Derived Algebraic Geometry
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Autumn Semester 2023-2024

Lecture 9: Yoneda

December 7th 2023

1 Motivation

The main goal of this weeks lecture is to say something about the Yoneda embedding for ∞-categories. For simplicial categories, Yoneda's lemma is an exercise, similar to the classical case.

Exercise 1. Let C be a simplicial category.

1. Show that for each object X the assignment $Y \mapsto \mathrm{Map}(Y, X)$ defines a morphism of simplicial categories $jX : C^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$.
2. Show that the assignment $X \mapsto jX$ also defines a morphism of simplicial categories $j : C \to \mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$.
3. Show that the canonical maps $\mathrm{Map}_C(X, X') \to \mathrm{Map}_{\mathrm{Fun}(C^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)}(jX, jX')$ are *isomorphisms*.

For quasi-categories, Yoneda's lemma is a theorem. The obstacles are easy to explain.

Obstacle 1: Functoriality. We would like to define a functor

$$j : C \xrightarrow{?} \mathrm{Fun}_{\mathcal{Q}\mathrm{Cat}}(C^{\mathrm{op}}, N\mathcal{K}\mathrm{an})$$

such that for each $X, Y \in C$, there are equivalences

$$j(X)(Y) \cong \mathrm{Map}^R_C(Y, X),$$

and then show that j is fully faithful. However, $\mathrm{Map}^R_C(Y, X)$ does not have obvious compositions. For example, we can assign the Kan complex $\mathrm{Map}^R_C(Y, X)$ to $X, Y \in C_0$, but given morphisms $f : X \to X'$ or $g : Y' \to Y$ in C_1 it is unclear what the corresponding morphisms

$$\mathrm{Map}^R_C(Y, X) \xrightarrow{?} \mathrm{Map}^R_C(Y, X'), \qquad \mathrm{Map}^R_C(Y, X) \xrightarrow{?} \mathrm{Map}^R_C(Y', X).$$

of $(N\mathcal{K}\mathrm{an})_1$ should be, let alone higher simplicies in C_n.

Obstacle 2: Non-cofibrant presheaves. Recall the adjunction

$$\mathfrak{C}[-] : \mathcal{Q}\mathrm{Cat} \rightleftarrows \mathcal{C}\mathrm{at}_\Delta : N$$

from Lecture 7. This gives another model for the mapping spaces of a quasi-category C. Namely, the mapping spaces $\mathrm{Map}_{R\mathfrak{C}[C]}(Y,X) = \mathrm{Sing}\,|\mathrm{Map}_{\mathfrak{C}[C]}(Y,X)|$ of the fibrant simplicial category $R\mathfrak{C}[C]$ associated to C. Moreover, we have an easy fully faithful embedding

$$j : R\mathfrak{C}[C] \to \mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R\mathfrak{C}[C]^{\mathrm{op}}, \mathcal{K}\mathrm{an})$$

into the simplicial category of simplicial functors from Exercise 1. Additionally, there is a canonical comparison functor

$$\Phi : N\Big(\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R\mathfrak{C}[C]^{\mathrm{op}}, \mathcal{K}\mathrm{an})\Big) \to \mathrm{Fun}_{\mathcal{Q}\mathrm{Cat}}(C^{\mathrm{op}}, N\mathcal{K}\mathrm{an})$$

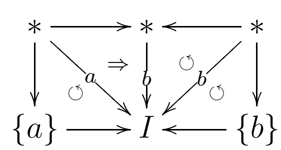
which sends a functor $f : R\mathfrak{C}[C]^{\mathrm{op}} \to \mathcal{K}\mathrm{an}$ to the adjoint $f^\dagger : C^{\mathrm{op}} \to N\mathcal{K}\mathrm{an}$ of the composition $\mathfrak{C}[C]^{\mathrm{op}} \to R\mathfrak{C}[C]^{\mathrm{op}} \to \mathcal{K}\mathrm{an}$. The problem is that:

$$\boxed{\Phi \text{ is not a categorical equivalence.}}$$

Example 2. Consider $C = \Lambda^2_2$. Since $\Lambda^2_2 = \Delta^1 \sqcup_{\Delta^0} \Delta^1$ and $\mathfrak{C}$ preserves colimits, one sees that $\mathfrak{C}[C]$ is the small category $\{0 \to 2 \leftarrow 1\}$ considered as a simplicial category. Since all mapping spaces are $\varnothing$ or $*$ we have $R\mathfrak{C}[C] = \mathfrak{C}[C]$. Choose any contractible Kan complex I with at least two distinct points $a, b \in I$, e.g., $\mathrm{Sing}\,\Delta^1_{\mathrm{top}}$, and consider the morphism of diagrams

$$\begin{array}{ccccc} \{a\} & \longrightarrow & I & \longleftarrow & \{b\} \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & * & \longleftarrow & * \end{array} \tag{1}$$

