Derived Algebraic Geometry
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Lecture 8: Limits in ∞-categories

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In this lecture we discuss limits in ∞-categories. In the first section we define limits in quasi-categories. Many quasi-categories of interest come from simplicial model categories, and as such, we also discuss weighted limits in simplicial categories. We finish with a list of properties with reference to [HTT]. The comparison between limits in quasi-categories and simplicial categories will be done next week.

1. limits in quasi-categories,
2. weighted limits in simplicial categories,
3. properties.

In the examples, we will usually focus on pullbacks, but the reader may like to keep in mind the following commonly occurring limits.

Limit	Indexing category	
Terminal object	$\varnothing$	
Product	$\{i_0, i_1, i_2, \dots\}$	a discrete category[1]
Pullback	$\{{}^0 \searrow_2 \swarrow {}^1\}$	$= \Lambda^2_2$
Equaliser	$\{0 \rightrightarrows 1\}$	$= \Delta^1 \sqcup_{\partial\Delta^1} \Delta^1$
Sequential limits	$\{\dots \to 2 \to 1 \to 0\}$	$= \mathbb{N}^{\mathrm{op}}$
Fat totalisation	$\{[0] \rightrightarrows [1] \substack{\rightarrow\\\rightarrow\\\rightarrow} [2] \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \dots\}$	$= \Delta^{monic}$
Global sections		$\mathcal{O}\mathrm{pen}(X)$ for some $X \in \mathrm{Top}$.

1 Limits in quasi-categories

Recall that for, classical category theory, the limit of a diagram $p : I \to C$ is a final object in the over category $C_{/p}$.[2] As such, we begin with final objects.

Definition 1 ([HTT, Prop.1.2.12.4])**.** Let C be a quasi-category. An object $X \in C_0$ is *final* if $\hom^R_C(Y, X)$ is contractible for all $Y \in C_0$.

Remark 2 ([HTT, Prop.1.2.12.9])**.** Let C be a quasi-category, and C' the full subcategory of C spanned by the final vertices of C. Then C' is either empty, or is a contractible Kan complex. That is, any two final objects are equivalent, and any two equivalences are equivalent, and any two equivalences of equivalences are equivalent, and...

[2] Objects of $C_{/p}$ are pairs (X, ξ) consisting of an object $X \in C$ and a natural transformation $\gamma X \to p$ where γX is the constant diagram $i \mapsto X$. Morphisms $(X, \xi) \to (Y, \eta)$ are those morphisms $X \to Y$ of C which make a commutative triangle $\gamma X \to \gamma Y \to p$.

Example 3. The final objects of $N\mathcal{K}\text{an}$ are the contractible Kan complexes. The final objects of $N\mathcal{R}\text{ing}_\Delta^{\text{cof}}$ are those cofibrant simplicial rings whose homology groups are zero.

Exercise 4. Let C be a small category. Show that $X \in NC_0$ is final in the quasi-categorical sense if and only if it is final in the classical sense. I.e., there exists a unique morphism $Y \to X$ for every $Y \in C_0$.

Exercise 5. Recall that an exercise in Lecture 3 was to construct an isomorphism of simplicial sets $\text{Map}^R_{\text{Sing}\,X}(x,y) \cong \text{Sing}\,PX(x,y)$ associated to a topological space X where $PX(x,y) \subseteq \hom_{\text{Top}}(\Delta^1_{\text{top}}, X)$ is the subspace of paths from x to y. Using the facts[3] that:

1. for any topological space Y the natural transformation $|\,\text{Sing}\,Y| \to Y$ is always a weak equivalence, and
2. a Kan complex is contractible if and only if all its homotopy groups are trivial,
3. there exist isomorphisms $\pi_n(PX(x,y),\gamma) \cong \pi_{n+1}(X,x)$ for all $n \geq 0$, $x, y \in X$, $\gamma \in PX(x,y)$,

show that $\text{Sing}\,X$ admits a final object if and only if X is weakly equivalent to a point $*$, in which case every object of $\text{Sing}\,X$ is final.

Now we want to define the category $C_{/p}$ over a diagram $p : I \to C$. The n-simplicies of $C_{/p}$ will be maps from $\Delta^n \star I$, where $- \star -$ is a kind of cylinder construction.

Note that Δ is equipped with an operation

$$\sqcup : \Delta \times \Delta \to \Delta$$

that sends finite linearly ordered sets $I = \{i_0 < \cdots < i_n\}$ and $I' = \{i'_0 < \cdots < i'_{n'}\}$ to $I \sqcup I' := \{i_0 < \cdots < i_n < i'_0 < \cdots < i'_{n'}\}$.

