(Pro)étale cohomology
Shane Kelly, UTokyo
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Comments welcome!: shanekelly64[at]gmail[dot]com.

Lecture 2: Algebra I

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Last week we used the Weil conjectures to motivate the search for a cohomology theory for $\mathbb{F}_q$-varieties having analogous properties to $H^n_{sing}(X(\mathbb{C}), \mathbb{Q})$, where X is a smooth $\mathbb{C}$-variety, and $X(\mathbb{C})$ is the set of complex points considered as a differentiable manifold via the isomorphism $\mathbb{C} \cong \mathbb{R}^2$.

Just looking at the definition of singular cohomology it is not clear at all how this could be done algebraically. As a first step, notice that for "nice" topological spaces M, the first cohomology group $H^1(M, \mathbb{Q})$ can be defined in terms of local homeomorphisms; that is, morphisms $f : N \to M$ such that for all $n \in N$ there exists an open neighbourhood $n \in U \subseteq N$ such that $f(U) \subseteq M$ is open and $U \to f(U)$ is a homeomorphism.

So a first step towards developing a cohomology theory for varieties would be to develop a robust algebraic version of local homeomorphisms. This is achieved by *étale morphisms* which we study this week. A morphism is étale if it is *flat* and *unramified*. We begin with flatness.

1 Flatness

Definition 1. *Let A be a ring. An A-module M is* flat *if for every monomorphism of A-modules $N \subseteq N'$, the morphism $M \otimes_A N \to M \otimes_A N'$ is also a monomorphism. Thats is, if $M \otimes_A -$ preserves monomorphisms. An A-algebra is flat if it is flat when considered as an A-module.*

Exercise 1. Show that if k is a field, every k-module (and therefore k-algebra) is flat.

Exercise 2. Show that if A is a ring and $S \subseteq A$ a multiplicatively closed subset, $A \to A[S^{-1}]$ is flat.

Example 2. In fact, an A-module M is flat if and only if for every prime $\mathfrak{p} \subseteq A$ the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is flat. See Milne, Prop.I.2.2.

Example 3. If $M_1, \dots, M_n$ is a finite sequence of flat A-modules then since $(\bigoplus_{i=1}^n M_i) \otimes_A \phi \cong \bigoplus_{i=1}^n (M \otimes_A \phi)$ for any morphism $\phi : N \to N'$, the A-module $M_1 \oplus \cdots \oplus M_n$ is again a flat A-module. In particular, since $M_1 \oplus \cdots \oplus M_n = M_1 \times \cdots \times M_n$ in the category of A-modules, for any sequence $B_1, \dots, B_n$ of flat A-algebras, the A-algebra $B_1 \times \cdots \times B_n$ is again a flat A-module. So for any sequence $f_1, \dots, f_n \in A$ of elements, the A-module $\prod_{i=1}^n A_{f_i}$ is flat.

Example 4. The blowup of $k[x,y]$ at (x,y) is covered by the two open affines $\mathrm{Spec}(k[x,\frac{y}{x}])$ and $\mathrm{Spec}(k[\frac{x}{y},y])$. Neither of $k[x,y]\to k[x,\frac{y}{x}], k[\frac{x}{y},y]$ are flat.[1]

— draw picture of blowup —

Exercise 3. Let A be a ring and $I\subseteq A$ an ideal.

1. Show that if A/I is a flat A-algebra then $I=I^2$.
2. (Advanced) Show that if I is finitely generated and $I=I^2$ then A/I is a flat A-algebra.
3. (Advanced) Give an example of an ideal I of a ring A such that $I=I^2$ but A/I is not a flat A-algebra.

— draw picture of closed immersion, and inclusion of a connected component —

Flatness is a "local uniformity" condition.

Exercise 4.

1. If $A\to B$ and $B\to C$ are flat ring morphisms, show that the composition $A\to C$ is flat.
2. If $A\to B$ is flat and $A\to D$ is any ring morphism, show that $D\to D\otimes_A B$ is a flat ring morphism.

