(Pro)étale cohomology
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Lecture 5: Homological Algebra II

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References.

1. Grothendieck "Sur quelques points d'algèbre homologique"
2. Weibel, "An introduction to homological algebra"
3. Beke, "Sheafifiable homotopy modeal categories"
4. The Stacks Project (online)

1 Motivation

Last week we defined the derived functors $R^n\Phi$ of a functor $\Phi : \mathscr{A} \to \mathscr{B}$ between Grothendieck abelian categories as $R^n\Phi(A) := H^n(\Phi(Q_A^\bullet))$ where $A \to Q_A^\bullet$ is any quasi-isomorphism towards a fibrant complex.

We also showed that there exists a functor $Q : Ch(\mathscr{A}) \to Ch(\mathscr{A})$ equipped with a natural isomorphism $\mathrm{id} \to Q$ such that each $C^\bullet \to QC^\bullet$ is a monomorphic quasi-isomorphism, and each $QC^\bullet$ is fibrant. This shows that the $R^n\Phi(A)$ exist, and are functorial in A.

It remains to show:

1. The $R^n\Phi$ are independent of the choice of Q.
2. The $R^\bullet\Phi$ send short exact sequences to long exact sequences.

One can show the above two points directly, but we will use the derived category in the second half of this course, so we develop it now. We start with the homotopy category.

2 The homotopy category

Consider the complex of abelian groups

$$\Delta^1 = [\cdots \to 0 \to \underbrace{\mathbb{Z}}_{-1} \stackrel{diag}{\to} \underbrace{\mathbb{Z} \oplus \mathbb{Z}}_{0} \to 0 \to 0 \to \dots]$$

where $diag(m) = (n, n)$. Note that the morphisms $\mathbb{Z} \rightrightarrows \mathbb{Z} \oplus \mathbb{Z}$; $n \mapsto (n, 0), (0, -n)$ and the projection $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$; $(n, m) \mapsto n - m$ define morphisms

$$\mathbb{Z} \underset{\iota_1}{\overset{\iota_0}{\rightrightarrows}} \Delta^1 \xrightarrow{\pi} \mathbb{Z}$$

such that $\pi \circ \iota_0 = \pi \circ \iota_1 = \mathrm{id}_{\mathbb{Z}}$.

Remark 2.1. This is an algebraic analogue of $\Delta^1_{\mathrm{top}} \cong \{t \in \mathbb{R} \mid 0 \leq t \leq 1\}$ with inclusions $0, 1 \in \Delta^1_{\mathrm{top}}$ and projection $\Delta^1_{\mathrm{top}} \to \{*\}$.

Definition 2.2. Suppose that $f, g : A^\bullet \rightrightarrows B^\bullet$ are two morphisms between complexes. A *chain homotopy* from f to g is a morphism $h : A^\bullet \otimes \Delta^1 \to B^\bullet$ such that $h \circ \iota_0 = f$ and $h \circ \iota_1 = g$. In this case we write $f \sim g$.

Exercise 2.3. Show that if f and g are homotopic then they induce the same morphism on cohomology $H^n(f) = H^n(g)$.

Exercise 2.4. A morphism $f : A^\bullet \to B^\bullet$ is a *homotopy equivalence* if there exists a morphism $g : B^\bullet \to A^\bullet$ such that fg and gf are both homotopic to the identity. Show that every homotopy equivalence is a quasi-isomorphism.

Exercise 2.5. Suppose that $f_1, f_2, f_3 : A^\bullet \Rrightarrow B^\bullet$ are three morphisms such that $f_1 \sim f_2$ and $f_2 \sim f_3$. Show that $f_1 \sim f_3$. Deduce that chain homotopy is an equivalence relation. Hint.[1]

Exercise 2.6. Suppose we have morphisms $A^\bullet \xrightarrow{e} B^\bullet \underset{g}{\overset{f}{\rightrightarrows}} C^\bullet \xrightarrow{h} D^\bullet$ such that $f \sim g$. Show that $fe \sim ge$ and $hf \sim hg$. Deduce that there exists a category

$$K(\mathscr{A})$$

whose objects are the same as $Ch(\mathscr{A})$ but whose morphisms are given by $\hom_{Ch(\mathscr{A})}$ modulo homotopy. That is,

$$\hom_{K(\mathscr{A})}(A^\bullet, B^\bullet) = \frac{\hom_{Ch(\mathscr{A})}(A^\bullet, B^\bullet)}{\{f \ : \ f \sim 0\}}.$$

Remark 2.7. If we are thinking of $Ch(\mathrm{Ab})$ as analogous to the category of topological spaces, the category homotopy category $K(\mathrm{Ab})$ is analogous to the topological homotopy category H_{top}, whose objects are topological spaces and morphisms are modulo homotopy.[2] We have the following analogies.