Since $a \neq b$, this morphism of diagrams has no inverse in $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{K}\mathrm{an})$. However, in $\mathrm{Fun}(C, N\mathcal{K}\mathrm{an})$ we do have an inverse. Note that a natural transformation is a map $C \times \Delta^1 \to N\mathcal{K}\mathrm{an}$ whose end points are our two diagrams. The quasi-category $\Lambda^2_2 \times \Delta^1$ is built from four nondegenerate two simplices. We send three of these to the obvious commutative triangles. For the fourth one, choose a map $\Delta^1 \to I$ with end points a and b (this exists because I is a contractible Kan complex). Then this defines a simplicial homotopy from $a \in I$ to $b \in I$, and therefore a two cell in $N\mathcal{K}\mathrm{an}$.



(2)

Apparent problem. The problem seems to be that $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{K}\mathrm{an})$ consists of functors which are strictly compatible with composition but $\mathrm{Fun}(C, N\mathcal{K}\mathrm{an})$ allows functors which are only preserve composition up to coherent homotopy.

$$\mathrm{Fun}(\mathfrak{C}[C], \mathcal{K}\mathrm{an})\text{``}=\text{''}\Big\{F \;\Big|\; F(g\circ f) = F(g)\circ F(f)\Big\}$$

$$\mathrm{Fun}(C, N\mathcal{K}\mathrm{an}) \text{“}=\text{”} \left\{ F \;\middle|\; F(g \circ f) \sim F(g) \circ F(f) \right\}$$

That is, we suspect that $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{K}\mathrm{an})$ is too small. However, this is not the true problem. It turns out that every functor in $\mathrm{Fun}(C, N\mathcal{K}\mathrm{an})$ can be *rigidified* to a functor which strictly preserves composition. For example, if I is a classical category, then for every morphism of quasi-categories $F : NI \to N\mathcal{K}\mathrm{an}$ is a morphism of simplicial categories $G : I \to N\mathcal{K}\mathrm{an}$ and an equivalence $F \cong NG$. This phenomenon is well studied in SGA the case of groupoids.

Actual problem. The actual problem is that not all equivalences in $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{K}\mathrm{an})$ are equivalences in $\mathrm{Fun}(C, N\mathcal{K}\mathrm{an})$. That is, $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{K}\mathrm{an})$ is too big!

Remark 3. For the reader who knows some homological algebra, we point out now that this is exactly the same phenomenon that there is a fully faithful inclusion

$$D(\mathbb{Z}) \cong K(\mathrm{Ch}(\mathbb{Z})_{\mathrm{proj}}) \subseteq K(\mathrm{Ch}(\mathbb{Z}))$$

of the derived category of abelian groups (considered as a triangulated category) into homotopy category of all chain complexes.

To describe this more accurately we introduce the following definition.

Definition 4. A morphism $\alpha : F \to G$ in $\mathrm{Fun}(I, \mathcal{K}\mathrm{an})$ is a weak equivalence if for each $i \in I$ the morphism of Kan complexes $F(i) \to G(i)$ is a weak equivalence.

Since $\mathrm{Fun}(I, \mathcal{K}\mathrm{an})$ is a simplicial category we also have a notion of *homotopy equivalence*, namely, a map which becomes an isomorphism in the homotopy category $h\,\mathrm{Fun}(I, \mathcal{K}\mathrm{an})$. Example 2 is an example of a general phenomenon that even though all homotopy equivalences are weak equivalences, not all weak equivalences are homotopy equivalences.

$$\{\text{ homotopy equivalence }\} \subsetneq \{\text{ weak equivalence }\}$$

Example 5.

