Derived Algebraic Geometry
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Lecture 10: Derived schemes

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1 Some commutative algebra

A derived scheme should be something like a collection of derived affine schemes glued together along open immersions. To talk about open immersions of derived affine schemes we first need to understand localisation of derived rings. To do this we begin with quotients.

Definition 1. Given a simplicial ring $A \in \mathcal{R}\mathrm{ing}_\Delta$ and an element $f \in A_0$ we define $A//f$ as the pushout

$$\begin{array}{ccc} \mathbb{Z}[\partial\Delta^1] & \xrightarrow{(0,f)} & A \\ \downarrow & & \downarrow \\ \mathbb{Z}[\Delta^1] & \longrightarrow & A//f \end{array}$$

Remark 2. The idea is that instead of formally setting $f = 0$ we have instead freely attached a homotopy $f \sim 0$. If we are thinking of A as a Kan complex, we think of this as an isomorphism $f \cong 0$.

Remark 3. Recall that since $\partial\Delta^1 \cong * \sqcup *$, the simplicial ring $\mathbb{Z}[\partial\Delta^1]$ is the constant[1] polynomial ring $\mathbb{Z}[x_0, x_1]$. The upper horizontal map is $x_0 \mapsto 0$, $x_1 \mapsto f$. Since the pushout in $\mathcal{R}\mathrm{ing}_\Delta$ is $\otimes$ we have $(A//f)_n = \mathbb{Z}[\Delta^1]_n \otimes_{\mathbb{Z}[\partial\Delta^1]_n} A_n$. Explicitly, let's write x_i for the variable corresponding to the unique epimorphism $\sigma_i : [n] \to [1]$ which satisfies $\sigma_i(i-1) = 0$ and $\sigma_i(i) = 1$. Then we have

$$(A//f)_n = A_n[x_1, \dots, x_n].$$

The face morphisms $d_i : (A//f)_n \to (A//f)_{n-1}$ are

$$d_i : x_j \mapsto \begin{cases} f & (i,j) = (0,1) \\ x_{j-1} & i < j; (i,j) \neq (0,1) \\ x_j & j \leq i; (i,j) \neq (n,n) \\ 0 & (i,j) = (n,n) \end{cases}$$

Exercise 4. Show that if A is cofibrant then so is $A//f$.

[1] "Constant" means constant as a functor $\Delta^{op} \to \mathcal{R}\mathrm{ing}$.

Exercise 5. Suppose that A is a constant simplicial ring (so $A_i = A_{i+1}$ for all i) and $f \in A$. Show that

$$\pi_n(A//f) = 0 \text{ for } n > 0$$
$$\pi_0(A//f) = A/\langle f \rangle.$$

In other words, $A//f \to A/\langle f \rangle$ is a quasi-isomorphism.

Definition 6. Let $A \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring. Define

$$A[{}^1\!/\!/_f] := A[t]//(1 - ft).$$

Here $A[t]_n = A_n[t]$.

Remark 7. We have freely attached an element to A_0 (namely t) and then a freely attached a homotopy from 0 to $1 - ft$, or equivalently, from 1 to ft.

Exercise 8. Show that if A is cofibrant then so is $A[{}^1\!/\!/_f]$.

Exercise 9. Given a morphism of simplicial rings $\phi : A \to B$ and $f \in A_0$, show that there is an isomorphism of simplicial rings $B \otimes_A A[{}^1\!/\!/_f] \cong B[{}^1\!/\!/_{\phi f}]$.

Exercise 10.

1. Show that there is an isomorphism of simplicial rings

$$A[{}^1\!/\!/_f] \cong A \otimes_{A_0} (A_0[{}^1\!/\!/_f])$$

where we consider A_0 as a constant simplicial ring and the morphism of simplicial rings $A_0 \to A$ is the canonical one induced by the degeneracy morphisms.

2. Using the fact that if A is cofibrant then $A \otimes_{A_0} -$ preserves quasi-isomorphisms, show that there is a quasi-isomorphism

$$A[{}^1\!/\!/_f] \xrightarrow{q.i} A \otimes_{A_0} A_0[f^{-1}]$$

where $A_0[f^{-1}]$ on the right hand side is the usual localisation of classical rings.

