(Pro)étale cohomology
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Lecture 3: Topology I

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Classically, there are various sheaf theoretic ways to calculate singular cohomology. For example, if X is a triangularisable space then

$$H^*_{sing}(X,\mathbb{Q}) \cong \check{H}^*(X,\mathbb{Q}),$$

we always[1] have $\check{H}^*(X,\mathbb{Q}) \otimes \mathbb{R} \cong \check{H}^*(X,\mathbb{R})$ and if X is a manifold,

$$\check{H}^*(X,\mathbb{R}) \cong H^*_{dR}(X)$$

[Bott, Tu, "Differential forms in algebraic topology", Thm.II.10.6, Thm.III.15.8].

In this lecture we define étale sheaves on a scheme X using the étale topology on the category of étale X-schemes.

1 Topologies

We begin with the general theory of Grothendieck sites. This is a generalisation of the notion of a topological space, which allows us to use more general morphisms in place of open immersions.

Example 1.1. Let X be a topological space. Consider the assignment which sends an open subset $U \subset X$ to the set

$$F(U) := \{f : U \to \mathbb{R}\}$$

of continuous functions. Given an inclusion of open subsets $V \subseteq U$ we have an induced "restriction" morphism $F(U) \to F(V); (U \xrightarrow{f} \mathbb{R}) \mapsto (V \to U \xrightarrow{f} \mathbb{R})$. Furthermore, given any set of inclusions $\{U_i \subseteq U\}_{i\in I}$, if $\cup_{i\in I} U_i = U$, then any function $(U \xrightarrow{f} \mathbb{R}) \in F(U)$ on U can be recovered from its restrictions $f_i := (U_i \xrightarrow{f|_{U_i}} \mathbb{R}) \in F(U_i)$ together with the "glueing information"

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}.$$

In other words, F is a sheaf (we will define sheaves formally below). We want to be able to discuss such things when open subsets $U \subseteq X$ are replaced by more general morphisms $Y \to X$. The first step is to define abstractly what "covering" means, in a more general setting.

[1] This is because $\mathbb{Q} \to \mathbb{R}$ is flat.

Definition 1.2. *A (Grothendieck) topology on a category $\mathcal{C}$ is the data of: for every object $U \in \mathcal{C}$, a collection of families of morphisms $\{U_i \to U\}_{i\in I}$. The families in these collections are called* coverings *of U. This data is required to satisfy the following axioms:*

1. *$\{U \xrightarrow{\text{id}} U\}$ is a covering, for every object U.*
2. *If $\{U_i \to U\}_{i\in I}$ is a covering of U, and $V \to U$ is a morphism, then each fibre product $U_i \times_U V$ exists, and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering of V.*
3. *If $\{U_i \to U\}_{i\in I}$ is a covering of U, and for each $i \in I$ we have a covering $\{U_{ij} \to U_i\}_{j\in J_i}$ of U_i, then $\{U_{ij} \to U\}_{i\in I, j\in J_i}$ is a covering of U.*

A category equipped with a Grothendieck topology is called a site.

Remark 1.3.

1. The hypothesis that the $U_i \times_U V$ exist in condition 2 is actually unnecessary to get a good theory of sheaves (cf. *coverage* in [Johnstone, Sketches of an Elephant, §C2.1]. However, it is standard to ask this.
2. In fact, what we have defined is a *pre*topology, not a topology. But this is a standard misuse of terminology.

Exercise 1.4. Show that any two covering families of the same object admit a common *refinement*. That is, show that if $\mathcal{C}$ is a site, and $\{U_i \to Y\}_{i\in I}$ and $\{V_j \to Y\}_{j\in J}$ are covering families, then there exists a covering family $\{W_k \to Y\}_{k\in K}$, maps of sets $u : K \to I$ and $v : K \to J$, and factorisations $W_k \to U_{i(k)} \to Y$ and $W_k \to V_{j(k)} \to Y$.

Exercise 1.5. Suppose that X is a topological space in the conventional sense.[2] Define $Op(X)$ to be the category whose objects are open sets of X, and morphisms are inclusions. For $U \in Op(X)$, define the coverings of U to be the families $\{U_i \to U\}_{i\in I}$ such that $\cup_{i\in I} U_i = U$. Show that this defines a Grothendieck topology on $Op(X)$. (Note that in this category $V \times_U W = V \cap W$.)

