(Pro)étale cohomology
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Lecture 4: Homological Algebra I

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References.

1. Grothendieck "Sur quelques points d'algèbre homologique"
2. Weibel, "An introduction to homological algebra"
3. Beke, "Sheafifiable homotopy modeal categories"
4. The Stacks Project (online)

1 Motivation

In this lecture we work with a Grothendieck abelian category $\mathscr{A}$. We have in mind the following examples:

1. $\mathscr{A} = R$-mod, the category of R-modules for some ring R (for example, $R = \mathbb{Z}$ or a field K),
2. $\mathscr{A} = \mathrm{Shv}(X)$ sheaves of abelian groups on a topological space,
3. $\mathscr{A} = \mathrm{Shv}_{\mathrm{et}}(X)$, étale sheaves of abelian groups on a variety X,
4. $\mathscr{A} = G$-mod, the category of discrete G-modules for some profinite group[1] (such as $G = \mathrm{Gal}(k^{sep}/k)$ for some field k).

We are interested in functors $\Phi : \mathscr{A} \to \mathscr{B}$ which preserve limits. Such functors send exact sequences[2] of the form

$$0 \to A \to B \to C \to 0$$

to exact sequences of the form

$$0 \to \Phi(A) \to \Phi(B) \to \Phi(C). \tag{1}$$

Examples of such functors are:

[1] A group is *profinite* if it is of the form $\lim_{\lambda \in \Lambda} F_\lambda$ for some filtered category Λ and finite groups F_λ. The canonical example is $\mathbb{Z}_p = \lim(\dots \to \mathbb{Z}/p^3 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p)$. A module over a profinite group is *discrete* if all orbits are finite.

[2] Recall that a sequence $\dots \to K^{n-1} \stackrel{d^{n-1}}{\to} K^n \stackrel{d^n}{\to} K^{n+1} \to \dots$ is *exact* if for every n we have $\ker(d^n) = \mathrm{im}(d^{n-1})$.

1. If $\mathscr{A} = R$-mod, the functors $F(-) = \hom_R(M, -)$ for some fixed $M \in R$-mod.
2. If $\mathscr{A} = \mathrm{Shv}(X)$ the functor $\Phi(F) = F(X)$,
3. If $\mathscr{A} = \mathrm{Shv}_{et}(X)$ the functor $\Phi(F) = F(X)$,
4. If $\mathscr{A} = G$-mod, the functor $\Phi(M) = M^G = \{m \in M : gm = m \ \forall \ g \in G\}$.

Problem. Extend the exact sequence Eq.(1) to the right.

2 Chain complexes

Throughout this section $\mathscr{A}$ is always a Grothendieck abelian category. Note that there is a unique colimit preserving functor

$$- \otimes - : \mathscr{A} \times \mathrm{Ab} \to \mathscr{A}$$

such that $A \otimes \mathbb{Z} = A$ for each object A. So for example, $A \otimes (\bigoplus_{i \in I} \mathbb{Z}) = \bigoplus_{i \in I} A$, $A \otimes (\mathbb{Z}/n) = \mathrm{coker}(A \xrightarrow{n} A)$, and $A \otimes \mathbb{Q} = \varinjlim_{n \in \mathbb{N}} (A \xrightarrow{n} A \xrightarrow{n'} \dots)$.

Definition 2.1. A *chain complex* $C^\bullet$ is a sequence of morphisms

$$\dots \xrightarrow{d} C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \xrightarrow{d} \to \dots$$

in $\mathscr{A}$ such that $d \circ d = 0$.

A *morphism* of chain complexes $C^\bullet \to D^\bullet$ is a sequence of morphisms $f^n : C^n \to D^n$ such that all squares

$$\begin{tikzcd}
\dots \arrow[r] & C^n \arrow[r, "d"] \arrow[d, "f^n"'] & C^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \dots \\
\dots \arrow[r] & D^n \arrow[r, "d"] & D^{n+1} \arrow[r] & \dots
\end{tikzcd}$$

are commutative. The category of chain complexes is denoted $Ch(\mathscr{A})$.