Definition 5. *Let A be a ring. An A-module M is* faithfully flat *if it is flat, and given any morphism of A-modules $\phi: N\to N'$ such that $M\otimes_A N\to M\otimes_A N'$ is a monomorphism, the morphism ϕ is a monomorphism. An A-algebra B is faithfully flat if it is faithfully flat when considered as an A-module.*

Example 6. One can show quite easily (see Milne Prop.I.2.7) that a flat ring homomorphism $A\to B$ is faithfully flat if and only if $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ is surjective. Consequently, if k is a field, every k-algebra is faithfully flat. If A a ring and $f_1,\dots,f_n\in A$ generate the unit ideal then $\{\mathrm{Spec}\, A_{f_i}\to\mathrm{Spec}\, A\}_{i=1}^n$ is an open cover, so $A\to\prod A_{f_i}$ is flat.

Exercise 5. Suppose that M is an A-module and $A\to B$ is a faithfully flat morphism. Show that if $M\otimes_A B$ is a flat B-module, then M is a flat A-module.

[1] Consider the proper inclusions $(x)\subset(x,y)\subset k[x,y]$. Since $(x)k[x,\frac{y}{x}]=(x,y)k[x,\frac{y}{x}]$, the images of $\alpha:(x)\otimes_{k[x,y]}k[x,\frac{y}{x}]\to k[x,\frac{y}{x}]$ and $\beta:(x,y)\otimes_{k[x,y]}k[x,\frac{y}{x}]\to k[x,\frac{y}{x}]$ are equal. So if α and β are both monomorphisms, then we must have $(x)\otimes_{k[x,y]}k[x,\frac{y}{x}]\cong(x,y)\otimes_{k[x,y]}k[x,\frac{y}{x}]$, but this is not the case: Consider the exact sequence $(x)\to(x,y)\to(x,y)/(x)$. If $(x)\otimes_{k[x,y]}k[x,\frac{y}{x}]\cong(x,y)\otimes_{k[x,y]}k[x,\frac{y}{x}]$, then $((x,y)/(x))\otimes_{k[x,y]}k[x,\frac{y}{x}]=0$. But $k[x,y]/(x)\to(x,y)/(x); f\mapsto yf$ is an isomorphism of $k[x,y]$-modules, so

$$((x,y)/(x))\otimes_{k[x,y]}k[x,\tfrac{y}{x}]\cong(k[x,y]/(x))\otimes_{k[x,y]}k[x,\tfrac{y}{x}]\cong k[x,\tfrac{y}{x}]/(x)k[x,\tfrac{y}{x}]$$

which is not zero.

Exercise 6. Let M be a flat A-module.

1. Show that M is faithfully flat if and only if for every A-module N such that $M \otimes_A N \cong 0$, we have $N \cong 0$.
2. Show that if M is faithfully flat, then given any morphism of A-modules $\phi : N \to N'$ such that $M \otimes_A N \to M \otimes_A N'$ is a surjection, the morphism ϕ is a surjection.
3. Deduce that if M is faithfully flat, then a sequence of A modules is exact if it is exact after applying $M \otimes_A -$.

The following theorem will be used to show that $\mathcal{O}, \mathcal{O}^*, \mu_n, GL_n, \Omega^1, \ldots$ are étale sheaves.

Theorem 7 (See Milne I.2.17)**.** *Suppose that $f : A \to B$ is a faithfully flat ring morphism. Then*

$$0 \to A \xrightarrow{f} B \xrightarrow{d} B \otimes_A B \xrightarrow{d} B \otimes_A B \otimes_A B \xrightarrow{d} \ldots$$

is an exact sequence of A-modules. Here we define

$$e_i : b_0 \otimes \cdots \otimes b_{r-1} \quad \mapsto \quad b_0 \otimes \ldots b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1},$$

$$d = \sum (-1)^i e_i.$$

Exercise 7. Show that $d \circ f = 0$ and $d \circ d = 0$.

Proof. First suppose that there is a ring homomorphism $r : B \to A$ such that $r \circ f = \mathrm{id}$. Next define

$$s : b_0 \otimes b_1 \otimes \cdots \otimes b_{r+1} \mapsto r(b_0) b_1 \otimes \cdots \otimes b_{r+1}$$

and check that $s \circ d + f \circ r = \mathrm{id}$ and $s \circ d + d \circ s = \mathrm{id}$. In other words, we have constructed a chain complex homotopy between id and 0. Consequently, the cohomology groups of the chain complex are zero. In other words, the sequence is exact. More explicitly, if $a \in \ker d$, then $a = sda + dsa = 0 + dsa$, so $a \in \operatorname{im} d$.