1. Recall that in Lecture 6 we had the topologists sin curve X, and an inclusion $\{a, b\} \subseteq X$ which was a weak equivalence, but not a homotopy equivalence. Conversely, it can be show that every weak equivalence of CW complexes is a homotopy equivalence. Moreover, every topological space is weakly equivalent to a CW complex
$$|\,\mathrm{Sing}\, X| \overset{w.e.}{\to} X.$$
2. It can be shown that every weak equivalence of Kan complexes is a homotopy equivalence, however, there are certainly (Quillen) weak equivalences of simplicial sets which are not homotopy equivalences. On the other hand, every simplicial set is weakly equivalent to a Kan complex
$$K \overset{w.e.}{\to} \mathrm{Sing}\,|K|.$$

3. It can be shown that every weak equivalence of cofibrant simplicial rings is a homotopy equivalence, but in Lecture 7 we saw an example of weak equivalence of simplicial rings that had no inverse. We also saw in Lecture 7 that every simplicial ring is weakly equivalent to a cofibrant simplicial ring

$$P^{\Delta}(A) \stackrel{w.e.}{\to} A.$$

4. We have not introduced the category of chain complexes yet, but this would also be an example.

Question 6. What is the corresponding entry in the following table?

hom.equiv. $\subsetneq$ w.e.	hom.equiv = w.e.	resolution functor
Top	CW	$\lvert \operatorname{Sing} - \rvert$
$\mathcal{S}\mathrm{et}_{\Delta}$	$\mathcal{K}\mathrm{an}$	$\operatorname{Sing} \lvert - \rvert$
$\mathcal{R}\mathrm{ing}_{\Delta}$	$\mathcal{R}\mathrm{ing}_{\Delta}^{\mathrm{cof}}$	$P^{\Delta}(-)$
$\mathrm{Ch}(\mathbb{Z})$	$\mathrm{Ch}(\mathbb{Z})_{\mathrm{proj}}$	projective resolution
$\mathrm{Fun}(I, \mathcal{K}\mathrm{an})$	?	?

2 Model categories

One language used to work in the above situation is the language of *model categories*.

Definition 7 (Quillin 1967, [Hirschhorn, Def.7.1.3, Def.9.1.6], [HTT, Def.A.3.1.5])**.** A *model category* is a category $\mathcal{M}$ equipped with three classes of morphisms $\mathcal{C}, \mathcal{W}, \mathcal{F}$ called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying five axioms which we will introduce as we need them. An object X is called *fibrant* if $X \to *$ is a fibration, and *cofibrant* if $\varnothing \to X$ is a cofibration.

Example 8. In the Quillen model structure on the category $\mathcal{S}\mathrm{et}_{\Delta}$ of simplicial sets:

($\mathcal{W}$) The weak equivalences are weak equivalences, i.e., morphisms $K \to L$ such that $\pi_0|K| \cong \pi_0|L|$ and $\pi_n(|K|, k) \cong \pi_n(|L|, fk)$ for all n, k.

($\mathcal{C}$) The cofibrations are monomorphisms.

($\mathcal{F}$) The fibrations are Kan fibrations.

($\mathcal{M}^{\mathrm{cf}}$) All objects are cofibrant. The fibrant objects are Kan complexes.

Example 9. In a canonical model structure on the category $\mathcal{R}\mathrm{ing}_{\Delta}$ of simplicial rings:

($\mathcal{W}$) The weak equivalences are weak equivalences, i.e., morphisms $A \to B$ such that the underlying morphism of simplicial sets $UA \to UB$ is a weak equivalence.

($\mathcal{C}$) A morphism is called *cellular* if it is a colimit of the form

$$A \to B = \operatorname{colim}(A = A(-1) \to A(0) \to A(1) \to \dots)$$

were each morphism is of the form

$$\begin{array}{ccc} \bigotimes_{I_n} \mathbb{Z}[\partial \Delta^n] & \xrightarrow{g_n} & A(n-1) \\ \downarrow & & \downarrow \\ \bigotimes_{I_n} \mathbb{Z}[\Delta^n] & \longrightarrow & A(n) \end{array}$$

for some set I_n and morphism g_n. Cofibrations are retracts of cellular morphisms.