In the case $\mathscr{A} = R$-mod we will often write $Ch(R) := Ch(R\text{-mod})$. Sometimes we think of $\mathscr{A}$ as a subcategory of $Ch(\mathscr{A})$ by identifying $A \in \mathscr{A}$ with

$$\dots \to \underbrace{0}_{-1} \to \underbrace{A}_{0} \to \underbrace{0}_{1} \to \dots$$

in $Ch(\mathscr{A})$.

Example 2.2.

1. The category $Ch(\mathscr{A})$ is again a Grothendieck abelian category. Kernels, cokernels, images, and exact sequences of chain complexes are calculated degreewise. That is, $(\varprojlim_\lambda C^\bullet_\lambda)^n = \varprojlim_\lambda C^n_\lambda$ resp. $(\varinjlim_\lambda C^\bullet_\lambda)^n = \varinjlim_\lambda C^n_\lambda$.

2. We have the canonical chain complexes

$$S^n = [\cdots \to \underbrace{0}_{-n-2} \to \underbrace{0}_{-n-1} \to \underbrace{\mathbb{Z}}_{-n} \to \underbrace{0}_{-n+1} \to \dots]$$

$$D^{n+1} = [\cdots \to \underbrace{0}_{-n-2} \to \underbrace{\mathbb{Z}}_{-n-1} \overset{=}{\to} \underbrace{\mathbb{Z}}_{-n} \to \underbrace{0}_{-n+1} \to \dots]$$

in $Ch(\mathbb{Z})$. There is a canonical short exact sequence

$$0 \to S^n \to D^{n+1} \to S^{n+1} \to 0.$$

3. We can extend $\otimes$ to a functor

$$- \otimes - : Ch(\mathscr{A}) \times Ch(\mathrm{Ab}) \to Ch(\mathscr{A})$$

by setting

$$(C^\bullet \otimes A^\bullet)^n = \bigoplus_{i+j=n} C^i \otimes A^j$$

the differentials $(C^\bullet \otimes A^\bullet)^n \to (C^\bullet \otimes A^\bullet)^{n+1}$ are sums of the morphisms

$$d \otimes \mathrm{id} + (-1)^i \, \mathrm{id} \otimes d : C^i \otimes A^j \to (C^{i+1} \otimes A^j) \oplus (C^i \otimes A^{j+1}).$$

4. Given any chain complex $C^\bullet$ and $i \in \mathbb{Z}$ the *shift* is

$$C^\bullet[i] = C^\bullet \otimes S^i.$$

5. Given any morphism of chain complexes $f : A^\bullet \to B^\bullet$ we can form the *cone* as the pushout

$$\begin{tikzcd}
A^\bullet = & A^\bullet \otimes S^0 \arrow[r] \arrow[d] & A^\bullet \otimes D^1 \arrow[d] \\
B^\bullet = & B^\bullet \otimes S^0 \arrow[r] & \mathrm{Cone}(f)
\end{tikzcd}$$

Note that $\mathrm{Cone}(A^\bullet \to 0) = A^\bullet[1]$.

6. Given a morphism $f : A^\bullet \to B^\bullet$ the *mapping cylinder* is defined as the pushout

$$\begin{tikzcd}
A^\bullet \arrow[r] \arrow[d] & A^\bullet \otimes \Delta^1 \arrow[d] \\
B^\bullet \arrow[r] & \mathrm{Cyl}(f)
\end{tikzcd}$$

where $\Delta^1 = \mathrm{Cone}(\mathbb{Z} \overset{diag.}{\to} \mathbb{Z} \oplus \mathbb{Z})$ and $A \to A \otimes \mathrm{Cyl}$ is inclusion to the left component.

7. For any two chain complexes $A^\bullet$, $B^\bullet$ we define

$$\mathrm{Map}(A^\bullet, B^\bullet)^n = \prod_{i\in\mathbb{Z}} \hom_{\mathscr{A}}(A^i, B^{i-n})$$

in $Ch(\mathrm{Ab})$. This is equipped with differentials

$$df = d \circ f - (-1)^n f \circ d.$$

Exercise 2.3.