Exercise 1.6. Let X be a topological space, and define $LH(X)$ to be the category whose objects are local homeomorphisms[3] $Y \to X$ and morphisms are commutative triangles $\begin{smallmatrix} Y' \to Y \\ \searrow \swarrow \\ X \end{smallmatrix}$. Show that this category has fibre products.

For $Y \in LH(X)$, define the coverings of Y to be the families $\{f_i : Y_i \to Y\}_{i\in I}$ such that $\cup_{i\in I} f_i(Y_i) = Y$. Show that this defines a Grothendieck topology on $LH(X)$.

[2] I.e., a set equipped with a collection of subsets of X declared to be *open*, preserved by finite intersection, arbitrary union, and containing X and $\varnothing$.

[3] A morphism $f : Y \to X$ is a local homeomorphism if it is continuous and for every point $y \in Y$, there is an open neighbourhood $V \ni y$ such that $F(V) \subseteq X$ is open and $f : V \to F(V)$ is a homeomorphism.

Exercise 1.7. Recall that a morphism $f : Y \to X$ of schemes is étale if it is locally of finite presentation, and for every $y \in Y$, the ring morphism $\mathcal{O}_{X,f(x)} \to \mathcal{O}_{Y,y}$ is étale. Let $Et(X)$ denote the category whose objects are étale morphisms $Y \to X$, and morphisms are commutative triangles. Do Exercise 1.6 with $Et(X)$ instead of $LH(X)$.

Explicitly:

Definition 1.8. *Let X be a scheme. The* étale topology *on the category $Et(X)$ of étale X-schemes is the Grothendieck topology whose coverings are families $\{f_i : Y_i \to Y\}_{i\in I}$ of morphisms in $Et(X)$ such that $\cup_{i\in I} f_i(Y_i) = Y$.*

Definition 1.9. *A* presheaf *F on a category $\mathcal{C}$ is just a functor $\mathcal{C}^{op} \to \mathrm{Set}$. A morphism of presheaves $F \to G$ is just natural transformation of functors $F \to G$.*

Remark 1.10.

1. A presheaf of abelian groups, K-vector spaces, Λ-modules, etc is a functor from $\mathcal{C}^{op}$ to the category of abelian groups, K-vector spaces, Λ-modules, etc.

2. Morphisms $F \to G$ of presheaves of abelian groups, etc, are again natural transformations, but now since F, G take values in abelian groups, etc, the morphisms $F(U) \to G(U)$ must be group homomorphisms, etc.

3. Given a morphism $f : V \to U$ in $\mathcal{C}$ and a presheaf F we often write $(-)|_V : F(U) \to F(V)$ or $f^* : F(U) \to F(V)$ for the morphism $F(f) : F(U) \to F(V)$.

2 Sheaves

Definition 2.1. *If $\mathcal{C}$ is a site, then a presheaf F is called a* sheaf *if for any object U and any covering $\{U_i \to U\}_{i\in I}$ we have*

$$F(U) = \mathrm{eq}\left(\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)\right). \tag{Ш}$$

A morphism of sheaves is just a morphism of presheaves. A sheaf on $Et(X)$ for some scheme X is called an étale sheaf on X.

Remark 2.2. If A is a ring, we will write $Et(A)$ instead of $Et(\mathrm{Spec}(A))$ and if $A \to B$ is an étale algebra, and F a presheaf on $Et(A)$ we will write $F(B)$ instead of $F(\mathrm{Spec}(B))$. Since all the information about an étale sheaf is contained in affine schemes (cf. Exercise 4.4), we will sometimes pretend that all schemes in $Et(\mathrm{Spec}(A))$ are affine.

Example 2.3. Let X be a topological space and consider the $Op(X)$ from Exercise 1.5. Then the assignment $U \mapsto \hom_{cont.}(U, \mathbb{R})$ from Example 1.1 is a sheaf. More generally, for any topological space Y, the assignment $U \mapsto \hom_{cont.}(U, Y)$ is a sheaf on $Op(X)$.

If Y is equipped with a map $Y \to X$, there is also a relative version $U \mapsto \hom_{/X}(U, Y)$ where $\hom_{/X}(U, Y)$ is the set of commutative triangles of continuous maps $\begin{smallmatrix} U \longrightarrow Y \\ \searrow \swarrow \\ X \end{smallmatrix}$

Exercise 2.4. Show that every sheaf on a topological space X is of the form $\hom_{/X}(-, Y)$ for some $Y \to X$. Hint.[4]

Example 2.5. Let A be a ring. We have the following important examples of étale sheaves on $Et(A)$, cf. Exercise **??**.