3. Using the fact that localisation $-[f^{-1}] : A_0\text{-mod} \to A_0\text{-mod}$ of classical modules is an exact functor, show that for each n, we have

$$\pi_n(A[{}^1\!/\!/_f]) = (\pi_n A)[f^{-1}]$$

where on the right we consider f as an element of $\pi_0 A$ and $\pi_n A$ as a $\pi_0 A$-module.[2]

[2]Note that the ring structure on A_0 really does induce a ring structure on $\pi_0 A$ thanks to the common section $A_0 \to A_1$ of the face maps $A_1 \rightrightarrows A_0$.

Proposition 11. *Suppose $A \in \mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cof}}$ and that $f, g \in A_0$ are two elements generating the unit ideal in $\pi_0 A$. The square*

$$\begin{array}{ccc} A & \longrightarrow & A[{}^1/\!/_f] \\ \downarrow & & \downarrow \\ A[{}^1/\!/_g] & \longrightarrow & A[{}^1/\!/_{fg}] \end{array}$$

is cartesian in the quasicategory $N\mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cof}}$ of simplicial rings.

Sketch of proof. Any choice of pullback A' comes equipped with a morphism $A \to A'$. We want to show that this is an equivalence. Consider the adjunction[3]

$$\mathcal{K}\mathrm{an} \rightleftarrows \mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cof}}$$

of simplicial categories. The right adjoint detects weak equivalences and preserves limits. In $\mathcal{K}\mathrm{an}$ to any pullback square of pointed simplicial sets $(W, w) = (X, x) \times_{(Z,z)} (Y, y)$ we have an associated long exact sequence

$$\dots \to \pi_n W \to \pi_n X \times \pi_n Y \to \pi_n Z \to \pi_{n-1} W \to \dots$$

In the first part of the course we saw that for any $\pi_0 A$-module M

$$0 \to M \to M[f^{-1}] \oplus M[g^{-1}] \to M[(gf)^{-1}] \to 0$$

is a short exact sequence of $\pi_0 A$-modules. Applying this to each $\pi_n A$ shows that

$$\pi_n A' \cong ker(\pi_n A[f^{-1}] \oplus \pi_n A[g^{-1}] \to \pi_n A[(gf)^{-1}]) \cong \pi_n A.$$

□

2 Sheaves

Definition 12. A morphism of Kan complexes $f : K \to L$ is a *subobject* if it is the inclusion of a direct summand, so $L = K \sqcup K'$ for some K', [HTT, §6.1.6]. A morphism of Kan complexes is an *effective epimorphism* if the induced morphism $\pi_0 K \to \pi_0 L$ is surjective, [HTT, §6.2.3].

The following proposition is the higher version of the fact that given a morphism of sets $f : K \to L$ we have $\mathrm{coeq}(K \times_L K \rightrightarrows K) \cong im(f)$. The higher products are needed because we are not just enforcing a relation on points of K, but also on homotopies, homotopies between homotopies, etc.

[3]Instead of $\mathcal{K}\mathrm{an}$ on the left one could alternatively use simplicial abelian groups or, equivalently, homologically bounded below zero chain complexes.

Proposition 13. *Let $f : K \to L$ be a morphism in Kan and consider the sum $I \subseteq L$ of those connected components in the image of f. Then there is a canonical equivalence*

$$\operatorname{colim}_{n\in\Delta} K^{\times_L (n+1)} \xrightarrow{\sim} I$$

in the quasicategory $N\mathcal{K}\text{an}$.

In other words, $\operatorname{colim}_{n\in\Delta} K^{\times_L (n+1)}$ is (hoomotopic to) a direct summand of L, and is precisely the set of those connected components hit by K.

Recall that a sieve R on an object X in a classical category C is a subpresheaf $R \subseteq \hom(-, X)$.

Definition 14 ([HTT, Def.6.2.2.1, Lem.6.2.2.4, Prop.6.2.2.5)**.** Let C be a quasicategory and $X \in C$ an object. A *sieve* on X is a subobject $R \subseteq jX$ of the presheaf jX represented by X. That is, a sieve is a morphism of presheaves $R \to jX$ in $\mathrm{PSh}(C)$ such that for every object $Y \in C$ the map of Kan complexes $R(Y) \to jX(Y)$ is the inclusion of a direct summand.

Example 15.

1. The maximal sieve $R = j(X)$ and the minimal sieve $R = \varnothing$ are the easiest examples.
2. More interestingly, for any morphism $F \to jX$ of presheaves, one can take the image $im(F \to jX)$. Informally, this is the presheaf that sends an object Y to the connected components of $jX(Y)$ in the image of $F(Y) \to jX(Y)$. Formally, one can use $\operatorname{colim}_{n\in\Delta} F^{\times_{jX}(n+1)}$ since colimits and limits in presheaf categories are computed objectwise (see the lecture on limits).

