Notes on the [HTT] proof of sheafification

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Throughout we assume that C is a small[1] category equipped with a topology. We write

$$j : C \to \mathrm{PSh}(C)$$

for Yoneda. Indices are functorial in categories, not topoi. So for example, if $u : C \to D$ is a functor then $u^* : \mathrm{PSh}(D) \to \mathrm{PSh}(C)$ is the functor $F \mapsto F \circ u$, and its left and right Kan extensions $\mathrm{PSh}(C) \rightrightarrows \mathrm{PSh}(D)$ are written $u_!$ and u_* respectively.

Theorem 1 ([Lur06, Prop.6.2.2.7])**.** *The canonical inclusion* $\mathrm{Shv}(C) \to \mathrm{PSh}(C)$ *admits a left adjoint*

$$L : \mathrm{PSh}(C) \to \mathrm{Shv}(C)$$

which commutes with finite limits.

Sketch of proof. One defines $F^\dagger(X) = \mathrm{colim}_{R \to jX} \mathrm{Map}(R, F)$ and LF to be a transfinite composition of $(-)^\dagger$ applied κ times, where κ is any regular cardinal such that

($*$) $F \mapsto \mathrm{Map}(R, F)$ commutes with κ-filtered colimits for every covering seive R.

Then it suffices to prove:

1. For any presheaf F and sheaf G we have $\mathrm{Map}(F^\dagger, G) \cong \mathrm{Map}(F, G)$.
2. For any presheaf F the presheaf LF is a sheaf.
3. $(-)^\dagger$ commutes with finite limits.

The third part is obvious since the category of covering sieves on an object is filtered. The first and second parts are Lemma 3 and Lemma 6 below. □

1 Step 1

For functoriality reasons, it is nicer to express $(-)^\dagger$ using Kan extensions. Let $Cov(C) \subseteq \mathrm{Fun}(\Delta^1, \mathrm{PSh}(C))$ denote the full subcategory morphisms of the form $R \to jX$ with R a covering sieve of X. This comes equipped with a projection functor $(R \to jX) \mapsto jX$ which admits a right adjoint $s : X \mapsto (jX \to jX)$.

$$\pi : Cov(C) \rightleftarrows C : s$$

As such, we get four functors, each one left adjoint to the one below it.

$$\mathrm{PSh}(Cov(C)) \underset{\pi^!}{\overset{\pi_!}{\rightleftarrows}} \mathrm{PSh}(C), \qquad \pi^* : \mathrm{PSh}(C) \to \mathrm{PSh}(Cov(C)), \qquad \pi_* : \mathrm{PSh}(Cov(C)) \to \mathrm{PSh}(C)$$

[1]This is, for example, so that we can sensibly take colimits over the category of sieves on an object, and so that the over categories $C_{/R}$ are uniformly bounded in size, implying that we can find κ satisfying Hypothesis ($*$).

Explicitly, $\pi^* = (-) \circ \pi$ and $\pi_* = (-) \circ s$ and $\pi_!$, resp. $\pi^!$, is the left resp. right, Kan extension along π, resp. s. Informally,

$$\begin{aligned}
\operatorname{colim}_{R \to j(Y)} F(R \to jY) &= \pi_! F(Y) \\
\pi^* F(R \to jY) &= F(Y) \\
F(jY \to jY) &= \pi_* F(Y) \\
\pi^! F(R \to jY) &= \operatorname{Map}(R, F)
\end{aligned}$$

Remark 2. Note the last equation is actually expressing the formal fact that $\pi^!$ is the functor associated to

$$\begin{aligned}
\operatorname{PSh}(C) \times Cov(C) &\to \mathcal{S} \\
(F, (R \to jY)) &\mapsto \operatorname{Map}(R, F)
\end{aligned}$$

Lemma 3. *For any presheaf F and sheaf G the canonical morphism*

$$\operatorname{Map}(F^\dagger, G) \to \operatorname{Map}(F, G)$$

is an equivalence.