$Ch(\mathscr{A})$	Top
$K(\mathscr{A})$	H_{top}
quasi-isomorphisms	weak equivalences[3]
fibrant complexes	CW complexes
$A^\bullet \to QA^\bullet$	$\lvert Sing_* X\rvert \to X$

[1] Use a map of the form $\Delta^1 \to \Delta^1 \sqcup_{\mathbb{Z}} \Delta^1$.

[2] Two morphisms $f, g : X \to Y$ are homotopy if there is a morphism $h : X \times \Delta^1_{top} \to Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

3 The derived category

Definition 3.1. An object $Q^\bullet$ of $K(\mathscr{A})$ is called *q.i.-local* if for every quasi-isomorphism $A^\bullet \to B^\bullet$ the induced morphism

$$\hom(B^\bullet, Q^\bullet) \to \hom(A^\bullet, Q^\bullet) \tag{1}$$

is an isomorphism (hom's in $K(\mathscr{A})$). The full subcategory of q.i.-local objects is denoted

$$D(\mathscr{A}) \subseteq K(\mathscr{A}).$$

Exercise 3.2. Using Yoneda's Lemma, show that if $P^\bullet \to Q^\bullet$ is a quasi-isomorphism between $q.i.$-local objects then f is an isomorphism in $K(\mathscr{A})$.

Proposition 3.3. *Every fibrant complex is q.i.-local.*

Proof. It suffices to check on monomorphic quasi-isomorphisms. Suppose f is an arbitrary quasi-isomorphism. Factor it as $A^\bullet \to \mathrm{Cyl}(f) \to B^\bullet$. Here, $\mathrm{Cyl}(f)$ is the pushout in the square

$$\begin{tikzcd}
A^\bullet \arrow[r, "\iota_0"] \arrow[d] & A^\bullet \otimes \Delta^1 \arrow[d] \\
B^\bullet \arrow[r, "\sigma"'] & \mathrm{Cyl}(f).
\end{tikzcd}$$

Then the composition $A^\bullet \xrightarrow{\iota_1} A^\bullet \otimes \Delta^1 \to \mathrm{Cyl}(f)$ is a monomorphic quasi-isomorphism, and the morphism[4] $\mathrm{Cyl}(f) \to B^\bullet$ is a quasi-isomorphism admitting a section $\sigma : B^\bullet \to \mathrm{Cyl}(f)$, which is therefore a monomorphic quasi-isomorphism. So it suffices to show that $\hom(-, Q^\bullet)$ sends ι and σ to isomorphisms. That is, we can assume f is a monomorphism.

The case the f is a monomorphism. By the definition of fibrant the map Eq.(1) is surjective when $f : A^\bullet \to B^\bullet$ is a monomorphic quasi-isomorphism. Suppose that $g_0, g_1 : B^\bullet \rightrightarrows Q^\bullet$ are two morphisms which become homotopic on $A^\bullet$. That is, there exists a morphism $h : A^\bullet \otimes \Delta^1 \to Q^\bullet$ with $h\iota_\epsilon = g_\epsilon f$ for $\epsilon = 0, 1$. Consider the diagram:

$$\begin{tikzcd}
A^\bullet \oplus A^\bullet \arrow[r, "\iota_0+\iota_1"] \arrow[d, "f+f"'] & A^\bullet \otimes \Delta^1 \arrow[d, "f\otimes\mathrm{id}"] \arrow[dr, bend left, "h"] & \\
B^\bullet \oplus B^\bullet \arrow[r, "\iota_0+\iota_1"] \arrow[rr, bend right, "g_0+g_1"'] & B^\bullet \otimes \Delta^1 & Q^\bullet
\end{tikzcd}$$

Giving rise to the diagram

$$\begin{tikzcd}
(B^\bullet \oplus B^\bullet) \amalg_{A^\bullet \oplus A^\bullet} (A^\bullet \otimes \Delta^1) \arrow[r, "(**)"'] \arrow[rr, bend left, "(*)"] & B \otimes \Delta^1 \arrow[r, dashed] & Q^\bullet
\end{tikzcd}$$