1. $\mathcal{O} : B \mapsto (B, +)$.
2. $\mathcal{O}^* : B \mapsto (B^*, *)$.
3. $\mu_n : B \mapsto \{b \in B^* : b^n = 1\}$.
4. $GL_n : B \mapsto \{$ invertible $n \times n$ matrices with coefficients in $B\}$.
5. $\Omega^n : B \mapsto \Omega^n_{B/k}$ (if A is a k-algebra for some ring k).

Remark 2.6. If a presheaf takes values in the category of abelian groups, then the sheaf condition (Ш) is equivalent to asking that the sequence

$$0 \to F(U) \to \prod_{i \in I} F(U_i) \to \prod_{i,j \in I} F(U_i \times_U U_j)$$

be exact, where the last morphism is the difference of the two morphisms induced by the two projections $U_i \times_U U_j \rightrightarrows U_i, U_j$.

Exercise 2.7. Let X be a topological space in the conventional sense. Consider the Grothendieck topology defined on $Op(X)$ in Exercise 1.5. Show that a presheaf on X is the same thing as a presheaf on $Op(X)$, and a preseheaf on X is a sheaf if and only if its associated presheaf on $Op(X)$ is a sheaf. That is, Definition 2.1 is an honest generalisation of the classical notion of a sheaf.

3 Galois modules

Exercise 3.1. Let $\mathrm{Spec}(L) \to \mathrm{Spec}(L')$ be a morphism in $Et(k)$ such that L/L' is Galois with Galois group $G = Aut(L/L')$. Recall that there is a canonical isomorphism

$$L \otimes_{L'} L \cong \prod_G L$$

[4] Given a sheaf F on X, define Y_F to be the quotient of the set of triples (U, s, x) consisting of an open $U \subseteq X$, an element $s \in F(U)$, and a point $x \in U$, subject to the equivalence relation: $(U, s, x) \sim (V, t, y)$ if $x = y$ and $s|_{U \cap V} = t|_{U \cap V}$. Give the set Y_F the coarsest topology such that for every $U \subseteq X$ and $s \in F(U)$, the morphism $U \to Y_F$; $x \mapsto (U, s, x)$ is continuous. In other words, $Y_F = \varinjlim_{U \subseteq X} \amalg_{s \in F(U)} U$. Show that $F(-) = \hom_{/X}(-, Y_F)$.

where two morphisms $L \rightrightarrows L \otimes_{L'} L; a \mapsto 1 \otimes a, a \otimes 1$ are identified with $a \mapsto (a, a, \dots, a)$ and $a \mapsto (a^{g_1}, \dots, a^{g_n})$ where g_i are the elements of G. Show that if F is an étale sheaf on $\mathrm{Spec}(k)$, then $F(\prod_G L) \cong \prod_G F(L)$, and

$$F(L') = F(L)^G$$

where $F(L)^G = \{s \in F(L) : g^* s = s \ \forall \ g \in G\}$. Deduce that if $F_1 \to F_2$ is a morphism of étale sheaves such that $F_1(L) \cong F_2(L)$ for every Galois extension L/k, then $F_1 \cong F_2$.

Remark 3.2. We will be able to show later on that a presheaf F on $Et(k)$ is a sheaf if and only if

1. $F(\amalg_{i \in I} U_i) \cong \prod_{i \in I} F(U_i)$ for any collection $U_i, i \in I$, and
2. $F(L) = F(L')^{Aut(L'/L)}$ for every Galois extension L'/L.

Theorem 3.3 (cf.Milne, Thm.II.1.9)**.** *Suppose that k is a field, k^{sep}/k is a separable closure, and $G = Gal(k^{sep}/k)$. Then there is a canonical equivalence between the category* G-Set *of discrete*[5] *G-sets*[6] *and the category* $\mathrm{Shv}(Et(k))$ *of étale sheaves on k.*

Remark 3.4. An easy case of the above theorem is $k = \mathbb{R}$. In this case the equivalence $\mathrm{Shv}(Et(k)) \to G$-Set is given by $F \mapsto F(\mathbb{C})$. In general, however, k^{sep}/k will not be finite, and therefore $\mathrm{Spec}(k^{sep})$ is not in $Et(k)$. This "problem" will go away next quarter when we discuss the pro-étale topology.

Proof. The association $F \mapsto X_F$. For $F \in \mathrm{Shv}(Et(k))$ we define

$$X_F = \varinjlim_{k^{sep}/L/k} F(L) \tag{1}$$

as the colimit over all subfields L of k^{sep} which are finite Galois extensions of k.