Definition 16. A *topology* on a quasicategory C is the data of: for each object X a collection $J(X)$ of sieves called *covering sieves*. These covering sieves are required to satisfy the following axioms.

(T1) If $R \subseteq jX$ is a covering sieve and $Y \to X$ is any morphism in C then $jY \times_{jX} R$ is a covering sieve of Y.

$$\begin{array}{ccc} jY \times_{jX} R & \longrightarrow & R \\ \cap\!\mid & & \cap\!\mid \\ jY & \longrightarrow & jX. \end{array}$$

(T2) If $R \subseteq jX$ is a covering sieve and $R' \subseteq jX$ is any other sieve satisfying: for every morphism $jY \to R \to jX$ in $R(Y)$, the pullback $jY \times_{jX} R' \subseteq jY$ is a covering sieve, then R' is also a covering sieve.

$$\begin{array}{ccccc} jY \times_{jX} R' & \longrightarrow & R \cap R' & \longrightarrow & R' \\ \cap\!\mid & & \cap\!\mid & & \cap\!\mid \\ jY & \longrightarrow & R & \longrightarrow & jX \end{array}$$

(T3) For every object X, the maximal sieve

$$R = jX \subseteq jX$$

is a covering sieve.

Example 17. A topology on any small classical category $C \in \mathcal{C}\mathrm{at}$ is the same thing as a topology on its associated quasicategory $NC \in \mathcal{Q}\mathrm{Cat}$.

Example 18. A sieve $R \subseteq j\,\mathrm{Spec}\,A$ is a covering sieve for the *Zariski topology* on

$$\mathrm{dAff} := (N\mathcal{R}\mathrm{ing}_{\Delta}^{\mathrm{cof}})^{\mathrm{op}}$$

if there exists a sequence of elements $f_1, \dots, f_n \in A_0$ which generate the unit ideal of $\pi_0 A$, and factorisations $j\,\mathrm{Spec}\,A[{}^{1}/\!\!/_{f_i}] \to R \to j\,\mathrm{Spec}\,A$.

Exercise 19. Using the exercises in the first section, show that the covering sieves for the Zariski topology do satisfy the axioms for a Grothendieck topology on dAff.

Definition 20. Let C be a quasicategory equipped with a topology τ. A presheaf $F : C^{\mathrm{op}} \to N\mathcal{K}\mathrm{an}$ is a *sheaf* if for every covering sieve $R \subseteq jX$ the induced map

$$\mathrm{Map}(jX, F) \to \mathrm{Map}(R, F)$$

is a weak equivalence of Kan complexes. The quasicategory of τ-sheaves is the fullsubcategory $\mathrm{Shv}_\tau(C) \subseteq \mathrm{PSh}(C)$ consisting of those presheaves F which are τ-sheaves.

Theorem 21. *The canonical inclusion* $\mathrm{Shv}(C) \to \mathrm{PSh}(C)$ *admits a left adjoint.*

Proof. The proof from Lecture 2 works verbatim. □

We would like to know that representable presheaves on dAff are sheaves in the Zariski topology. For this we use the following proposition.

Proposition 22. *A presheaf F on* dAff *is a Zariski sheaf if and only if for every* $\mathrm{Spec}(A)$ *and $f, g \in A_0$ generating the unit ideal of $\pi_0 A$, the square*

$$\begin{array}{ccc} F(\mathrm{Spec}(A)) & \longrightarrow & F(\mathrm{Spec}(A[{}^{1}/\!\!/_{f}])) \\ \downarrow & & \downarrow \\ F(\mathrm{Spec}(A[{}^{1}/\!\!/_{g}])) & \longrightarrow & F(\mathrm{Spec}(A[{}^{1}/\!\!/_{fg}])) \end{array}$$

is cartesian in $N\mathcal{K}\mathrm{an}$.

Proposition 22 will be proven in a later version of these lecture notes.