[4] This is induced by the isomorphism $\mathrm{Cyl}(f) \sqcup_{A^\bullet \otimes \Delta^1} A^\bullet = (B^\bullet \sqcup_{A^\bullet} A^\bullet \otimes \Delta^1) \sqcup_{A^\bullet \otimes \Delta^1} A^\bullet = B^\bullet \sqcup_{A^\bullet} A^\bullet = B^\bullet$.

where $(*)$ comes from the outside square and $(**)$ from the inside square. Once one checks that $(**)$ is a monomorphic quasi-isomorphism, we get the desired factorisation by virtue of the fact that $Q^\bullet$ is fibrant. $\square$

Exercise 3.4 (Harder.). Prove the claim in the above proof that $(**)$ is a monomorphic quasi-isomorphism. Hint.[5] Hint.[6]

Corollary 3.5. *Any choice of fibrant replacement* $\mathrm{id} \to Q$ *induces a left adjoint*

$$L : K(\mathscr{A}) \to D(\mathscr{A})$$

to the canonical inclusion

$$K(\mathscr{A}) \supseteq D(\mathscr{A}) : \iota$$

Proof. Since every fibrant object is q.i.-local, the functor Q we developed last week induces a functor $L : K(\mathscr{A}) \to D(\mathscr{A})$ equipped with a natural transformation $\eta : \mathrm{id}_{K(\mathscr{A})} \to \iota L$ which is termwise a quasi-isomorphism. Hence, for any $C^\bullet \in K(\mathscr{A})$ and $q.i.$-local $D^\bullet \in D(\mathscr{A})$ we have $\hom_{K(\mathscr{A})}(\iota LC^\bullet, \iota D^\bullet) \cong \hom_{K(\mathscr{A})}(C^\bullet, \iota D^\bullet)$ by definition of $q.i.$-local. Since ι is a fully faithful inclusion, we also have $\hom(LC^\bullet, D^\bullet) = \hom(LC^\bullet, \iota D^\bullet)$. Hence,

$$\hom(LC^\bullet, D^\bullet) = \hom(\iota LC^\bullet, \iota D^\bullet) \xrightarrow{\sim} \hom(C^\bullet, \iota D^\bullet).$$

$\square$

Corollary 3.6. *The derived functors $R^n\Phi$ are independent of the choice of Q.*

Proof. By Corollary 3.5 the derived functors can be defined as

$$R^n\Phi(A) = H^n\Phi(\iota L(A)).$$

Since adjoints are unique up to unique isomorphism, this is independent of any particular construction of L. $\square$

4 Identifying $R^0\Phi$

Proposition 4.1. *Suppose that $A \in \mathscr{A}$ is an object thought of as a chain complex concentrated in degree zero. Then*

$$R^n\Phi(A) = 0$$

for all $n < 0$. If Φ preserves finite limits, then

$$R^0\Phi(A) \cong \Phi(A).$$

[5] For monomorphism, write down precisely the terms of $(B \oplus B) \amalg_{A \oplus A} (A \otimes \Delta^1)$.

[6] For quasi-isomorphism, first prove that $A \otimes \Delta^1 \to (B \oplus B) \amalg_{A \oplus A} (A \otimes \Delta^1)$ is a quasi-isomorphism.

Proof. Let $A \to Q^\bullet$ be a quasi-isomorphism with $Q^\bullet$ a $q.i.$-local object. Define $Q^\bullet_{\geq 0}$ as the complex

$$Q^\bullet_{\geq 0} := [\cdots \to 0 \to 0 \to Q^0/B^0 \to Q^1 \to Q^2 \to \dots].$$

Note that this is not necessarily $q.i.$-local anymore, however it is equipped with a morphism $Q^\bullet \to Q^\bullet_{\geq 0}$ which is a quasi-isomorphism since $A \to Q^\bullet$ is a quasi-isomorphism. By definition of $q.i.$-local, there exists a retraction $Q^\bullet \to Q^\bullet_{\geq 0} \dashrightarrow Q^\bullet$ in $Ch(\mathscr{A})$, and therefore in $K(\mathscr{A})$. Any functor preserves retractions so

$$\begin{aligned} \Phi Q^\bullet &\to \Phi(Q^\bullet_{\geq 0}) \\ H^n \Phi Q^\bullet &\to H^n \Phi(Q^\bullet_{\geq 0}) \end{aligned}$$

also admit retractions. Therefore they are monomorphisms. Since $\Phi(Q^\bullet_{\geq 0})$ is zero in degrees < 0 it follows that the $H^n\Phi(Q^\bullet) \subseteq H^n\Phi(Q^\bullet_{\geq 0})$ are zero in degrees < 0. If Φ preserves finite limits, then $\Phi(A) \to H^0\Phi(Q^\bullet_{\geq 0})$ is an isomorphism. This factors through $H^0\Phi(Q^\bullet) \subseteq H^0\Phi(Q^\bullet_{\geq 0})$ so $\Phi(A) \to H^0\Phi(Q^\bullet)$ is also an isomorphism. □

Exercise 4.2. Prove the claim in the above proof that $\Phi(A) \cong H^0\Phi(Q^\bullet_{\geq 0})$.