Proof.

$$\begin{aligned}
\operatorname{Map}_{\operatorname{PSh}(C)}(F^\dagger, G) &= \operatorname{Map}_{\operatorname{PSh}(C)}(\pi_! \pi^! F, G) && \text{by definition} \\
&= \operatorname{Map}_{\operatorname{PSh}(Cov(C))}(\pi^! F, \pi^* G) && \text{by adjunction} \\
&\overset{Lem.5}{=} \operatorname{Map}_{\operatorname{PSh}(Cov(C))}(\pi^! F, \pi^! G) && G \text{ is a sheaf} \\
&\overset{Lem.4}{=} \operatorname{Map}_{\operatorname{PSh}(C)}(F, G) && \pi^! \text{ is fully faithful}
\end{aligned}$$

□

Lemma 4. *The unit and two counits*

$$\pi_! \pi^* \xrightarrow{\sim} \mathrm{id}$$

$$\mathrm{id} \xrightarrow{\sim} \pi_* \pi^*$$

$$\pi_* \pi^! \xrightarrow{\sim} \mathrm{id}$$

are equivalences. Equivalently, the two functors

$$\pi^*, \pi^! : \operatorname{PSh}(C) \to \operatorname{PSh}(Cov(C))$$

are fully faithful.

Proof. Since π admits a right adjoint $s : C \to Cov(C)$ satisfying $\pi \circ s = \mathrm{id}$, we have $\pi_* \pi^* = (-) \circ \pi \circ s = \mathrm{id}$. So π^* is fully faithful, so we also have $\pi_! \pi^* \cong \mathrm{id}$. Since s is fully faithful, $s^* s_* = \mathrm{id}$, or in other words, $\pi_* \pi^! \cong \mathrm{id}$ since $s^* = \pi_*$ and $s_* = \pi^!$. □

Lemma 5. *A presheaf G is a sheaf if and only if the composition of the two counits $\pi^* G \xleftarrow{\sim} \pi^* \pi_* \pi^! G \to \pi^! G$ is an equivalence.*

Proof. By definition, G is a sheaf if and only if $\operatorname{Map}(jX, G) \to \operatorname{Map}(R, G)$ is an equivalence for every covering sieve. This map is precisely the composition $\pi^* G \xleftarrow{\sim} \pi^* \pi_* \pi^! G \to \pi^! G$ evaluated on $R \to jX$ in $Cov(C)$. □

2 Step 2

We defined LF as a transfinite composition of $(-)^\dagger$ applied κ times. Explicitly,

$$T_0F := F,$$

for successor ordinals

$$T_{\lambda+1}F := (T_\lambda F)^\dagger,$$

and for limit ordinals

$$T_\beta F := \operatorname{colim}_{\alpha<\beta} T_\alpha F.$$

Then

$$LF := T_\kappa F.$$

Lemma 6. *For any presheaf F the presheaf LF is a sheaf.*

Proof. We want to show that $\mathrm{Map}(jX, LF) \to \mathrm{Map}(R, LF)$ is an equivalence for any covering sieve. By the definition of L and the hypothesis $(*)$ it suffices to show that the canonical morphism of morphisms

$$\begin{array}{ccccc} \mathrm{Map}(jX,G) & \longrightarrow & \bullet & \longrightarrow & \mathrm{Map}(jX,G^\dagger) \\ \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow \\ \mathrm{Map}(R,G) & \longrightarrow & \bullet & \longrightarrow & \mathrm{Map}(R,G^\dagger) \end{array}$$

factor through an equivalence Φ.