X_F is a discrete G-set. For any Galois L/k and any $\sigma \in G$ we have $\sigma(L) = L$ so σ restricts to a (finite) automorphism of L/k (and hence an automorphism of $F(L)$) via the canonical map $G \to Gal(L/k) \cong G/Aut(k^{sep}/L)$ where $Aut(k^{sep}/L) = \{g \in G : g(a) = a \ \forall \ a \in L\}$. These actions are compatible with inclusions $L \subseteq L'$ (and hence, the morphisms $F(L) \to F(L')$), hence we get an action of G on X_F. Moreover, every $x \in X_F$ is the image of some $y \in F(L)$, so X_F is a discrete G-set. The assignment $F \mapsto X_F$ is clearly natural in F, that is, it defines a functor.

For future reference, we note that since F is an étale sheaf, for each extension L'/L, the morphism $F(L) \to F(L')$ is injective, and moreover, for any two Galois extensions $L'/L/k$ of k, by Exercise 3.1 we have $F(L) = F(L')^{Aut(L'/L)}$. Since

[5] Here *discrete* means that for every $x \in X$, there is a finite Galois extension L/k with stabiliser $Stab(L) \subseteq G$ such that $x \in X^{Stab(L)}$.

[6] That is, a set X equipped with an action of G.

the action of G commutes with the colimit (1), we get

$$\begin{aligned} X_F^{Aut(k^{sep}/L)} = \varinjlim_{k^{sep}/L'/L/k} F(L')^{Aut(k^{sep}/L)} &= \varinjlim_{k^{sep}/L'/L/k} F(L')^{Aut(L'/L)} \\ &= \varinjlim_{k^{sep}/L'/L/k} F(L) = F(L). \end{aligned}$$

The association $X \mapsto F_X$. Now suppose we have a discrete G-set X. Recall that every étale k-algebra is of the form $\prod_{i=1}^n L_i$ for some finite separable field extensions L_i. We would like to define a presheaf which sends $\prod_{i=1}^n L_i$ to $\prod_{i=1}^n X^{Aut(k^{sep}/L_i)}$ but this depends on a choice of embeddings $L_i \to k^{sep}$. One way to get a choice-independent construction is to define a presheaf on $Et(k)$ as

$$F_X(\prod_{i=1}^n L_i) = \hom_G\Big(\hom_k\big(\prod_{i=1}^n L_i, k^{sep}\big), X\Big)$$

where $\hom_G$ means G-equivariant morphisms, and $G = Gal(k^{sep}/k) = \hom_k(k^{sep}, k^{sep})$ acts on $\hom_k(-, k^{sep})$ by composition.

F_X *is an étale sheaf.* Cf. Milne, Lem.I.1.8. By Remark 3.2, to show F_X is a sheaf, it suffices to check that

$$F_X(L) = F_X(L')^{Aut(L'/L)}$$

for finite Galois extensions L'/L. Note that for any Galois extension L'/k and any subextension $L'/L/k$ we have

$$\hom_k(L', k^{sep})_{Aut(L'/L)} \xrightarrow{\sim} \hom_k(L, k^{sep}).$$

Where Y_Γ is the set of Γ-orbits in Y for a group Γ acting on a set Y. It follows from this that $F_X(L) = F_X(L')^{Aut(L'/L)}$. Note that for any finite Galois subextension $k^{sep}/L/k$ we have $\hom_k(L, k^{sep}) = \mathrm{Gal}(L/k)$. So

$$F_X(L) = \hom_G(\mathrm{Gal}(L/k), X) = X^{Aut(k^{sep}/L)}. \tag{2}$$

Combining (1) and (2) we get

$$X_{F_X} = \varinjlim_L F_X(L) = \varinjlim_L X^{Aut(k^{sep}/L)} = X.$$

On the other hand, by (2) we get

$$F_{X_F}(L) = X_F^{Aut(k^{sep}/L)} = F(L)$$

for Galois extensions L/k. Then by Exercise 3.1 we have $F_{X_F} = F$.

So the assignements $X \mapsto F_X$ and $F \to X_F$ are inverse equivalences. □

4 Some covering combinatorics

In this section we give some conditions for a presheaf on $LH(X)$, resp. $Et(X)$, to be a sheaf.