Corollary 23. *Every representable presheaf is a Zariski sheaf.*

3 Schemes

Proposition 24. *Suppose that C is a quasicategory equipped with a topology and $f : F \to G$ is a morphism in $\mathrm{Shv}_\tau(C)$. The following are equivalent.*

1. *For every $X \in C$ and $s : jX \to G$ there exists a covering sieve R and a commutative square in $\mathrm{PSh}(C)$*

$$\begin{array}{ccc} R & \longrightarrow & F \\ \downarrow & & \downarrow \\ jX & \longrightarrow & G \end{array}$$

2. *The morphism $\mathrm{colim}_{n\in\Delta} F^{\times_G (n+1)} \to G$ is an equivalence in $\mathrm{Shv}_\tau(C)$.*

Remark 25. One way to prove Proposition 24 is by looking at the construction of the sheafification funtor.

Definition 26. A morphism satisfying the equivalent conditions of Proposition 24 is called an *effective epimorphism*.

Exercise 27. Using the fact that colimits are universal in a quasicategory of the form $\mathrm{Shv}(C)$, show that the pullback of an effective epimorphism is an effective epimorphism.

In this section sheaf means object of Shv(dAff).

Definition 28. A subsheaf $U \subseteq j\,\mathrm{Spec}(A)$ is an *open immersion* if there exists an effective epimorphism *of sheaves* of the form

$$\coprod_{\lambda\in\Lambda} j\,\mathrm{Spec}(A[{}^1\!/\!\!/_{f_\lambda}]) \to U.$$

In other words, if it is the sheafification of a union of opens $\mathrm{Spec}(A[{}^1\!/\!\!/_{f_\lambda}]) \subseteq \mathrm{Spec}(A)$.

Exercise 29. Show that if $U \subseteq j\,\mathrm{Spec}(A)$ is an open immersion and $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ any morphism then $U \times_{j\,\mathrm{Spec}(A)} j\,\mathrm{Spec}(B) \subseteq j\,\mathrm{Spec}(B)$ is an open immersion.

Definition 30. A morphism $F \to G$ in Shv(dAff) is an *open immersion* if for every $j\,\mathrm{Spec}(A) \to G$ the pullback $j\,\mathrm{Spec}(A) \times_G F \to j\,\mathrm{Spec}(A)$ is an open immersion.

Exercise 31.
1. Suppose $F \overset{\circ}{\hookrightarrow} G$ is an open immersion and $H \to G$ any morphism. Show that $F \times_G H \to H$ is an open immersion.
2. Suppose that $F \overset{\circ}{\hookrightarrow} G$ and $G \overset{\circ}{\hookrightarrow} H$ are open immersions. Show that the composition $F \to H$ is an open immersion.
3. Suppose that $F \to G$ is a morphism, $H \twoheadrightarrow G$ is an epimorphism, and that $F \times_G H \overset{\circ}{\hookrightarrow} H$ is an open immersion. Show that $F \to G$ is an open immersion.

Definition 32. A *derived scheme* is a sheaf $X \in \mathrm{Shv}(\mathrm{dAff})$ which admits an effective epimorphism of sheaves of the form

$$\coprod_{\lambda \in \Lambda} j\,\mathrm{Spec}(A_\lambda) \to X$$

such that each $j\,\mathrm{Spec}(A_\lambda) \to X$ is an open immersion. Such a morphism is called an *open affine covering.* The category of derived schemes is the full subcategory

$$\mathrm{dSch} \subseteq \mathrm{Shv}(\mathrm{dAff})$$

whose objects are schemes.

Remark 33. Given a derived scheme X, the collection of open immersions $U \to X$ are the opens of a topological space X_{top}. Furthermore, restricting $\mathrm{Map}(-, \mathrm{Spec}\,\mathbb{Z}[t])$ to these opens we obtain a sheaf $\mathcal{O}_X$ of spaces on X_{top} which has a canonical structure of sheaf of derived rings. Conversely, one can reconstruct the derived scheme X from this structure.

Exercise 34. Suppose $U \subseteq j\,\mathrm{Spec}(A)$ is an open immersion. Show that U is a derived scheme.

Exercise 35. Suppose that $X \to Y$ is a morphism of derived schemes and $\{V_\mu \to Y\}_{\mu \in M}$ an open affine covering of Y. Show that there exists an open affine covering $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ of X such that for each λ there is some μ_λ admitting a commutative square

$$\begin{array}{ccc} U_\lambda & \longrightarrow & V_{\mu_\lambda} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Theorem 36. *The category of derived schemes admits finite limits and the canonical inclusion* $\mathrm{dSch} \subseteq \mathrm{PSh}(\mathrm{dAff})$ *preserves those limits.*

Proof. The proof from Lecture 3 works verbatim. □