Now consider some A-algebra A', let $B' = A' \otimes_A B$, and let $f' = A' \otimes_A f$. Since

$$A' \otimes_A (B \otimes_A \cdots \otimes_A B) \cong (A' \otimes_A B) \otimes_{A'} \cdots \otimes_{A'} (A' \otimes_A B)$$

applying $A' \otimes_A -$ to the sequence for f produces the sequence for f'. So by Exercise 6, if we can find some faithfully flat $A \to A'$ such that f' has a retraction, then the theorem is proven. Taking $A' = B$ with the retraction $B \otimes_A B \to B; b_1 \otimes b_2 \mapsto b_1 b_2$ finishes the proof. ∎

2 Unramified morphisms

Recall that by definition, the residue field $k(\mathfrak{p})$ at a prime $\mathfrak{p}$ of a ring A is $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Definition 8. *A morphism of rings $\phi : A \to B$ is* unramified *at a prime $\mathfrak{q} \subseteq B$ if $k(\mathfrak{q})$ is a finite separable field extension of $k(\mathfrak{p})$ where $\mathfrak{p} = \phi^{-1}\mathfrak{q}$, and $\mathfrak{q}B_\mathfrak{q} = \phi(\mathfrak{p})B_\mathfrak{q}$. It is* unramified *if it is of finite presentation and unramified at every prime.*

Example 9. The morphisms $k[x] \to k[x,y]/(x+y)(x-y)$ and $k[x] \to k[x]; x \mapsto x^2$ are unramified everywhere except at the origin.

— draw pictures —

Suppose $k = \mathbb{F}_p(t)$. The morphism $k[x] \to k[x,y]/y^p - xy - t$ is unramified every except at $(x, y^p - t)$ where it becomes the inseparable extension $\mathbb{F}_p(t) \to \mathbb{F}_p(t^{1/p})$.

Remark 10. Cf.Milne Prop.I.3.5. A morphism of finite presentation is unramified if and only if the diagonal morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes_A B)$ is an open immersion.[2] Since the diagonal is always a closed immersion, this is equivalent to it being flat, [Stacks project, 0819]. That is, a morphism $A \to B$ of finite presentation is unramified if and only if

$$B \otimes_A B \to B; \qquad (b_1 \otimes b_2) \mapsto b_1 b_2$$

is flat. This is a much more robust definition and will be heavily used in the proétale cohomology section.

Exercise 8 (Advanced)**.** Using Remark 10 (i.e., that the diagonal is flat) show that: if $\phi : A \to B$ is unramified, and $\sigma : B \to A$ any retract (i.e., $\sigma \circ \phi = \mathrm{id}$) Then $\sigma : B \to A$ is flat. Hint.[3]

Exercise 9 (cf. Exercise 4)**.** Let $A \to B \to C$ and $A \to D$ be ring homomorphisms. Show the following.

1. If $A \to B$ and $B \to C$ are unramified, then so is $A \to C$.
2. (a) If k is a field, a finite presentation k-algebra $k \to S$ is unramified if and only if $k \to S_\mathfrak{q}$ is a finite separable field extension for every prime $\mathfrak{q} \subseteq S$.
   (b) A finite presentation morphism $R \to S$ is unramified if and only if $k(\mathfrak{p}) \to k(\mathfrak{p}) \otimes_R S$ is unramified for every prime $\mathfrak{p} \in R$.
   (c) If $A \to B$ is unramified then so is $D \to D \otimes_A B$.

Remark 11. Milne uses finite type instead of finite presentation, but all Milne's schemes and rings are noetherian, so its the same thing.

[2]This uses: a morphism of finite presentation is unramified if and only if $\Omega_{B/A} = 0$, and the identification $I/I^2 \cong \Omega_{B/A}$ where $I = \ker(B \otimes_A B \to B)$.

[3]Apply $A \otimes_B -$ to $B \to B \otimes_A B \to B$.