Exercise 4.1. Suppose that F is a presheaf on a site $\mathcal{C}$, and $\{U_i \to U\}_{i\in I}$ and $\{U_{ij} \to U_i\}_{j\in J_i}$ are coverings. Using the diagram

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d, equal] & \prod_{i\in I} F(U_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,i'\in I} F(U_i \times_U U_{i'}) \arrow[d] \\
F(U) \arrow[r] & \prod_{i\in I, j\in J_i} F(U_{ij}) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,i'\in I, j\in J_i, j'\in J_{i'}} F(U_{ij} \times_U U_{i'j'})
\end{tikzcd}$$

show that if

1. F satisfies the sheaf condition (Ш) for $\{U_{ij} \to U\}_{i\in I, j\in J_i}$ and
2. each $F(U_i) \to \prod_{J_i} F(U_{ij})$ is injective,

then F satisfies the sheaf condition for $\{U_i \to U\}_{i\in I}$.

Deduce that a presheaf F on $LH(X)$ from Exercise 1.6 is a presheaf if and only if $F|_{Op(Y)}$ is a sheaf on $Op(Y)$ from Exercise 1.5 for every $Y \in LH(X)$.

Exercise 4.2. Suppose that F is a presheaf on a category $\mathcal{C}$ with fibre products and $\{V \to U\}$ and $\{U \to X\}$ are families consisting of single morphisms. Using the diagram

$$\begin{tikzcd}
F(V \times_X V) \arrow[d] & & F(U \times_X U) \arrow[ll] \\
F(V \times_U V) & F(V) \arrow[l, shift left] \arrow[l, shift right] \arrow[ul, shift left] \arrow[ul, shift right] & F(U) \arrow[l] \arrow[u, shift left] \arrow[u, shift right] \\
& & F(X) \arrow[ul] \arrow[u]
\end{tikzcd}$$

show that if

1. F satisfies the sheaf condition (Ш) for $\{V \to U\}$ (cf.middle row) and $\{U \to X\}$ (cf.right column), and
2. each $F(U \times_X U) \to F(V \times_X V)$ is injective (cf. top row),

then F satisfies the sheaf condition for $\{V \to X\}$ (cf. diagonal).

Exercise 4.3 (Harder)**.** Do Exercise 4.2 for coverings $\{U_i \to X\}_{i\in I}$ and $\{V_{ij} \to U_i\}_{j\in J_i}$ containing more than one element.

Exercise 4.4. Let $\mathcal{C}$ be a category with fibre products.

1. Suppose that $\mathcal{Y} = \{Y_i \to X\}_{i \in I}$ is a family in $\mathcal{C}$ such that $Y_{i_0} \to X$ is an isomorphism for some $i_0 \in I$. Show that every presheaf on $\mathcal{C}$ satisfies the sheaf condition for $\mathcal{Y}$.

2. Suppose $\mathcal{U} = \{U_i \to X\}_{i \in I}$ and $\mathcal{V} = \{V_j \to X\}_{j \in J}$ are families in $\mathcal{C}$ and for each j there exists $i_j \in I$ and a factorisation $V_j \to U_{i_j} \to X$. Using Part 1, Exercise 4.1, and Exercise 4.3 show that if a presheaf F satisfies the sheaf condition for $\mathcal{V}$, then it satisfies the sheaf condition for $\mathcal{U}$.

3. Using Lemma 4.5 below, deduce that a presheaf F on $Et(X)$ is a sheaf if and only if

   (a) $F|_{Op(Y)}$ is a sheaf for every $Y \in Et(X)$, and

   (b) F satisfies the sheaf condition (Ш) for every covering $\{Y' \to Y\}$ containing a single element such that Y and Y' are affine schemes.

Lemma 4.5. *Let Y be a scheme, $\{f_i : Y_i \to Y\}_{i \in I}$ be an étale covering, and $y \in Y$ a point. Then there exists an open neighbourhood $y \in U \subseteq Y$ and a surjective étale morphism $V \to U$ such that $V \to Y$ factors through $V \to Y_i \to Y$ for some $i \in I$, and U and V are affine.*

Proof. Choose any $i \in I$ such that there is a point $y' \in Y_i$ with $f_i(y') = y$. Let $\mathrm{Spec}(A) \subseteq Y$ and $\mathrm{Spec}(B) \subseteq Y_i$ be open affine neighbourhoods of y and y' such that $Y_i \to Y$ induces a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Since flat morphisms of finite presentation are open, [Stacks Project, 01UA], the image $f_i(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(A)$ is an open subscheme. By construction this image contains y. Choose an open affine neighbourhood $y \in \mathrm{Spec}(A') \subseteq f_i(\mathrm{Spec}(B)) \subseteq \mathrm{Spec}(B)$. Then $U := \mathrm{Spec}(A')$ and $V := \mathrm{Spec}(A' \otimes_A B)$ satisfy the requirements of the statement. □