Definition 6 ([HTT, Def.1.2.8.1])**.** Let K, L be simplicial sets. For any linearly ordered set J we define

$$(K \star L)_J := \coprod_{J = I \sqcup I'} K_I \times L_{I'}$$

In the case I or I' is empty, we set $K_\varnothing = \{*\} = L_\varnothing$ to be a single element set.

Given a morphism $p : J \to J'$ of linearly ordered sets and a decomposition $J' = I \sqcup I'$, there is an induced decomposition $J = p^{-1}I \sqcup p^{-1}I'$, and an induced morphism

$$K_I \times L_{I'} \to K_{p^{-1}I} \times L_{p^{-1}I'}.$$

These fit together to define morphisms

$$p^* : (K \star L)_{J'} \to (K \star L)_J$$

giving $K \star L$ the structure of a simplicial set. In the case $K = \Delta^0$ we write

$$L^{\triangleleft} := \Delta^0 \star L.$$

[3] The first two facts are theorems. The third is possible to prove directly.

Exercise 7.

1. Show that $K \star \varnothing = K = \varnothing \star K$ for any $K \in \mathcal{S}\mathrm{et}_\Delta$.
2. Show that $\Delta^0 \star \Delta^n \cong \Delta^{n+1} \cong \Delta^n \star \Delta^0$. More generally, show that
$$\Delta^{i-1} \star \Delta^{j-1} \cong \Delta^{(i+j)-1}.$$
3. Suppose that P, Q are partially ordered sets. Consider the coarsest partial order on $P \sqcup Q$ such that $P, Q \to P \sqcup Q$ are both morphisms of partially ordered sets, and such that $p \leq q$ for all $(p, q) \in P \times Q$. Show that
$$N(P \sqcup Q) = N(P) \star N(Q).$$
4. Deduce that there is a pushout square in $\mathcal{S}\mathrm{et}_\Delta$
$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{d_{n+1}} & \Delta^{n+2} \\ {\scriptstyle d_{n+1}}\downarrow & & \downarrow \\ \Delta^{n+2} & \longrightarrow & \Delta^n \star \Lambda^2_2 \end{array}$$

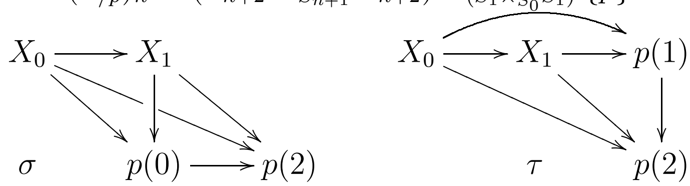
Definition 8 (Joyal, [HTT, Prop.1.2.9.2])**.** Let $p : I \to S$ be a morphism of simplicial sets, define
$$(S_{/p})_n = \{f : \Delta^n \star I \to S \ : \ f|_I = p\}.$$
Note, this is functorial in $[n] \in \Delta$, so defines a simplicial set $S_{/p}$. Moreover, there is a canonical projection morphism $S_{/p} \to S$.

Exercise 9.

1. Let X be a topological space, and give $\hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ the compact-open topology. Let $x \in X$ be a point and consider the subspace $X_{/x} \subseteq \hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ of those $\gamma : \Delta^1_{\mathrm{top}} \to X$ such that $\gamma((1,0)) = x$. Show that
$$\mathrm{Sing}(X_{/x}) = (\mathrm{Sing}\, X)_{/x}.$$
2. Let $p : I \to C$ be a functor between 1-categories. Show that $C_{/p}$ is the 1-category of cones over p. That is, the category whose objects are collections of morphisms $(\xi_i : X \to p(i))_{i \in ObI}$ such that the triangles
$$\begin{array}{ccc} & X & \\ {\scriptstyle \xi_j}\swarrow & & \searrow{\scriptstyle \xi_i} \\ p(i) & \xrightarrow[p(u)]{} & p(j) \end{array}$$
commute for each $i \xrightarrow{u} j$ and whose morphisms $(X, \xi) \to (Y, \eta)$ are those morphisms $f : X \to X'$ such that the triangles
$$\begin{array}{ccc} X & \xrightarrow[f]{} & X' \\ {\scriptstyle \xi_i}\searrow & & \swarrow{\scriptstyle \eta_i} \\ & p(i) & \end{array}$$
commute for each i.