($\mathcal{F}$) Recall that $\pi_0 A = \mathrm{coker}(A_1 \stackrel{d_0-d_1}{\to} A_0)$. A morphism $A \to B$ is a fibration if

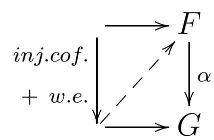
$$A_n \to \pi_0 A \times_{\pi_0 B} B_n$$

is surjective for all n.

($\mathcal{M}^{\mathrm{cf}}$) Every simplicial ring is fibrant. Cofibrant simplicial rings were introduced last week.

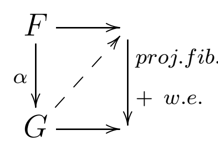
Example 10 (cf. [HTT, Def.A.3.3.1])**.** Suppose $\mathcal{M}$ is a model category and I a small category. A morphism $\alpha : F \to G$ in $\mathrm{Fun}_{\mathcal{C}\mathrm{at}}(I, \mathcal{M})$ is called:

1. a *weak equivalence* if $\alpha(i) : F(i) \to G(i)$ is a weak equivalence for each $i \in I$,
2. an *injective cofibration* if $\alpha(i) : F(i) \to G(i)$ is a cofibration for each $i \in I$,
3. an *injective fibration* if it has the right lifting property with respect to all γ which are both a weak equivalence and injective cofibration, that is, for every commutative square



there exists a diagonal morphism making two commutative triangles.

4. a *projective fibration* if $\alpha(i) : F(i) \to G(i)$ is a fibration for each $i \in I$.
5. an *projective cofibration* if it has the left lifting property with respect to all γ which are both weak equivalence and projective fibration, that is, for every commutative square



there exists a diagonal morphism making two commutative triangles.

Remark 11. The above example has a simplicial version too, where I and $\mathcal{M}$ are simplicial categories and we consider simplicial functors $\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(I, \mathcal{M})$.

Remark 12. All of the above model categories are combinatorial model categories, cf. [HTT, A.3.3.2]. Indeed, every model category we see in this course will be combinatorial.

Remark 13. There is a very large amount of abstract homotopy theory that we are not mentioning here. Basically, there notion of a model category is an abstraction of many situations where one can "do homotopy theory". In particular, in model categories there is a very robust notion of *homotopy equivalence*. One consequence of the lifting properties described above is that weak equivalence and homotopy equivalence will agree on the subcategory $\mathcal{M}^{\mathrm{cf}}$ of fibrant-cofibrant objects.

Clearly, in general, the cofibrant-fibrant objects[1] in $\mathrm{Fun}(I, \mathcal{M})$ will be difficult to describe, but in some nice cases we can give a complete characterisation.

Exercise 14. Let I be a small category and $\mathcal{M}$ a model category. An easy consequence of the axioms of a model category is :

($*$) an object is cofibrant if and only if for every $f : X \to Y$ in $\mathcal{W} \cap \mathcal{F}$, and morphism $A \to Y$, there exists a factorisation $A \dashrightarrow X \to Y$.

For $i \in I$ and a cofibrant object $A \in \mathcal{M}^{\mathrm{cof}}$ let $i_! A \in \mathrm{Fun}(I, \mathcal{M})$ be the functor $j \mapsto \sqcup_{\hom(i,j)} A$. Show that $i_! A$ is projectively cofibrant. Hint.[2]

Exercise 15. Recall that for a simplicial set K we wrote $\mathrm{sk}_n K \subseteq K$ for the smallest subsimplicial set containing all non-degenerate i-simplicies for $i \leq n$. Admitting the fact that pushouts of cofibrations are cofibrations, and coproducts of cofibrations are cofibrations, show that the canonical morphism

$$\mathrm{sk}_{n-1} N(I_{/-}) \to \mathrm{sk}_n N(I_{/-})$$

in $\mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)$ is a projective cofibration. Hint.[3] Hint.[4]

Exercise 16. Admitting that a colimit $\mathrm{colim}(\varnothing \to A(0) \to A(1) \to \dots)$ of cofibrations is cofibrant, using the previous exercises show that for any small category I, the diagram $i \mapsto N(I_{/i})$ in $\mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)$ is projectively cofibrant.