5 Exact sequences

Now we are finally in a position to attack the motivating question. Namely, to extend $0 \to \Phi A \to \Phi B \to \Phi C$ to a long exact sequence. We first develop some basic pieces of homological algebra.

Exercise 5.1 (The Five Lemma (Harder))**.** Suppose that we have a commutative diagram of abelian groups

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

such that the rows are exact and f_1, f_2, f_4, f_5 are isomorphisms. Show that f_3 is an isomorphism.

Recall that for complexes $A^\bullet, B^\bullet \in Ch(\mathscr{A})$ we defined the mapping complex $\mathrm{Map}(A^\bullet, B^\bullet) \in Ch(\mathrm{Ab})$ by

$$\mathrm{Map}(A^\bullet, B^\bullet)^n = \prod_{i\in\mathbb{Z}} \hom(A^i, B^{i-n})$$

with differential $f \mapsto df - (-1)^n fd$.

Exercise 5.2 (Harder)**.** Choose an object $A^\bullet$ and a morphism $f : B^\bullet \to C^\bullet$ of $Ch(\mathscr{A})$.

1. Show that $\mathrm{Map}(A^\bullet, \mathrm{Cone}(f)) \cong \mathrm{Cone}(\mathrm{Map}(A^\bullet, f))$ in $Ch(\mathrm{Ab})$.
2. Show that for any $i \in \mathbb{Z}$ we have $\hom_{K(\mathscr{A})}(A^\bullet, B^\bullet[i]) = H^{-i} \mathrm{Map}(A^\bullet, B^\bullet)$
3. Deduce that there is a long exact sequence

$$
\begin{aligned}
\cdots \to \hom(A^\bullet, B^\bullet[i]) \to \hom(A^\bullet, C^\bullet[i]) \\
\to \hom(A^\bullet, \mathrm{Cone}(f)[i]) \to \hom(A^\bullet, B^\bullet[i+1]) \to \dots
\end{aligned} \tag{2}
$$

   where the homs are in $K(\mathscr{A})$.

Definition 5.3. The *cofibre sequences* of $K(\mathscr{A})$ are those isomorphic (in $K(\mathscr{A})$) to the sequences

$$A^\bullet \xrightarrow{f} B^\bullet \to \mathrm{Cone}(f).$$

The cofibre sequences of $D(\mathscr{A}) \subseteq K(\mathscr{A})$ are those which are cofibre sequences in $K(\mathscr{A})$.

Exercise 5.4.

1. Using the long exact sequence Eq.(2) and the Five Lemma, show that if $B^\bullet$ and $C^\bullet$ are $q.i.$-local, then so is $\mathrm{Cone}(f)$.
2. Deduce that the canonical morphism $\mathrm{Cone}(Lf) \to L\,\mathrm{Cone}(Lf)$ is an isomorphism in $D(\mathscr{A})$.
3. Using Yoneda and the Five Lemma show that the canonical morphism $L\,\mathrm{Cone}(f) \to L\,\mathrm{Cone}(Lf)$ is also an isomorphism.
4. Deduce that $L : K(\mathscr{A}) \to D(\mathscr{A})$ sends cofibre sequence to cofibre sequences.

Proposition 5.5. *If $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathscr{A}$ and $\Phi : \mathscr{A} \to \mathscr{B}$ is an additive functor, then*

$$R\Phi A \to R\Phi B \to R\Phi C$$

is a cofibre sequence of $\mathscr{B}$. If Φ preserves finite limits, then we obtain a long exact sequence

$$
\begin{tikzcd}
 & R^2\Phi(A) \arrow[r] & R^2\Phi(B) \arrow[r] & R^2\Phi(C) \arrow[r] & \dots \\
 & R^1\Phi(A) \arrow[r] & R^1\Phi(B) \arrow[r] & R^1\Phi(C) \arrow[ull, rounded corners] & \\
0 \arrow[r] & \Phi(A) \arrow[r] & \Phi(B) \arrow[r] & \Phi(C) \arrow[ull, rounded corners] &
\end{tikzcd} \tag{3}
$$