3 Étale morphisms

Definition 12. *A morphism of finite presentation of rings is* étale *if it is flat and unramified.*

Remark 13. It is equivalent to define an étale morphism as a smooth morphism of relative dimension zero. That is, a flat morphism $A \to B$ of finite presentation such that $\Omega^1_{B/A} = 0$. In practice the above definition is often easier to use.

Example 14. Let k be a field and $k \to A$ a finitely presented k-algebra. Then A is étale if and only if $A \cong L_1 \times \cdots \times L_n$ for some finite separable field extensions L_i/k.

Exercise 10 (cf. Exercises 4 and 9)**.** Let $A \to B \to C$ and $A \to D$ be ring homomorphisms. Show the following.

1. If $A \to B$ and $B \to C$ are étale, then so is $A \to C$.
2. If $A \to B$ is étale then so is $D \to D \otimes_A B$.

Example 15. Suppose $Y \to X$ is a morphism of smooth affine $\mathbb{C}$-varieties, say $Y = \operatorname{Spec}(B)$ and $\operatorname{Spec}(A)$. Then $A \to B$ is étale if and only if $Y(\mathbb{C}) \to X(\mathbb{C})$ is a local homeomorphism of topological spaces.

To see this, one can use the Jacobian criterion for étale morphisms: a morphism $A \to B$ is étale if and only if for some n there is an isomorphism $B \cong A[x_1, \dots, x_n]/\langle f_1, \dots, f_n\rangle$ such that $det(\partial f_i/\partial x_j)$ is invertible in A, [Stacks project, 00U9]. Then the manifold $Y(\mathbb{C})$ is the set of points in $X(\mathbb{C}) \times \mathbb{C}^n$ where the f_i vanish[4] The same criterion ($det(\partial f_i/\partial x_j)$ is invertible in A) assures that $Y(\mathbb{C}) \to X(\mathbb{C})$ is a smooth morphism of complex manifolds of relative dimension zero, hence, a local homeomorphism. Cf. [Milne, Example I.3.4, pp.22-23].

— draw picture of a local homeomorphism —

We will see later that this "local homeomorphism" description is true in an algebraic setting too.

4 Hensel rings

Definition 16. *A local ring A with maximal ideal $\mathfrak{m}$ is* henselian *if for every étale morphism $\phi : A \to B$, and every prime $\mathfrak{q} \subset B$ such that $\phi^{-1}\mathfrak{q} = \mathfrak{m}$ and $k(\mathfrak{m}) = k(\mathfrak{q})$, there exists a ring homomorphism $\sigma : B \to A$ such that $\sigma^{-1}\mathfrak{m} = \mathfrak{q}$ and $\sigma \circ \phi = \mathrm{id}$.*

cf. "Inverse function theorem"

$$\begin{array}{ccc} \operatorname{Spec}(k(\mathfrak{q})) & \longrightarrow & \operatorname{Spec}(B) \\ \| & & \downarrow \\ \operatorname{Spec}(k(\mathfrak{m})) & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

[4]Note elements of a are identified with polynomial functions on $X(\mathbb{C})$.

Theorem 17. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = A/\mathfrak{m}$. The following are equivalent.*

1. *Hensel's Lemma holds: If $f \in A[t]$ is a monic such that $\overline{f} \in \kappa[t]$ factors as $\overline{f} = g_0 h_0$ with g_0, h_0 monic and coprime, then f factors as $f = gh$ with g and h monic and such that $\overline{g} = g_0$ and $\overline{h} = h_0$.*

2. *Any finite A-algebra B is a direct product of local rings $B = \prod B_i$.*

3. *If $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is quasi-finite and finite type, then $B = B_0 \times \cdots \times B_n$ where $\kappa \otimes_A B_0 = 0$, and for $i > 0$, each $A \to B_i$ is finite and B_i is a local ring.*

4. *A is henselian.*

Proof. For the omitted steps, see Milne, Étale cohomology, Theorem I.4.1.

(1) ⇒ (2). First note: by the going up theorem for any finite A-algebra B, all maximal ideals of B lie over $\mathfrak{m}$. So B is local if and only if $B/\mathfrak{m}B$ is local.