1. Describe the terms and differentials of $C^\bullet[i]$ and $\mathrm{Cone}(f)$ explicitly.
2. Show that there is a canonical exact sequence

$$0 \to B^\bullet \to \mathrm{Cone}(f) \to A^\bullet[1] \to 0.$$

3. Using the canonical inclusion $D^1 \to \Delta^1$ and an appropriate pushout square, show that the canonical morphism $B^\bullet \to \mathrm{Cyl}(f)$ admits a retraction. Hint.[3]

Remark 2.4. The S and D in S^n and D^n are for *sphere* and *disc*. They are algebraic analogues of

$$S^n_{top} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1\}$$

$$D^{n+1}_{top} = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i^2 \leq 1\}.$$

The cone is an algebraic version of the topological cone construction which sends a continuous morphism of topological spaces $X \to Y$ to the topological space $Y \sqcup_{(X\times\{0\})} (X \times [0,1]) \sqcup_{X\times\{0\}} \{0\}$ where $[0,1] \subseteq \mathbb{R}$ is the unit interval.

$$- - - - \textit{picture} - - -$$

The exact sequence $0 \to S^n \to D^{n+1} \to S^{n+1} \to 0$ is the algebraic analogue of the homeomorphism $S^n_{top} \cong D^n_{top}/S^{n-1}_{top}$.

3 Quasi-isomorphisms

Definition 3.1. Let $C^\bullet$ be a chain complex. We define

$$\text{the group of } n\text{-cycles as} \qquad Z^n = \ker(C^n \to C^{n+1}),$$

$$\text{the group of } n\text{-boundaries as} \qquad B^n = \mathrm{im}(C^{n-1} \to C^n), \text{ and}$$

$$\text{the } n\text{-th cohomology group as} \qquad H^n = Z^n/B^n.$$

[3] First show that $\Delta^1 \cong D^1 \oplus S^0$.

Exercise 3.2. Show that $H^n \cong \operatorname{coker}\Big(\operatorname{Map}(D^{-n}, C^\bullet) \to \operatorname{Map}(S^{-n}, C^\bullet)\Big)$.

Exercise 3.3. Show that a morphism of chain complexes $C^\bullet \to D^\bullet$ induces a morphism on cohomology groups $H^n C \to H^n D$.

In a normal first course on homological algebra the following would be the first main theorem.

Exercise 3.4 (Snake Lemma. (Harder))**.** Show that for any exact sequence $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ in $Ch(\mathscr{A})$, there is an associated long exact sequence

$$\cdots \to H^n(A) \to H^n(B) \to H^n(C) \to H^{n+1}(A) \to \dots .$$

Deduce that for any morphism $A^\bullet \to B^\bullet$ there is a long exact sequence

$$\cdots \to H^n(A) \to H^n(B) \to H^n(\operatorname{Cone}(f)) \to H^{n+1}(A) \to \dots .$$

Exercise 3.5. Using three cocartesian squares and the Snake Lemma, show that $\operatorname{Cone}(B^\bullet \to \operatorname{Cone}(A^\bullet \to B^\bullet)) \to A^\bullet$ is a quasi-isomorphism. Hint.[4]

Exercise 3.6. Using the Snake Lemma, show that for morphism $f : A^\bullet \to B^\bullet$ the morphism $B^\bullet \to \operatorname{Cyl}(f)$ is a quasi-isomorphism. Hint.[5]

Definition 3.7. A morphism of chain complexes $f : C^\bullet \to D^\bullet$ is called a *quasi-isomorphism* if the induced maps $H^n f : H^n C \to H^n D$ are isomorphisms $\forall\ n$. Two chain complexes $C^\bullet$, $D^\bullet$ are said to be *quasi-isomorphic* if there exists a sequence of quasi-isomorphisms

$$\begin{tikzcd}
& {}^{(1)}C^\bullet \arrow[dl] \arrow[dr] & & {}^{(3)}C^\bullet \arrow[dl] \arrow[dr, dashed] & & & & {}^{(n-1)}C^\bullet \arrow[dl, dashed] \arrow[dr] & \\
C^\bullet = {}^{(0)}C^\bullet & & {}^{(2)}C^\bullet & & {} & \cdots & {} & & {}^{(n)}C^\bullet = D^\bullet
\end{tikzcd}$$

Example 3.8.

$$\begin{array}{c}
C^\bullet = (\cdots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \dots) \\
\downarrow \\
D^\bullet = (\cdots \to 0 \to 0 \to \mathbb{Z}/2 \to 0 \to \dots)
\end{array}$$

is a quasi-isomorphism. Here, "2" means the morphism $n \mapsto 2n$. Note that there are no nonzero morphisms $D^\bullet \to C^\bullet$.

