F_2, \phi) := \ker\left(i_*F_1 \oplus j_*F_2 \xrightarrow{i_*\phi \,+\, \eta} i_*i^*j_*F_2 \right).$$

Here, $\eta : \mathrm{id} \to i_*i^*$ is the unit of the adjunction (i^*, i_*). Notice that every morphism of $T(X)$ induces a morphism in $\mathrm{Shv}_{\mathrm{et}}(X)$ in a way that defines a functor

$$s : T(X) \to \mathrm{Shv}_{\mathrm{et}}(X).$$

So it suffices to check that $st \cong \mathrm{id}$ and $ts \cong \mathrm{id}$. Consider stF. By definition, this is

$$stF = \ker\left(i_*i^*F \oplus j_*j^*F \xrightarrow{i_*\phi \,+\, \eta} i_*i^*j_*j^*F \right).$$

This comes equipped with a canonical morphism $F \to stF$. This morphism is an isomorphism if and only if the sequence

$$0 \to F \to i_*i^*F \oplus j_*j^*F \xrightarrow{i_*\phi \,+\, \eta} i_*i^*j_*j^*F \tag{4}$$

is exact. By Exercise 3.2 it suffices to check exactness after applying j^* and i^*. After j^* we obtain

$$0 \to j^*F \to \underbrace{j^*i_*i^*F}_{\cong 0} \oplus \underbrace{j^*j_*j^*F}_{\cong j^*F} \to \underbrace{j^*i_*i^*j_*j^*F}_{\cong 0}$$

where the underbraces follows from Exercise 3.3. Applying i^* we obtain

$$0 \to i^*F \to \underbrace{i^*i_*i^*F}_{\cong i^*F} \oplus \underbrace{i^*j_*j^*F}_{\cong 0} \to \underbrace{i^*i_*i^*j_*j^*F}_{\cong 0}.$$

where the underbraces follows from Exercise 3.3 and Proposition 3.1. Hence, Eq.(4) is exact, so $F \xrightarrow{\sim} stF$.

Now consider $ts(F_1, F_2, \phi)$. We have

$$\begin{aligned} i^*s(F_1, F_2, \phi) &= i^* \ker\left(i_*F_1 \oplus j_*F_2 \longrightarrow i_*i^*j_*F_2 \right) \\ &= \ker\left(i^*i_*F_1 \oplus i^*j_*F_2 \longrightarrow i^*i_*i^*j_*F_2 \right) \\ &\overset{\text{Prop.3.1}}{=} \ker\left(F_1 \oplus i^*j_*F_2 \longrightarrow i^*j_*F_2 \right) \\ &= F_1 \end{aligned}$$

One similarly checks that $j^*s(F_1, F_2, \phi) \cong F_2$, and that the canonical morphism $i^*s(F_1, F_2, \phi) \to i^*j_*j^*s(F_1, F_2, \phi)$ is none-other-than ϕ, under these identifications. Hence, $ts(F_1, F_2, \phi) = (F_1, F_2, \phi)$. ∎

Remark 4.2. Under the identification $\mathrm{Shv}_{\mathrm{et}}(X) \cong T(X)$, the functors $j_!, j^*, j_*, i^*, i_*$ correspond to:

$$\begin{array}{ccc c rcl}
F_1 & \overset{i^*}{\mapsfrom} & (F_1, F_2, \phi) & \qquad & (0, F_2, 0) & \overset{j_!}{\mapsfrom} & F_2 \\
F_1 & \overset{i_*}{\mapsto} & (F_1, 0, 0) & \qquad & (F_1, F_2, \phi) & \overset{j^*}{\mapsto} & F_2 \\
 & & & \qquad & (i^* j_* F_2, F_2, \mathrm{id}) & \overset{j_*}{\mapsfrom} & F_2
\end{array}$$

Exercise 4.3 (Harder.)**.** Suppose that $i : Z \to X$ is a closed immersion and $j : U \to X$ the open complement.

1. Show that
$$i^! := i^* \ker(\mathrm{id} \to j_* j^*)$$
defines a right adjoint to the functor i_*.
2. Show the identities
$$i^! i_* = \mathrm{id}, \qquad i^! j_! = 0, \qquad i^! j_* = 0.$$
3. Show that for any sheaf $F \in \mathrm{Shv}_{\mathrm{et}}(X)$ the following sequence is exact.
$$0 \to i_* i^! F \to F \to j_* j^* F$$
4. Given an example of closed immersion $Z \to X$ and a sheaf F for which the cokernel $\mathrm{coker}(F \to j_* j^* F)$ is not zero.

5 Curves

Example 5.1 (Milne, Exam.II.3.12)**.** Let A be a discrete valuation ring (e.g., $\mathbb{C}[[z]], \mathbb{F}_p[[z]], \mathbb{Z}_p, \dots$). Let

$$K = \mathrm{Frac}(A), \qquad k = A/\mathfrak{m}$$

$$G_K = Gal(K^{sep}/K), \qquad G_k = Gal(k^{sep}/k)$$

Since A is a discrete valuation ring, $X = \mathrm{Spec}(A)$ has one open point, and one closed point. Let $U = \mathrm{Spec}(K), Z = \mathrm{Spec}(k)$ be the corresponding open and closed subschemes. Recall that the category of étale sheaves over a field is equivalent to the category of discrete Galois modules. That is, $\mathrm{Shv}_{et}(Z) \cong G_k$-mod and $\mathrm{Shv}_{et}(U) \cong G_K$-mod. We can give an analogous description of $\mathrm{Shv}_{et}(X)$ using a similar construction to $T(X)$. It suffices to work out what functor $G_K\text{-mod} \to G_k\text{-mod}$ corresponds to $i^* j_* : \mathrm{Shv}_{\mathrm{et}}(U) \to \mathrm{Shv}_{\mathrm{et}}(Z)$.

Let A^h be the henselisation of A, and A^{sh} a strict henselisation. Since K^{sep} is separable closed, there are factorisations $A \to A^h \to A^{sh} \to K^{sep}$ which are actually inclusions. The choice of A^{sh} and the inclusion define subgroups $I = \mathrm{Gal}(K^{sep}/\,\mathrm{Frac}(A^{sh}))$ and $D = \mathrm{Gal}(K^{sep}/\,\mathrm{Frac}(A^h))$, with $I \subseteq D \subseteq G_K$. The identifications $A^{sh}/\mathfrak{m}_A \cong k^{sep}$ and $A^h/\mathfrak{m}_A \cong k$ induce a group homomorphism

$D/I \to G_k$, and it turns out this is an isomorphism. In particular, given any G_K-module M, the subset M^I of I-invariant elements admits a canonical action of $G_k \cong D/I$. We claim that the functor $i^* j_* : \mathrm{Shv}_{\mathrm{et}}(U) \to \mathrm{Shv}_{\mathrm{et}}(Z)$ corresponds to the functor of I-invariants.

$$(-)^I : G_K\text{-mod} \to G_k\text{-mod}.$$

Hence, the category $\mathrm{Shv}_{\mathrm{et}}$ is equivalent to the category of triples (M_1, M_2, ϕ) where $M_1 \in G_k$-mod, $M_2 \in G_K$-mod, and $\phi : M_1 \to M_2$ is compatible with the actions of $G_k \cong D/I$ and G_K.

Example 5.2. Example 5.1 can be generalised to any normal curve, see Milne Exer.II.3.16 for details.