Proof. By Exercise 5.4 the functor $L : K(\mathscr{A}) \to D(\mathscr{A})$ preserves cofibre sequences, by definition the inclusion $D(\mathscr{A}) \to K(\mathscr{A})$ preserves cofibre sequences, and by additivity $\Phi : K(\mathscr{A}) \to K(\mathscr{B})$ preserves cofibre sequences. Hence,

$$R\Phi := \Phi\iota L$$

preserves cofibre sequences, and therefore, sends $A \to B \to \mathrm{Cone}(A{\to}B)$ to a cofibre sequence. The functor L sends the quasi-isomorphism $\mathrm{Cone}(A{\to}B) \to C$ to an isomorphism in $D(\mathscr{A})$, so $A \to B \to C$ is also sent to a cofibre sequence by $R\Phi$.

We saw last week that cofibre sequences give rise to long exact sequences on cohomology, so the last thing to check is that $R^n\Phi(-) : \mathscr{A} \to \mathrm{Ab}$ is zero for $n < 0$ and Φ for $n = 0$. This is Proposition 4.1. $\square$

Remark 5.6. The derived category is not necessary to prove Proposition 5.5. However, we want to have access to the derived category in the second part of the course.

Here is an argument not using derived categories. Choose fibrant replacements $B \to Q_B^\bullet$ and $C \to Q_C^\bullet$ fitting into a commutative square and set $Q_A^\bullet := \mathrm{Cone}(Q_B^\bullet \to Q_C^\bullet)[-1]$. Since the compositions $A \to Q_C^0$ and $A \to Q_B^1$ are zero, there exists a factorisation $A \to Q_A^\bullet$. So we get the desired long exact sequence as soon as we know that $A \to Q_A^\bullet$ is a fibrant replacement. Proving that $A \to Q_A^\bullet$ is a quasi-isomorphism can be done using the Snake Lemma. Proving that $Q_A^\bullet$ is fibrant is harder. We include a proof at the end. See Proposition B.1.

A Universal properties

In this section we show that both $D(\mathscr{A})$ and $R\Phi$ satisfy universal properties. This material holds for more general localisations, so we develop it in this greater level of generalilty.

Exercise A.1. Suppose f is a morphism of $K(\mathscr{A})$. Show that $L(f)$ is an isomorphism if and only if f is a quasi-isomorphism. Hint.[7] Hint.[8]

Exercise A.2. Suppose that $\iota : \mathcal{D} \subseteq \mathcal{C}$ is a fully faithful functor admitting a left adjoint L (e.g., $\iota : D(\mathscr{A}) \subseteq K(\mathscr{A})$). Let

$$S = \{f : A \to B \in \mathcal{C} \mid L(f) \text{ is an isomorphism }\}.$$

Show that for any category $\mathcal{E}$, composition with L induces an equivalence of categories

$$L_* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \overset{\cong}{\to} \mathrm{Fun}^S(\mathcal{C}, \mathcal{E})$$
$$\Psi \mapsto \Psi \circ L$$

[7]For ($\Leftarrow$) use Yoneda.

[8]For ($\Rightarrow$) use the square $\begin{matrix} A^\bullet & \to & B^\bullet \\ \downarrow & & \downarrow \\ LA^\bullet & \to & LB^\bullet \end{matrix}$.

where $\mathrm{Fun}^S(\mathcal{C},\mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{E})$ is the fullsubcategory of those functors which send all elements of S to isomorphisms. Hint.[9]

Deduce that any functor $\Psi : K(\mathscr{A}) \to D(\mathscr{B})$ which sends quasi-isomorphisms to isomorphisms factors uniquely (up to unique natural isomorphism) through $D(\mathscr{A}) \to D(\mathscr{B})$.