Corollary 4.6. *The following representable presheaves are étale sheaves.*

1. $\hom(-, \mathbb{A}^1)$; $X \mapsto \Gamma(X, \mathcal{O}_X)$,

2. $\hom(-, \mathbb{G}_m)$; $X \mapsto \Gamma(X, \mathcal{O}_X^*)$,

3. $\mu_n = \hom(-, \mathrm{Spec}(\frac{\mathbb{Z}[T]}{T^n - 1}))$; $X \mapsto \{a \in \Gamma(X, \mathcal{O}_X^*) : a^n = 1\}$,

4. $GL_n = \hom\left(-, \mathrm{Spec}\left(\frac{\mathbb{Z}[U, T_{ij} : 1 \leq i,j \leq n]}{U \cdot \det T_{ij} - 1}\right)\right)$; $X \mapsto GL_n(\Gamma(X, \mathcal{O}_X))$,

5 Sheafification

Proposition 5.1. *Suppose that $\mathcal{C}$ is a small site such that all covering families are finite.*[7] *Then the inclusion $\mathrm{Shv}(\mathcal{C}) \subseteq \mathrm{PSh}(\mathcal{C})$ admits a left adjoint*

$$L : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C}).$$

That is, for every presheaf F, there exists a morphism $F \to LF$ such that:

[7] See Remark 5.4

1. *LF is a sheaf, and*
2. *for any sheaf G we have*

$$\hom(LF, G) = \hom(F, G).$$

Definition 5.2. *The sheaf LF in Proposition 5.1 is called the* sheafification *or associated sheaf of F.*

We give the proof from Higher Topos Theory. For the classical proof see [Artin, Grothendieck topologies, 1962, Lemma.2.1.2(ii)].

Proof. It suffices to find a functor L and natural transformation $\mathrm{id} \to L$ such that:

1. for every sheaf G, the morphism $G \to LG$ is an isomorphism, and
2. for every presheaf F, the presheaf LF is a sheaf.

Cf.Exercise 5.3.

For an object $X \in \mathcal{C}$ we have the representable presheaves $h_X(-) := \hom(-, X)$. We extend this to families $\mathcal{U} = \{U_i \to X\}_{i \in I}$ by constructing the presheaves $h_{\mathcal{U}}(-) := \mathrm{coeq}(\coprod_{i,j} h_{U_i \times_X U_j} \rightrightarrows \coprod_i h_{U_i})$. Then the condition for a presheaf F to be a sheaf is precisely that

$$\hom(h_X, F) \to \hom(h_{\mathcal{U}}, F) \tag{$\boxplus$}$$

is an *isomorphism* for every covering family $\mathcal{U}$. A first attempt to force F to satisfy ($\boxplus$) is to define

$$F^\dagger(Y) := \varinjlim_{\mathcal{V} = \{V_j \to Y\}_{j \in J}} \hom(h_{\mathcal{V}}, F)$$

where the colimit is over all coverings $\mathcal{V} = \{V_j \to Y\}_{j \in J}$ of Y. Note that this is functorial in $Y \in \mathcal{C}$, and that there is a canonical natural transformation

$$F \to F^\dagger$$

In general $F^\dagger$ will not be a sheaf, but we can repeatedly apply the construction, and define

$$LF := \varinjlim(F \to F^\dagger \to F^{\dagger\dagger} \to F^{\dagger\dagger\dagger} \to \dots).$$

We claim that $\mathrm{id} \to L$ satisfies the two desired conditions.

It is easy to see that if G is a sheaf then $G \to LG$ is an isomorphism. Let F be a presheaf. We want to show that LF is a sheaf. That is, given a covering $\mathcal{U} = \{U_i \to X\}_{i \in I}$, we want to show that the morphism

$$\hom(h_X, LF) \to \hom(h_{\mathcal{U}}, LF)$$

is an isomorphism. For this it will be enough to show that for all of the presheaves $G = F^{\dagger\cdots\dagger}$ we have factorisations

$$\begin{tikzcd} \hom(h_X, G) \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d, "\cong"] & \hom(h_X, G^\dagger) \arrow[d] \\ \hom(h_{\mathcal{U}}, G) \arrow[r] & {} \arrow[r] & \hom(h_{\mathcal{U}}, G^\dagger) \end{tikzcd} \tag{3}$$