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[4] Show that if $\begin{smallmatrix} A & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{smallmatrix}$ is a cocartesian square such that $A \to B$ is a monomorphism, then $0 \to A \to B \oplus C \to D \to 0$ is exact.

[5] Show that if $\begin{smallmatrix} A & \to & C \\ \downarrow & & \downarrow \\ B & \to & D \end{smallmatrix}$ is a cocartesian square such that $A \to B$ is a monomorphism, then $0 \to A \to B \oplus C \to D \to 0$ is exact.

Exercise 3.9. Let K be a field. Show that every chain complex of vector spaces is quasi-isomorphic to one of the form $(\cdots \xrightarrow{0} V^{n-1} \xrightarrow{0} V^n \xrightarrow{0} V^{n+1} \xrightarrow{0} \dots)$ in $Ch(K)$.

Exercise 3.10 (Harder)**.** Show that every chain complex of abelian groups is quasi-isomorphic to one of the form $(\cdots \xrightarrow{0} A^{n-1} \xrightarrow{0} A^n \xrightarrow{0} A^{n+1} \xrightarrow{0} \dots)$ in $Ch(\mathbb{Z})$. Hint.[6] Hint.[7] Hint.[8]

Exercise 3.11 (Difficult)**.** Let $R = \mathbb{Z}/4$ or $R = \mathbb{Q}[x,y]$. Find a chain complex of R-modules which is *not* quasi-isomorphic to one of the form $(\cdots \xrightarrow{0} M^{n-1} \xrightarrow{0} M^n \xrightarrow{0} M^{n+1} \xrightarrow{0} \dots)$ in $Ch(R)$, and prove that it is not.

Now we can formulate our strategy for the question at the beginning.

Strategy 3.12. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathscr{A}$. Suppose that we can find a quasi-isomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Q_A^\bullet \arrow[r] & Q_B^\bullet \arrow[r] & Q_C^\bullet \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "q.i."'] & B \arrow[r] \arrow[u, "q.i."'] & C \arrow[r] \arrow[u, "q.i."'] & 0
\end{tikzcd}$$

in $Ch(\mathscr{A})$ which is termwise split, i.e., $Q_B^n \cong Q_A^n \oplus Q_C^n$ for each $n \in \mathbb{Z}$, and with $Q_A^n = Q_B^n = Q_C^n = 0$ for $n < 0$. Since Φ preserves sums,

$$0 \to \Phi Q_A^\bullet \to \Phi Q_B^\bullet \to \phi Q_C^\bullet \to 0$$

remains exact. On the other hand, since Φ is left exact it follows that $H^0(\Phi Q_A^\bullet) \cong \Phi A$ and similar for B, C. So by the Snake Lemma, we obtain a long exact sequence

$$\begin{tikzcd}
 & H^2\Phi(Q_A^\bullet) \arrow[r] & H^2\Phi(Q_B^\bullet) \arrow[r] & H^2\Phi(Q_C^\bullet) \arrow[r] & \dots \\
 & H^1\Phi(Q_A^\bullet) \arrow[r] & H^1\Phi(Q_B^\bullet) \arrow[r] & H^1\Phi(Q_C^\bullet) \arrow[llu, rounded corners] & \\
0 \arrow[r] & \Phi A \arrow[r] & \Phi B \arrow[r] & \Phi C \arrow[llu, rounded corners] &
\end{tikzcd} \tag{2}$$

We must also show existence, functoriality, uniqueness (or at least canonicity) of the $Q_-^\bullet$.

[6] Use the fact that any subgroup of a free abelian group is again a free abelian group. That is, if I is a set and $M \subseteq \mathbb{Z}^{\oplus I}$ a subgroup, then $M \cong \mathbb{Z}^{\oplus J}$ for some set J.