Example 17. Consider the case $I{=}\Lambda^2_2$ and $\mathcal{M}{=}(\mathcal{S}\mathrm{et}_\Delta)_{\mathrm{Quillen}}$. An object

$$\begin{array}{ccc} & & X_1 \\ & & \downarrow \\ X_0 & \rightarrow & X_2 \end{array}$$

in $\mathrm{Fun}(\Lambda^2_2, \mathcal{S}\mathrm{et}_\Delta)$ is:

1. always *injectively cofibrant*,
2. *injectively fibrant*[5] if and only if $X_2 \in \mathcal{K}\mathrm{an}$ and both morphisms are Kan fibrations.
3. *projectively cofibrant*[6] if and only if $X_0 \sqcup X_1 \to X_2$ is a monomorphism.
4. *projectively fibrant* if and only if $X_0, X_1, X_2 \in \mathcal{K}\mathrm{an}$.

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[1] Recall that an object X is *cofibrant* if the canonical morphism $\varnothing \to X$ from the initial object is a cofibration and *fibrant* if the canonical morphism $X \to *$ to the terminal object is a fibration.

[2] Note that the functor $i_!$ is left adjoint to the evaluation-at-i functor $i^* : \mathrm{Fun}(I, \mathcal{M}) \to \mathcal{M}$; $p \mapsto p(i)$.

[3] Use Exercise 14 applied to the cofibrations $\partial\Delta^n \to \Delta^n$.

[4] Note also that every n-simplex $(i_n{\to}\dots{\to}i_0{\to}i)$ in $N(I_{/i})_n$ can be written uniquely as the image of a simplex of the form $(i_n{\to}\dots{\to}i_0{=}i_0)$ in $N(I_{/i_0})_n$. So in particular, $N(I_{/i})_n = \coprod_{\hom(i_0,i)} \{(i_n{\to}\dots{\to}i_0{=}i_0) \in N(I_{/i_0})_n\}$.

[5] More generally, a morphism $\alpha : X \to Y$ is an injective fibration if and only if $\alpha_2 : X_2 \to Y_2$ is a Kan fibration, and $X_\varepsilon \to Y_\varepsilon \times_{Y_2} X_2$ are Kan fibrations for $\varepsilon = 0, 1$.

[6] More generally, a morphism $\alpha : A \to B$ is a projective cofibration if and only if $\alpha_\varepsilon : A_\varepsilon \to B_\varepsilon$ are monomorphisms for $\varepsilon = 0, 1$ and $B_0 \sqcup_{A_0} A_2 \sqcup_{A_1} B_1 \to B_2$ is a monomorphism.

inj.cof.	no conditions
inj.fib.	$\substack{X\to Y\\ Z\to Y}$ Kan fibrations, Y Kan complex
proj.cof.	$X \sqcup Z \to Y$ monomorphism
proj.fib.	X, Y, Z Kan complexes

3 Yoneda

Now we want to construct the Yoneda functor

$$j : C \to \mathrm{Fun}(C^{\mathrm{op}}, N\mathcal{K}\mathrm{an})$$

associated to a quasi-category C.

Construction 18. To begin with note that for any simplicial sets K, L, applying $\mathfrak{C}$ to the canonical projections $K \times L \to K$ and $K \times L \to L$ produces a simplicial functor

$$\Phi : \mathfrak{C}[K \times L] \to \mathfrak{C}[K] \times \mathfrak{C}[L].$$

Construction 19 ([HTT, §5.1.3, pg.317])**.** Let C be a simplicial set and $\mathfrak{C}[C] \to R\mathfrak{C}[C]$ a categorical equivalence towards a fibrant simplicial category. Consider the simplicial Yoneda functor

$$j : R\mathfrak{C}[C] \to \mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R\mathfrak{C}[C]^{\mathrm{op}}, \mathcal{K}\mathrm{an})$$

Note that for any fixed object X, the functor $j(X)$ is projectively fibrant because $R\mathfrak{C}[C]$ is fibrant, and projectively cofibrant because it is representable.