through isomorphisms. Indeed, since coverings are finite, we have the identification $(*)$ in the diagram

$$\begin{tikzcd} \hom(h_X, LF) \arrow[r, phantom, "="] \arrow[d, "(**)"'] & \hom(h_X, \varinjlim F^{\dagger\cdots\dagger}) \arrow[r, phantom, "="] \arrow[d] & \varinjlim \hom(h_X, F^{\dagger\cdots\dagger}) \arrow[d] \\ \hom(h_{\mathcal{U}}, LF) \arrow[r, phantom, "="] & \hom(h_{\mathcal{U}}, \varinjlim F^{\dagger\cdots\dagger}) \arrow[r, phantom, "\underset{(*)}{=}"] & \varinjlim \hom(h_{\mathcal{U}}, F^{\dagger\cdots\dagger}) \end{tikzcd}$$

so if there are factorisations (3), the morphism $(**)$ is a colimit of isomorphisms.[8]

Notice that we have factorisations (3) if and only if we have the analogous factorisations for the restriction $G|_{C_{/X}} \in \mathrm{PSh}(C_{/X})$. Indeed, for any $Y \in C_{/X}$ we have $G^\dagger(Y) = G|_X^\dagger(Y)$, etc. So we can assume that X is a terminal object of $\mathcal{C}$.

Taking our fixed covering $\mathcal{U}$ of X we define

$$G^{\mathcal{U}}(Y) = \hom(h_Y \times h_{\mathcal{U}}, G).$$

This comes equipped with a factorisation

$$G(Y) \to \underset{\hom(h_Y \times h_{\mathcal{U}}, G)}{G^{\mathcal{U}}(Y)} \to \underset{\varinjlim_{\mathcal{V}} \hom(h_{\mathcal{V}}, G)}{G^\dagger(Y)}$$

functorial in Y and therefore a commutative diagram

$$\begin{tikzcd} \hom(h_X, G) \arrow[r] \arrow[d] & \hom(h_X, G^{\mathcal{U}}) \arrow[r] \arrow[d, "(*)"] & \hom(h_X, G^\dagger) \arrow[d] \\ \hom(h_{\mathcal{U}}, G) \arrow[r] & \hom(h_{\mathcal{U}}, G^{\mathcal{U}}) \arrow[r] & \hom(h_{\mathcal{U}}, G^\dagger) \end{tikzcd}$$

It remains only to show that $(*)$ is an isomorphism. This follows directly from the definitions of $G^{\mathcal{U}}$ and $h_{\mathcal{U}}$, cf.Exercise 5.3. □

Exercise 5.3.

[8] Note that if $E_0 \to E_1 \to \dots$ is any sequence of morphisms then $\varinjlim(E_0 \to E_2 \to E_4 \to \dots) = \varinjlim(E_1 \to E_3 \to E_5 \to \dots)$.

1. Prove that given a family $\{U_i \to X\}_{i\in I}$, the following two conditions are equivalent.

$$F(X) = \mathrm{eq}(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)) \tag{4}$$

$$\hom(h_X, F) = \hom(h_{\mathcal{U}}, F) \tag{5}$$

That is, we have Eq.(4) if and only if we have Eq.(5).

2. Suppose that $\iota : \mathcal{D} \to \mathcal{C}$ is a *fully faithful* functor, $L : \mathcal{C} \to \mathcal{D}$ another functor, and $\eta : \mathrm{id}_{\mathcal{C}} \to \iota L$ a natural transformation. Suppose further that for any F in $\mathcal{D}$, the morphism $\eta_F : F \to \iota L F$ is an isomorphism. Show that L is a left adjoint to ι.

3. Suppose that $\mathcal{C}$ is a category with a terminal object $X = *$ and that $\{U_i \to X\}_{i\in I}$ is a family of morphisms. Describe the set $h_{\mathcal{U}}(T)$ where $T \in \mathcal{C}$ is some other object. Using this prove that the two projections

$$h_{\mathcal{U}} \times h_{\mathcal{U}} \rightrightarrows h_{\mathcal{U}}$$

are isomorphisms. Deduce that $(*)$ in the above proof is an isomorphism.

Remark 5.4.

1. The assumption that coverings are finite in Proposition 5.1 can be replaced by the assumption that all coverings are smaller than κ for some regular cardinal κ. In this case, the colimit $\varinjlim F^{\dagger\cdots\dagger}$ is replaced by a transfinite composition. Since we are assuming that $\mathcal{C}$ is small, there is always such a κ. So there is actually no assumption necessary on covering size at all.