Now assume B is of the form $B = A[t]/(f)$ with f monic. If $\overline{f} = g_0^n$ for some $n \in \mathbb{Z}, g_0 \in \kappa[t]$ irreducible, then $B/\mathfrak{m}B$ is local, so B is local. If not, then by (1) we have $f = gh$ with g, h monic and $\overline{g}, \overline{h}$ coprime. Hence, $\kappa[t]/(g_0, h_0) = 0$, so $A[t]/(g, h) = 0$ (by Nakayama's Lemma). Since $(g) + (h) = (g, h) = 0$, and $(f) = (gh) = (g) \cap (h)$, it then follows from a version of the Chinese Remainder Theorem that $A[t]/g \times A[t]/h \xrightarrow{\sim} A[t]/f$. Iterating this process gives the result. The general case is omitted.

(2) ⇒ (3). Let A' be the integral closure of A in B, so we have morphisms $\operatorname{Spec}(B) \to \operatorname{Spec}(A') \to \operatorname{Spec}(A)$. A version of Zariski's Main Theorem says that $\operatorname{Spec}(B) \to \operatorname{Spec}(A')$ is an open immersion, and $\operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is finite. By (2), $A' \cong \prod A'_i$ for local rings A'_i (which are finite over A). The decomposition $\operatorname{Spec}(A') = \sqcup \operatorname{Spec}(A'_i)$ induces a decomposition $\operatorname{Spec}(B) = \sqcup \operatorname{Spec}(B'_i)$ (explicitly $\operatorname{Spec}(B'_i) = \operatorname{Spec}(B) \cap \operatorname{Spec}(A'_i)$). Let $\operatorname{Spec}(B_0)$ be the union of the $\operatorname{Spec}(B'_i)$ such that $\operatorname{Spec}(B'_i) \to \operatorname{Spec}(A'_i)$ is not surjective (i.e., such that $\operatorname{Spec}(B'_i) \subseteq \operatorname{Spec}(A'_i)$ does not contain the closed point), and let $\operatorname{Spec}(B_1), \ldots, \operatorname{Spec}(B_n)$ be the other connected components.

(3) ⇒ (4). Suppose $\mathfrak{q} \subseteq B$ is as in the definition of henselian. By (3) we can assume that $\phi : A \to B$ is finite and B is a local ring. Since B is a finite flat module over a local ring, it is free (Matsumura. Commutative ring theory, Thm.7.10). That is, $B \cong A^{\oplus d}$ as an A-module. But $B \otimes_A \kappa \cong B/\mathfrak{m}B \cong B/\mathfrak{q} = k(\mathfrak{q}) \cong k(\mathfrak{m}) = \kappa$ by assumption. So $d = 1$, and we find that $B \cong A$.

(4) ⇒ (1). Ommited. □

Example 18. Fields are henselian. Any complete local ring is henselian (see Milne, I.4.5).

Proposition 19. *For every local ring A, there exists a universal local morphism to a local henselian ring. That is, there exists a local morphism $A \to A^h$ to a local henselian ring A^h such that for every other local morphism $A \to B$ to a local henselian ring, there is a unique factorisation $A \to A^h \to B$.*

Proof. Consider the category of factorisations $A \xrightarrow{\phi} B \to A/\mathfrak{m}$ such that ϕ is étale. Morphisms in this category are commutative diamonds

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & \downarrow & \searrow & \\ A & & & & A/\mathfrak{m} \\ & \searrow & & \nearrow & \\ & & B' & & \end{array}$$

This category has an initial object ($A = A \to A/\mathfrak{m}$) as well as fibre coproducts ($A \to B' \otimes_B B'' \to A/\mathfrak{m}$), so it is filtered. Define

$$A^h = \varinjlim_{A \to B \to A/\mathfrak{m}} B.$$

The ring A^h is local with the same residue field as A: Note that the set of those $A \to B \to A/\mathfrak{m}$ such that $A \to B$ is a local homomorphism of local rings is cofinal. It follows that the set of non-units of A^h is the colimit of the sets of non-units of the B. From this it follows that the non-units of A^h are closed under addition. But a ring is local if and only if the set of non-units is closed under addition.