Exercise 10. Let $p : \Lambda^2_2 \to S$ be a morphism of simplicial sets. Using Exercise 7 show that $(S_{/p})_n$ can be identified with the set of pairs of $n{+}2$-simplicies $(\sigma, \tau) \in S^2_{n+2}$ whose $\{0, 1, 2, \ldots, n, n{+}2\}$-faces agree, and whose $\{n{+}1, n{+}2\}$-edges are the two edges of p. That is, such that $d_{n+1}\sigma = d_{n+1}\tau$, and $d_0^{n+1}\sigma = p(0{\to}2)$, $d_0^{n+1}\sigma = p(1{\to}2)$.

$$(S_{/p})_n = (S_{n+2} \times_{S_{n+1}} S_{n+2}) \times_{(S_1 \times_{S_0} S_1)} \{p\}$$



Definition 11 ([HTT, Def.1.2.13.4])**.** Let C be a quasi-category and $p : I \to C$ a morphism of simplicial sets. A *limit* for p is a final object in $C_{/p}$.

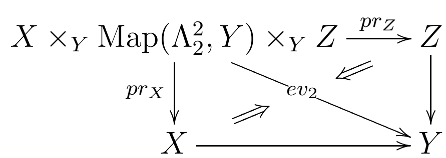
Remark 12 ([HTT, 1.2.13.5])**.** Note that an object in $C_{/p}$ is a map $\Delta^0 \star K = K^{\triangleleft} \to C$. Restricting to Δ^0, we obtain an object $\Delta^0 \to C$ of C. One says that $K^{\triangleleft} \to C$ is a *limit diagram* if it is a limit of p, and abuse terminology be referring to $\Delta^0 \to C$ as a limit of p. We use the notation

$$\lim(p)$$

By Remark 2 the space of limit diagrams is empty or contractible. That is, any two choices of limit are equivalent, and any two equivalences are equivalent, and etc, etc.

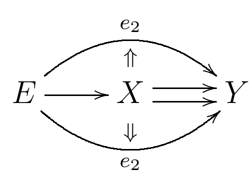
Example 13 (Products)**.** If $\{X_i\}_{i \in I}$ is a collection of Kan complexes then their product $\prod_{i \in I} X_i$ in $\mathcal{K}$an is a model for their product in the quasi-category $N\mathcal{K}$an. In particular, any contractible Kan complex is a terminal object of $N\mathcal{K}$an.

Example 14 (Pullbacks)**.** Suppose $X \xrightarrow{f} Y \leftarrow Z$ are two morphisms in $\mathcal{K}$an. The Kan complex $X \times_Y \mathrm{Map}(\Lambda^2_2, Y) \times_Y Z$, or more accurately, the obvious map $\Delta^1 \times \Delta^1 \to N\mathcal{K}$an, is a limit for the corresponding diagram $p : \Lambda^2_2 \to N\mathcal{K}$an in the quasi-category $N\mathcal{K}$an.



Notice that the outside square is not commutative in the classical category $\mathcal{K}$an. It is a commutative square in the quasi-category $N\mathcal{K}$an formed from two non-trivial 2-simplicies.

Example 15 (Equalisers)**.** Suppose that $f, g, : X \rightrightarrows Y \in \mathcal{K}$an are parallel morphisms between Kan complexes. Then $E = \mathrm{eq}(X \times \mathrm{Map}(\Lambda^2_2, Y) \rightrightarrows Y \times Y)$ is a limit for the corresponding diagram in $\mathcal{K}$an. Here, the two maps are $(x, \gamma) \mapsto (f(x), g(x))$ and $(x, \gamma) \mapsto (\gamma_0, \gamma_1)$.



The map e_2 is induced by the evaluation at $2 \in \Lambda^2_2$ map $\mathrm{Map}(\Lambda^2_2, Y) \to Y$.