Replacing C with $C_{/X}$ with the induced topology, we can assume that X is the final object, see Lemma 7. With this assumption we can use a modified version of the adjunction we described above. Let $R_0 \subseteq j*$ be our fixed covering sieve of X which is now the terminal object $*$. Let $Cov(C)_0 \subseteq Cov(C)$ denote the full subcategory of those $R \to jY$ such that $jY \times R_0 \subseteq R$, that is, those covering sieves R containing the pullback of our fixed covering sieve. The composition $\rho : Cov(C)_0 \subseteq Cov(C) \xrightarrow{\pi} C$ gives rise to four analogous functors

$$\mathrm{PSh}(Cov(C)_0) \;\begin{array}{c} \xrightarrow{\rho_!} \\ \xleftarrow{\rho^*} \\ \xrightarrow{\rho_*} \\ \xleftarrow{\rho^!} \end{array}\; \mathrm{PSh}(C)$$

also satisfying Lemma 4. Moreover, ρ has a left adjoint $z : Y \mapsto (jY \times R_0 \to jY)$ so $\rho_! = (-) \circ z$ or informally

$$F(jY \times R_0 \to jY) = \rho_! F(Y).$$

The transformation $\mathrm{id} \to \pi_!\pi^! = (-)^\dagger$ naturally factors as $\mathrm{id} \to \rho_!\rho^! \to \pi_!\pi^!$, Lemma 8. So to conclude our proof it remains only to show that

$$\Phi : \mathrm{Map}(jX, \rho_!\rho^!G) \to \mathrm{Map}(R_0, \rho_!\rho^!G)$$

is an equivalence. By what is essentially the cofinality argument aluded to in [Lur06, Rem.6.2.2.15], we have $\mathrm{Map}(R_0, \rho_* F) \cong \mathrm{Map}(R_0, \rho_! F)$, Lemma 9. Inputting this, we get

$$\begin{aligned}
\mathrm{Map}(R_0, \rho_!\rho^! G) &= \mathrm{Map}(R_0, \rho_*\rho^! G) && \text{Lemma 9} \\
&= \mathrm{Map}(R_0, G) && \text{Lemma 4} \\
&= \mathrm{Map}((R_0 \to *), \rho^! G) \\
&= \mathrm{Map}(*, \rho_!\rho^! G)
\end{aligned}$$

where the last two equalities are the definitions of $\rho^!$ and $\rho_!$. So Φ is indeed an equivalence. □

3 Lemmas used in the proof.

Let $p : C_{/X} \to C$ the canonical projection. Recall that in general there is a bijection[2] of sieves

$$p_! : \mathrm{Sub}(j(Y \to X)) \xrightarrow{\sim} \mathrm{Sub}(jY) \tag{1}$$

and the covering sieves of $C_{/X}$ are precisely those sieves $R \to j(Y \to X)$ of $C_{/X}$ such that $p_! R \to p_! j(Y \to X) = jY$ is a covering sieve of C.

Lemma 7. *There is a natural equivalence $p^*(-)^\dagger \cong (p^*-)^\dagger$. In other words, the square*

$$\begin{array}{ccc}
\mathrm{PSh}(C) & \xrightarrow{(-)^\dagger} & \mathrm{PSh}(C) \\
{\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle p^*} \\
\mathrm{PSh}(C_{/X}) & \xrightarrow[(-)^\dagger]{} & \mathrm{PSh}(C_{/X})
\end{array}$$

commutes up to natural isomorphism. Consequently, the square below on the left is equivalent to the square on the right (note that $jX = p_! j$)*

$$\begin{array}{ccc}
\mathrm{Map}(jX, G) & \longrightarrow & \mathrm{Map}(jX, G^\dagger) \\
\downarrow & & \downarrow \\
\mathrm{Map}(p_! R_0, G) & \longrightarrow & \mathrm{Map}(p_! R_0, G^\dagger)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Map}(*, p^* G) & \longrightarrow & \mathrm{Map}(*, (p^* G)^\dagger) \\
\downarrow & & \downarrow \\
\mathrm{Map}(R_0, p^* G) & \longrightarrow & \mathrm{Map}(R_0, (p^* G)^\dagger)
\end{array}$$

Proof. We seek natural isomorphisms $p^*\pi_! \cong \pi_! p^*$ and $p^*\pi^! \cong \pi^! p^*$ where the latter uses the functor $p : Cov(C_{/X}) \to Cov(C)$. For this latter we can calculate directly, cf.Remark 2,

$$\begin{aligned}
p^*\pi^! F(R \to j(Y \to X)) &= \pi^! F(p_! R \to jY) \\
&\cong \mathrm{Map}(p_! R, F) \\
&\cong \mathrm{Map}(R, p^* F) \\
&\cong \pi^! p^* F(R \to j(Y \to X)).
\end{aligned}$$

[2]The inverse sends $R \to jY$ to $jY \times_{p^* jY} p^* R \to jY$.