The ring A^h is henselian: Suppose $A^h \to C \to \kappa$ is a factorisation with C étale over A^h. As $A^h \to C$ is finite presentation, there is some $A \to B \to \kappa$ and a factorisation $B \to C_0 \to \kappa$ such that $C = A^h \otimes_B C$. In fact, $B \to C_0$ is étale, since $A \to A^h$ is faithfully flat, and faithfully flat morphisms detect étale morphisms.[5] But then the canonical morphism $C_0 \to A^h$ induces a morphism $C \to A^h$ with the required properties.

The morphism $A \to A^h$ satisfies the universal property: Suppose $A \to A'$ is a local homomorphism to a henselian local ring A'. Consider some $A \to B \to A/\mathfrak{m}$ in the system defining A^h. The morphism $A' \to A' \otimes_A B$ is étale, and there is a factorisation $A' \to A' \otimes_A B \to A'/\mathfrak{m}'$ by the universal property of the tensor product since $A \to A'$ is a local homomorphism. Now since A' is henselian, by definition we get a retraction $A' \leftarrow A' \otimes_A B$, and this induces a factorisation $A \to B \to A'$. Since we have such a factorisation for every $A \to B \to A/\mathfrak{m}$ in a compatible way, we get an induced factorisation $A \to \varinjlim B \to A'$. That is, a factorisation $A \to A^h \to A'$. □

Remark 20.
1. Let A be a noetherian local ring. Since $\widehat{A}$ is henselian there is a canonical factorisation $A \to A^h \to \widehat{A}$. In fact, both morphisms are monomorphisms. That is, A^h can be considered as a subring of the completion.

2. We will see in Lecture 8 that if A is a normal integral local ring, then A^h is isomorphic to a subring of the separable closure $\mathrm{Frac}(A)^s$ of $\mathrm{Frac}(A)$.

[5] A^h is a filtered colimit of flat A-modules so it is flat. Moreover, it is a filtered colimit of algebras which are surjective on spectra, so it is surjective on spectra. That is, it is faithfully flat. Faithfully flat morphisms detect flatness is Exercise 5. That they detect unramifiedness can be deduced from Exercise 9.

3. In fact, if A is a dvr, then $A^h = \mathrm{Frac}(A)^s \cap \widehat{A}$. That is, the henselisation is precisely the set of those power series which are "algebraic".

Definition 21. *A local henselian ring is* strictly local *if its residue field is separably closed.*

Exercise 11. Show that if A is a strictly local henselian ring, then every surjective étale morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ admits a section.

Proposition 22. *If A is a local ring and $\phi : \kappa \to \kappa^s$ is a separable closure of its residue field, then there exists a universal local homomorphism of local rings $A \to A^{sh}$ such that A^{sh} is strictly henselian, and the induced map on residue fields is ϕ.*

Proof. Run the proof of Proposition 19 with κ^s instead of κ. □

Definition 23. *The ring A^{sh} in the above proposition is called a* strict henselisation.

Exercise 12. Let $\phi : A \to B$ be a finite étale morphism. Then for each $\mathfrak{p} \subseteq A$, and each strict henselisation $A_\mathfrak{p}^{sh}$ we have $A_\mathfrak{p}^{sh} \otimes_A B \cong \prod_{i=1}^n A_\mathfrak{p}^{sh}$ for some n.

Note $\mathrm{Spec}(\prod_{i=1}^n A_\mathfrak{p}^{sh}) \cong \amalg_{i=1}^n \mathrm{Spec}(A_\mathfrak{p}^{sh})$. So by the above exercise, étale morphisms are local homeomorphisms if we consider $A_\mathfrak{p}^{sh}$ to be small neighbourhoods of $\mathfrak{p} \in \mathrm{Spec}(A)$.

Exercise 13 (Advanced)**.** Show that the necessary condition in the above exercise is also sufficient. That is, a finite morphism $\phi : A \to B$ is étale if and only if for each $\mathfrak{p}$ and $A_\mathfrak{p}^{sh}$ we have $A_\mathfrak{p}^{sh} \otimes_A B \cong \prod_{i=1}^n A_\mathfrak{p}^{sh}$ for some n.