Composing the adjoint $j^\dagger : R\mathfrak{C}[C]^{\mathrm{op}} \times R\mathfrak{C}[C] \to \mathcal{K}\mathrm{an}$ with $\mathfrak{C}[C] \to R\mathfrak{C}[C]$ and Φ we obtain three functors which determine each other by adjunction.

$$\begin{array}{rl} \mathfrak{C}[C^{op}\times C] \to \mathfrak{C}[C]^{op}\times\mathfrak{C}[C] \to R\mathfrak{C}[C]^{\mathrm{op}}\times R\mathfrak{C}[C] \to \mathcal{K}\mathrm{an} & \in \mathcal{C}\mathrm{at}_\Delta \\ C^{op}\times C \to N(\mathcal{K}\mathrm{an}) & \in \mathcal{C}\mathrm{at}_\infty \\ C \to \mathrm{Fun}_{\mathcal{S}\mathrm{et}_\Delta}(C^{op}, N\mathcal{K}\mathrm{an}) & \in \mathcal{C}\mathrm{at}_\infty. \end{array}$$

The last one of these is the quasi-categorical Yoneda functor.

Theorem 20 ([HTT, 5.1.3.1])**.** *Let C be a quasi-category. The Yoneda functor constructed above is fully faithful. Explicitly, for all objects X, Y of C the induced map*

$$\mathrm{Map}^R_C(X, Y) \to \mathrm{Map}^R_{\mathrm{Fun}(C^{\mathrm{op}}, N\mathcal{K}\mathrm{an})}(jX, jY)$$

is an equivalence.

Sketch of proof. If one follows the adjunctions around, one can see that the Yoneda functor factors as

$$C \xrightarrow{j'} N(\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R\mathfrak{C}[C]^{\mathrm{op}}, \mathcal{K}\mathrm{an})^{\mathrm{cf}}) \xrightarrow{j''} \mathrm{Fun}_{\mathcal{S}\mathrm{et}_\Delta}(C^{op}, N\mathcal{K}\mathrm{an}).$$

We claim that both of these are fully faithful. To show j' is fully faithful, it suffices to show that the adjoint

$$\mathfrak{C}[C] \to R\mathfrak{C}[C] \to \mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R\mathfrak{C}[C]^{\mathrm{op}}, \mathcal{K}\mathrm{an})^{\mathrm{cf}}$$

is fully faithful. The first map is a categorical equivalence by assumption, and the second map is the simplicial Yoneda. To show that j'' is an equivalence is a serious business contained in the following theorem.

Theorem 21 ([HTT, Prop.4.2.4.4])**.** *Let K be a simplicial set, and $u : \mathfrak{C}[C] \to R$ an equivalence of simplicial categories. Then the induced map*

$$N(\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R, \mathcal{K}\mathrm{an})^{\mathrm{cf}}) \to \mathrm{Fun}_{\mathcal{S}\mathrm{et}_\Delta}(K, N\mathcal{K}\mathrm{an})$$

is a categorical equivalence of simplicial sets.

□

Exercise 22. Prove the claim in the proof that the quasi-categorical Yoneda functor $j : C \to \mathrm{Fun}_{\mathcal{Q}\mathrm{Cat}}(C^{\mathrm{op}}, N\mathcal{K}\mathrm{an})$ factors through the nerve of the simplicial Yoneda functor $NR\mathfrak{C}[C] \to N\,\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(R\mathfrak{C}[C]^{\mathrm{op}}, \mathcal{K}\mathrm{an})^{\mathrm{cf}}$.

Remark 23. One case we are particularly interested in is the case $S = N\mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cf}}$. In this case we can take $u : \mathfrak{C}[S] \to C$ to be the canonical equivalence $\mathfrak{C}[N\mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cf}}] \to \mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cf}}$. Then the theorem says that

$$N\,\mathrm{Fun}_{\mathcal{C}\mathrm{at}_\Delta}(\mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cof}}, \mathcal{K}\mathrm{an})^{\mathrm{cof}} \to \mathrm{Fun}_{\mathcal{S}\mathrm{et}_\Delta}(N\mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cof}}, N\mathcal{K}\mathrm{an})$$

is a categorical equivalence. In particular, this says that any derived scheme (which we haven't defined yet) can be represented by a projectively cofibrant simplicial functor $X : \mathcal{R}\mathrm{ing}_\Delta^{\mathrm{cf}} \to \mathcal{K}\mathrm{an}$. Affine schemes are the corepresentable functors; $\mathrm{Map}_{\mathcal{R}\mathrm{ing}_\Delta}(A, -)$ with A cofibrant.