[7] Use the fact that morphisms from free abelian groups lift through surjections. That is, if $\mathbb{Z}^{\oplus I} \to C$ is any morphism and $B \to C$ is surjective, then there exists a factorisation $\mathbb{Z}^{\oplus I} \to D \to C$.

[8] Given an arbitrary chain complex of abelian groups $C^\bullet$ and $n \in \mathbb{Z}$, find a morphism of the form $P^\bullet = (\dots \to 0 \to P^{n-1} \to P^n \to 0 \to \dots) \to C^\bullet$ such that P^{n-1} and P^n are free, $H^n(P^\bullet) \cong H^n(C^\bullet)$ and $H^i(P^\bullet) = 0$ for $i \neq n$.

Remark 3.13. We will not carry out the exact strategy above. That is, our replacement $Q^\bullet_A \to Q^\bullet_B \to Q^\bullet_C$ will not be split exact. Instead, we will use an exact sequence of the form $0 \to Q^\bullet_C[-1] \to \mathrm{Cone}(Q^\bullet_B \to Q^\bullet_C)[-1] \to Q^\bullet_B \to 0$.

We will also be working with untruncated resolutions, so showing $H^0(\Phi Q^\bullet_A) \cong \Phi A$, etc., will require an additional argument.

4 Fibrant replacement

Definition 4.1. A chain complex $Q^\bullet$ is called *fibrant* if for any monomorphic quasi-isomorphism $f : C^\bullet \to D^\bullet$ in $Ch(\mathscr{A})$, every map $C^\bullet \to Q^\bullet$ factors through $D^\bullet$.

$$\begin{tikzcd} C^\bullet \arrow[r] \arrow[rr, bend left] & D^\bullet \arrow[r, dashed] & Q^\bullet \end{tikzcd}$$

Exercise 4.2. Let $Q^\bullet \in Ch(\mathrm{Ab})$ be a fibrant complex of abelian groups. Show that each Q^n is *divisible* in the sense that for every $m \in \mathbb{Z}$ and $q \in Q^n$ there is a $p \in Q^n$ with $mp = q$. Hint.[9]

Exercise 4.3 (Difficult)**.** Suppose that

$$\cdots \to 0 \to 0 \to 0 \to Q^0 \to Q^1 \to Q^2 \to \dots$$

is a chain complex in $Ch(\mathrm{Ab})$ such that each Q^n is divisible. Show that $Q^\bullet$ is fibrant.

Exercise 4.4. Suppose that X is a topological space and $Q^\bullet \in Ch(\mathrm{Shv}(X, \mathrm{Ab}))$ is fibrant. Show that for each n, and inclusion $V \subseteq U \subseteq X$ of open subsets, the map $Q^n(U) \to Q^n(V)$ is surjective. Show that each abelian group $Q^n(U)$ is divisible.

Definition 4.5. Suppose that $\Phi : \mathscr{A} \to \mathscr{B}$ is a functor between Grothendieck abelian categories which preserves limits. The *derived functors* of Φ are the cohomology

$$R^n\Phi(A) := H^n(\Phi Q^\bullet)$$

where $A \to Q^\bullet$ is any quasi-isomorphism towards a fibrant complex $Q^\bullet$. In particular, given $F \in \mathrm{Shv}_{\mathrm{et}}(X, \mathrm{Ab})$,

$$H^n_{\mathrm{et}}(X, F) = H^n(Q^\bullet(X))$$

where $F \to Q^\bullet$ is a quasi-isomorphism in $Ch(\mathrm{Shv}_{et}(X, \mathrm{Ab}))$ and $Q^\bullet$ is a fibrant complex of étale sheaves.

Claim 4.6.

1. *The $R^n\Phi(A)$ exist and are functorial in A.* In fact, there exists a functor $Q : Ch(\mathscr{A}) \to Ch(\mathscr{A})$ and a natural transformation $\mathrm{id} \to Q$ such that for each complex $C^\bullet$, the complex $QC^\bullet$ is fibrant, and the morphism $C^\bullet \to QC^\bullet$ is a monomorphism and a quasi-isomorphism.

[9] Use the D^n form Example 2.2.