2. Since we are working with presheaves of *sets* (i.e., not spaces), one can show that, in fact, $LF = F^{\dagger\dagger}$. So this discussion of transfinite composition and covering size is moot.

3. On the other hand, we do need some kind of assumption on the size of $\mathcal{C}$. There are examples of large categories with presheaves which do not admit a sheafification.

Corollary 5.5. *Let $\mathcal{C}$ be a small category equipped with a Grothendieck topology. Then the category* $\mathrm{Shv}(\mathcal{C}, Ab)$ *of sheaves of abelian groups is an abelian category.*

Sketch of proof. Limits (i.e., products and kernels) can be calculated sectionwise. E.g., $\ker(F \to G)(U) = \ker(F(U) \to G(U))$. Colimits (i.e., sums and cokernels) are calculated sectionwise, and then sheafified. E.g., the sheaf cokernel of $F \to G$ is the sheafification of the presheaf $U \mapsto \mathrm{coker}(F(U) \to G(U))$. □

6 Stalks

Definition 6.1. *A* geometric point *of a scheme X is a morphism $\overline{x} \to X$ such that $\overline{x} = \mathrm{Spec}(\Omega)$ for some separably closed field Ω.*

Definition 6.2. *Let F be a presheaf on $Et(X)$. For a geometric point $\overline{x} \to X$ we define the* stalk *at $\overline{x}$ as*

$$F_{\overline{x}} = \varinjlim_{\overline{x} \to Y \to X} F(Y)$$

where the colimit is over factorisations of $\overline{x} \to X$ via some $Y \in Et(X)$.

Remark 6.3. If X is a topological space, F is a sheaf on X, and $x \in X$ is a point, then classically, the stalk of F at x is defined as the colimit

$$F_x = \varinjlim_{x \in U \subseteq X} F(U)$$

over open subsets of X containing x. The above definition is the étale analogue of this classical definition.

Remark 6.4. If F is a presheaf defined on all schemes that commutes with filtered limits of affine schemes, then $F_{\overline{x}} = F(\mathcal{O}_{X,x}^{sh})$ where $x = \operatorname{im}(\overline{x}) \in X$ and $\mathcal{O}_{X,x}^{sh}$ is the strict henselisation of $\mathcal{O}_{X,x}$ defined by $k(\overline{x})/k(x)$. In particular, if $F = \mathcal{O} : Y \mapsto \Gamma(Y, \mathcal{O}_Y)$, then $F_{\overline{x}} = \mathcal{O}_{X,x}^{sh}$.

Remark 6.5. If k^{sep}/k is a separable closure, then $\overline{x} = \operatorname{Spec}(k^{sep}) \to \operatorname{Spec}(k)$ is a geometric point, and $F_{\overline{x}}$ is the G-set X_F defined above.

Proposition 6.6. *Suppose that F is a sheaf of abelian groups on $Et(X)$ and $Y \in Et(X)$. Then a section $s \in F(Y)$ is zero if and only if for any geometric point $\overline{x} \to Y$ its image in each $F_{\overline{x}}$ is zero.*

Proof. Since all sheaves are separated, it suffices to show that for every $s \in F(Y)$, there exists a covering $\{U_i \to Y\}_{i \in I}$ such that $s|_{U_i} = 0$ for all $i \in I$. For every point $x \in Y$, choose a separable closure $k(x)^s/k(x)$, and let $\overline{x} \to X$ be the corresponding geometric point. Since the image of s in $F_{\overline{x}}$ is zero, there is some $\overline{x} \to V \to Y$ such that $s|_V = 0$. Since V is associated to x, let us write $V_x = V$. We do this for every point $x \in Y$, and obtain a family $\{V_x \to Y\}_{x \in Y}$ of étale morphisms indexed by points of Y. Since $x \in \operatorname{im}(V_x \to Y)$ for each $x \in Y$, the family is surjective, and therefore is a covering. By construction $s|_{Y_x} = 0$ for each Y_x, so $s = 0$. □

Exercise 6.7. Show that a sheaf of abelian groups F on $Et(X)$ is zero if and only if $F_{\overline{x}} = 0$ for each $x \in X$.

Exercise 6.8. Show that a morphism of sheaves of abelian groups $\phi : F \to G$ is a monomorphism, (resp. epimorphism, resp. isomorphism) if and only if $\phi_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is for each geometric point $\overline{x} \to X$.