Definition A.3. Suppose $L_* : \hat{\mathcal{D}} \to \hat{\mathcal{C}}$ is a functor and $c \in \hat{\mathcal{C}}$ an object. The *slice category*

$$(c \downarrow L_*)$$

has as objects pairs $(d, \overset{c}{\underset{L_*d}{\downarrow}})$ where $d \in \hat{\mathcal{D}}$ and $\overset{c}{\underset{L_*d}{\downarrow}} \in \hat{\mathcal{C}}$. Morphisms are

$$\hom((d, \overset{c}{\underset{L_*d}{\downarrow}}), (d, \overset{c}{\underset{L_*d'}{\downarrow}})) = \left\{ g \in \hom(d,d') \mid \begin{tikzcd}[column sep=small] & c \arrow[dl] \arrow[dr] & \\ L_*d \arrow[rr] & & L_*d' \end{tikzcd} \text{ commutes} \right\}.$$

Exercise A.4. Suppose that L_* is fully faithful and admits a left adjoint $\iota_* : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ (e.g., $L_* : \mathrm{Fun}(D(\mathscr{A}), D(\mathscr{B})) \to \mathrm{Fun}(K(\mathscr{A}), D(\mathscr{B}))$. Show that for any $c \in \hat{C}$, the pair $(L_*\iota_*c, \overset{c}{\underset{L_*\iota_*c}{\downarrow}})$ is an initial object of $(c \downarrow L_*)$.

Deduce that, given a fixed functor $\Phi : K(\mathscr{A}) \to D(\mathscr{B})$, for every functor $\Psi : D(\mathscr{A}) \to D(\mathscr{B})$ equipped with a natural transformation $\eta : \Phi \Rightarrow \Psi L$ there exists a unique natural transformation $R\Phi \Rightarrow \Psi$ admitting a factorisation $\Phi \Rightarrow R\Phi L \Rightarrow \Psi L$.

B Cone preserves fibrancy

Proposition B.1. *Suppose that $f : Q^\bullet \to P^\bullet$ is a morphism between fibrant complexes. Then $\mathrm{Cone}(f)[-1]$ is also fibrant.*

Proof. We have a pullback square of the form

$$\begin{tikzcd} \mathrm{Cone}(f)[-1] \arrow[r] \arrow[d] & P^\bullet \otimes D^0 \arrow[d, "pr."] \\ Q^\bullet \otimes S^0 \arrow[r] & P^\bullet \otimes S^0 \end{tikzcd}$$

Suppose that $A^\bullet \to B^\bullet$ is a monomorphic quasi-isomorphism and we have a morphism $A^\bullet \to \mathrm{Cone}(f)[-1]$ we would like to factor through $B^\bullet$. Since $Q^\bullet$ is fibrant, we can extend the above square to the diagram on the left below. Since the square above is a pullback square, it suffices to find the diagonal morphism

[9] Show that for each $A \in \mathcal{C}$, the natural transformation $A \to \iota LA$ is in S. For this, use Yoneda and the hypothesis that ι is fully faithful.

in the square below on the right.

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d] & \mathrm{Cone}(f)[-1] \arrow[r] \arrow[d] & P^\bullet \otimes D^0 \arrow[d, "pr."] \\
B^\bullet \arrow[r] & Q^\bullet \otimes S^0 \arrow[r] & P^\bullet \otimes S^0
\end{tikzcd}
\qquad
\begin{tikzcd}
A^\bullet \arrow[r, "\alpha"] \arrow[d, "\iota"'] & P^\bullet \otimes D^0 \arrow[d, "pr."] \\
B^\bullet \arrow[r, "\beta"'] \arrow[ur, dashed] & P^\bullet \otimes S^0 \; (\cong P^\bullet)
\end{tikzcd}$$

Now in general, there is a canonical bijection

$$\hom(X^\bullet, Y^\bullet \otimes D^0) \cong \hom(X^\bullet \otimes D^1, Y^\bullet)$$

natural in $X^\bullet$ and $Y^\bullet$. So the right hand square above is equivalent to the diagram below, with the two dashed morphisms correspond to each other.

$$\begin{tikzcd}
A^\bullet \arrow[r, "inc."] \arrow[d, "\iota"'] & A^\bullet \otimes D^1 \arrow[d] \arrow[dr, "\alpha'"] & \\
B^\bullet \arrow[r, "inc."] \arrow[rr, bend right, "\beta"'] & B^\bullet \otimes D^1 \arrow[r, dashed] & P^\bullet
\end{tikzcd}$$

Now we play the same game as in the proof of Proposition 3.3. Namely, we show that the canonical morphism from the pushout factors as below, using the fact that $P^\bullet$ is fibrant, and (*) is a monomorphic quasi-isomorphism.

$$\begin{tikzcd}
B^\bullet \amalg_{A^\bullet} (A^\bullet \otimes D^1) \arrow[r, "(*)"] \arrow[rr, bend right] & B^\bullet \otimes D^1 \arrow[r, dashed] & P^\bullet
\end{tikzcd}$$

$\square$