Example 16 (Towers)**.** Suppose that $\cdots \to X(2) \to X(1) \to X(0)$ is a sequence of morphisms in $\mathcal{K}\mathrm{an}$. For $n \in \mathbb{N}$ consider the totally ordered set $\mathbb{N}_{\geq n} = \{m \in \mathbb{N} \mid m \geq n\}$ and the opposite of its nerve $N\mathbb{N}^{\mathrm{op}}_{\geq n}$. For each Kan complex $X(j)$ we can consider the mapping Kan complex $M_{i,j} := \mathrm{Map}(N\mathbb{N}^{\mathrm{op}}_{\geq i}, X(j))$. The maps $X(j{+}1) \to X(j)$ induce maps $M_{i,j+1} \to M_{i,j}$ and the canonical inclusions $\mathbb{N}_{\geq i} \subseteq \mathbb{N}_{\geq i-1}$ induce maps $M_{i-1,j} \to M_{i,j}$. Combining these we get an object

$$T = \cdots \times_{M_{3,2}} M_{2,2} \times_{M_{2,1}} M_{1,1} \times_{M_{1,0}} M_{0,0}$$

We claim that T is a limit for the diagram $X : \mathbb{N}^{\mathrm{op}} \to N\mathcal{K}\mathrm{an};\ n \mapsto X(n)$ in the quasi-category $N\mathcal{K}\mathrm{an}$.

In the coming sections we explain why the above examples calculate the limits.

2 Weighted limits in simplicial categories

Recall that to a 2-functor $p : I \to \mathcal{C}\mathrm{at}$ from a small category to the category of categories, we associated a category $2\text{-}\lim p$ called the *2-limit*. For example, given two functors $A \overset{f}{\to} B \overset{g}{\leftarrow} C$, objects of the *2-pullback* were tuples $(a, fa \overset{\sim}{\to} b \overset{\sim}{\leftarrow} gc, c)$ with a, b, c in A, B, C respectively. In this section we consider a version of this procedure where $\mathcal{C}\mathrm{at}$ is replaced by a simplicial category.[4]

Definition 17. A simplicial category C is said to be *powered* over $\mathcal{S}\mathrm{et}_\Delta$ if for every $X, Y \in C$, $K \in \mathcal{S}\mathrm{et}_\Delta$, the functor

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Map}_C(-, Y)) : C^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$$

is representable. That is, if for all $Y \in C, K \in \mathcal{S}\mathrm{et}_\Delta$ there exists an object Y^K, and a natural transformation

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Map}_C(-, Y)) \cong \mathrm{Map}_C(-, Y^K)$$

Example 18.

1. The simplicial category $\mathcal{S}\mathrm{et}_\Delta$ is powered over itself with $X^K = \mathrm{Map}(K, X)$.
2. The simplicial category $\mathcal{R}\mathrm{ing}_\Delta$ is powered over $\mathcal{S}\mathrm{et}_\Delta$ with $(X^K)_n = \prod_{k \in K_n} X_n$.

Exercise 19. Confirm that these really do define powerings.

Definition 20 (Cf.[Bousfield, Kan, XI.3.1, XII.2.1], [Hirschorn Def.18.1.2, Def.18.1.8])**.** Suppose that C is a $\mathcal{S}\mathrm{et}_\Delta$-powered simplicial category whose underlying category C_0[5] admits all small limits, let I be a small category and $p : I \to C_0$ a functor.

[4] Note that $\mathcal{C}\mathrm{at}$ has a structure of simplicial category if we set $\mathrm{Map}(C, D) = N\,\mathrm{Fun}(C, D)$. If we replace the category $\mathrm{Fun}(C, D)$ of functors and natural transformations with the groupoid $\mathrm{Fun}^{\cong}(C, D)$ of functors and natural *isomorphisms*, then $\mathcal{C}\mathrm{at}$ is a fibrant simplicial category.

[5] Recall that when C is a simplicial category, the notation C_0 means the classical category with $Ob\ C = Ob\ C_0$ and $\hom_{C_0}(X, Y) = \mathrm{Map}(X, Y)_0$.

The *weighted limit*, with respect to a functor $W : I \to \mathcal{S}\mathrm{et}_\Delta$, is defined as

$$\lim^W p = \mathrm{eq}\left(\prod_{i \in Ob\ I} p_i^{W_i} \rightrightarrows \prod_{\substack{i \xrightarrow{u} j \\ \in Arr\ I}} p_j^{W_i} \right)$$

where the two morphisms are induced by

$$\begin{aligned} p_u^{W_{\mathrm{id}}} &: p_i^{W_i} \to p_j^{W_i}, \\ p_{\mathrm{id}}^{W_u} &: p_i^{W_j} \to p_i^{W_i}. \end{aligned}$$

Exercise 21. Show that if W is the constant functor with value $* \in \mathcal{S}\mathrm{et}_\Delta$ then $\lim^W = \lim$. That is, in this case the weighted limit is the same as the usual (co)limit in the classical category $\mathcal{C}_0$.