2. *The $R^n\Phi$ are independent of $Q^\bullet$.* In fact, if $A \to Q^\bullet$, $A \to P^\bullet$ are two quasi-isomorphisms towards fibrant complexes, then there exists a quasi-isomorphism $\Phi Q^\bullet \overset{q.i.}{\to} \Phi P^\bullet$.

3. *The $R^n\Phi$ send short exact sequences to long exact sequences.* More precisely, if $0 \to A \to B \to C \to 0$ is an exact sequence in A, then there exist fibrant replacements $Q_A^\bullet, Q_B^\bullet, Q_C^\bullet$, such that $Q_A^\bullet = \mathrm{Cone}(Q_B^\bullet \to Q_C^\bullet)[-1]$.

5 Functorial fibrant replacement

We start with the first claim. The proof is known as the Small Object Argument and appears in many places and many guises throughout the homotopy theoretic literature.

Theorem 5.1. *Suppose $\mathscr{A}$ is a Grothendieck abelian category. Then there exists a functor $Q : Ch(\mathscr{A}) \to Ch(\mathscr{A})$ and a natural transformation* $\mathrm{id} \to Q$ *such that for each complex $C^\bullet$, the complex $QC^\bullet$ is fibrant, and the morphism $C^\bullet \to QC^\bullet$ is a monomorphism and a quasi-isomorphism.*

Remark 5.2. A proof of the case where $\mathscr{A}$ is the category of modules over a ring is in Hovey, "Model categories", see Theorem 2.3.13. For a more general version see Beke, "Sheafifiable homotopy model categories", Proposition 1.3.

Discussion of proof. The *idea* is as follows. Given $C^\bullet$, define $FC^\bullet$ as the pushout

$$\begin{tikzcd}
\oplus_\Lambda A_\lambda \arrow[r] \arrow[d, "\oplus_\Lambda \iota_\lambda"'] & C^\bullet \arrow[d] \\
\oplus_\Lambda B_\lambda \arrow[r] & FC^\bullet
\end{tikzcd}$$

where the sum is over the collection Λ of roofs $\begin{smallmatrix} A_\lambda & \to C \\ \iota_\lambda \downarrow & \\ B_\lambda & \end{smallmatrix}$ such that ι_λ is both a monomophism and a quasi-isomorphism. Then take

$$QC^\bullet = \varinjlim(C^\bullet \to FC^\bullet \to FFC^\bullet \to FFFC^\bullet \to \dots).$$

Functoriality of F comes from functoriality of Λ. By the Snake Lemma, the morphisms $C^\bullet \to FC^\bullet$ are quasi-isomorphisms, so $C^\bullet \to QC^\bullet$ is a filtered colimit of quasi-isomorphisms, and therefore a quasi-isomorphism. To see that $QC^\bullet$ is fibrant, we would like to argue that given an arbitrary monomorphic quasi-isomorphism $A \to B$ and a morphism $A \to QC^\bullet$, we can lift to some $A \to F^nC^\bullet$. Then by definition of F we automatically have a canonical commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & F^nC^\bullet \arrow[d] \\
B \arrow[r] & F^{n+1}C^\bullet
\end{tikzcd}$$

so we get the factorisation $A \to B \dashrightarrow QC^\bullet$.

There are two big problems with the above proof. The first is that Λ is a proper class, not a set. So the sums defining F will not exist in general. The second problem is that an arbitrary $A \to QC^\bullet$ (for the $QC^\bullet$ defined above) will not necessarily factor through some $F^n C^\bullet$.

The solution to these problems is to make Λ smaller and $\varinjlim F^n C^\bullet$ bigger. Since $\mathscr{A}$ is a Grothendieck abelian category, it can be shown that there is a *set* I of monomorphic quasi-isomorphisms which still detect fibrant objects, and which generate the set of all monomorphic quasi-isomorphisms. On the other hand, instead of defining $QC^\bullet$ as above, for ordinals γ one recursively defines

$$F^\gamma = \begin{cases} F^{\gamma+1} = F \circ F^\gamma & \text{successor ordinals} \\ \varinjlim_{\gamma' < \gamma} F^{\gamma'} & \text{limit ordinals} \end{cases}$$

Then, $QC^\bullet := F^\kappa C^\bullet$ for some appropriate κ. For details, see [Beke]. $\square$