For the former, it suffices to show the right adjoints are equivalent, $\pi^* p_* \cong p_* \pi^*$. Noting that $p_* F(Y) = \mathrm{Map}((jY \times jX \to jX), F)$, we calculate

$$\begin{aligned}
\pi^* p_* F(R \to jY) &\cong \mathrm{Map}((jY \times jX \to jX), F) \\
&\cong \mathrm{Map}\Big(j(R \to jY) \times j(jX \to jX) \to j(jX \to jX)), \pi^* F \Big) \\
&\cong p_* \pi^* F(R \to jY)
\end{aligned}$$

□

Lemma 8. *There is a canonical factorisation*

$$\mathrm{id} \longrightarrow \rho_! \rho^! \longrightarrow \pi_! \pi^!$$

Proof. One way to see this is to just evaluate on a presheaf F and an object Y and see that there is a canonical factorisation

$$F(Y) \to \lim_{jV \to R_0} F(R) \to \mathrm{colim}_{R \to jY} \lim_{jW \to R} F(W).$$

Alternatively, if we want to be careful about functorialities: The triangle

$$Cov(C)_0 \xrightarrow{\iota} Cov(C) \xleftarrow{s} C, \qquad s_0 \colon C \to Cov(C)_0$$

commutes on the nose so we have an equality of functors $(-) \circ \iota \circ s_0 = \rho_* \iota^* = \pi_* = (-) \circ s$. This equality of left adjoints corresponds to an equivalence of right adjoints $\iota_* \rho^! \cong \pi^!$ and since $\iota^* \iota_* \cong \mathrm{id}$ (ι is fully faithful) we obtain a further equivalence $\rho^! \cong i^* \pi^!$. This last equivalence gives rise to a morphism.

$$\rho_! \rho^! \cong \pi_! \iota_! i^* \pi^! \to \pi_! \pi^!. \tag{2}$$

We claim that this forms the commutative triangle in the statement. Checking this is a (annoying) exercise in adjunctions. □

Lemma 9. *Consider $R \subseteq j*$ as a full subcategory $R \subseteq C$. When restricted to this subcategory the composition $\rho_* \xleftarrow{\sim} \rho_! \rho^* \rho_* \to \rho_!$ becomes an equivalence. Consequently, for any presheaf $F \in \mathrm{PSh}(Cov(C)_0)$ we have*

$$Map(R_0, \rho_* F) \cong \mathrm{Map}(R_0, \rho_! F).$$

Proof. All three functors in question are compositions, namely, with the functors

$$Cov(C)_0 \xrightarrow{\rho} C, \qquad z, s_0 \colon C \to Cov(C)_0$$

Explicitly $\rho_!, \rho^*, \rho_*$ are respectively $(-) \circ z$, $(-) \circ \rho$, and $(-) \circ s_0$. So it suffices to observe that $\mathrm{id} \to \rho \circ s_0$ and $z \circ \rho \to \mathrm{id}$ are equivalences on R. The first one is always an equivalence, and the second one is the natural transformation $(jY \times R_0 \to jY) \to (R \to jY)$. This is an equivalence on R_0 by virtue of the fact that for any $jV \to R_0$ we have $jV \times R_0 = jV$. For the "Consequently", observe that $\mathrm{Map}(R_0, F') = \lim_{jY \to R_0} F'(jY)$. □

References

[Lur06] Jacob Lurie, *Higher Topos Theory*, arXiv Mathematics e-prints (2006), math/0608040.