Example 22. The canonical choice for the weighting $W : I \to \mathcal{S}\mathrm{et}_\Delta$ is to take the nerve of the over categories

$$W(i) = N(I_{/i}).$$

Given $i \to j$, the morphism $N(I_{/i}) \to N(I_{/j})$ sends $(i_n \to \dots \to i_0 \to i)$ to $(i_n \to \dots \to i_0 \to j)$ where the last morphism is the composition $i_0 \to i \to j$.

In the case of Λ^2_2 this gives the diagram $\{0\} \to \{0 \to 2 \leftarrow 1\} \leftarrow \{1\}$. So with this weighting, the weighted limit of a diagram $A \to B \leftarrow C$ is

$$\mathrm{eq}\left(A \times B^{\Lambda^2_2} \times C \rightrightarrows B \times B \right)$$

where one arrow is the combination of $A \to B$ and $B^{\Lambda^2_2} \to B^{\{0\}} \cong B$ and the other is the combination of $C \to B$ and $B^{\Lambda^2_2} \to B^{\{1\}} \cong B$.

Remark 23. We will see next week that weighted limits in the simplicial categories $\mathcal{K}\mathrm{an}$ and $\mathcal{R}\mathrm{ing}_\Delta$ can be used to calculate limits in the quasi-categories $N\mathcal{K}\mathrm{an}$ and $N\mathcal{R}\mathrm{ing}^{\mathrm{cof}}_\Delta$.

3 Main properties

We now summarise the main properties of (co)limits. All proofs are omitted but we give references to [HTT] for the interested reader.

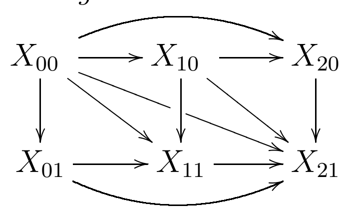
Proposition 24 ([HTT, Lem.4.3.2.12, Cor.4.3.2.16])**.** *If C, I be quasi-categories, and suppose C admits all I-shaped limits, then $\lim_I$ is functorial. More precisely, there exists a section*

$$\sigma : \mathrm{Fun}(I, C) \to \mathrm{Fun}(I^{\triangleleft}, C)$$

to the canonical restriction such that for each $p \in \mathrm{Fun}(I, C)$ the cone $\sigma(p)$ is a limit diagram for p.

Remark 25. The idea is to apply [HTT, Lem.4.3.2.12] to the inclusion $I \subseteq I^{\triangleleft}$ and the categorical fibration $C \to *$. Then let $K \subseteq \mathrm{Fun}(I^{\triangleleft}, C)$ be the full subcategory whose objects are limit diagrams and $K^0 \subseteq \mathrm{Fun}(I, C)$ the full subcategory whose objects admit limit diagrams. Then [HTT, Lem.4.3.2.12] says precisely that $K \to K^0$ is a weak Kan fibration of simplicial sets. Therefore it admits a section. If all I-diagrams admit limits, then $K^0 = \mathrm{Fun}(I, C)$ and we get the section $\sigma : \mathrm{Fun}(I, C) = K^0 \to K \subseteq \mathrm{Fun}(I^{\triangleleft}, C)$.

Proposition 26 ([HTT, Lem.4.4.2.1] 2-out-of-3 for Cartesian squares)**.** *Let $C \in \mathcal{Q}\mathrm{Cat}$ and $X : \Delta^2 \times \Delta^1 \to C$ a diagram:*



Suppose that the right square is a pullback in C. Then the left square is a pullback if and only if the outer square is a pullback.

Definition 27. We say a diagram $p : K \to C$ is *finite* or $\aleph_0$*-small* if the simplicial set K has finitely many non-degenerate[6] simplicies. More generally, if κ is an uncountable regular cardinal[7] a diagram is called κ*-small* if each K_n is in $\mathcal{S}\mathrm{et}_{<\kappa}$.

Proposition 28 ([HTT, Prop.4.4.2.6, Prop.4.4.3.2])**.** *Let C be a quasi-category. The following are equivalent.*

1. *C has all κ-small limits.*
2. *C has equalisers and all κ-small products.*
3. *C has pullbacks and all κ-small products.*

Remark 29. The main tools in the above proposition are:

1. If $L' \to L$ is a monomorphism of simplicial sets, $L' \to K'$ any morphism, and $p : K' \sqcup_{L'} L \to C$ a diagram, then $\lim p = \lim p|_{K'} \times_{\lim p|_{L'}} \lim p|_L$, assuming all these limits exist, [HTT, Prop.4.4.2.2].
2. If $\{K_\alpha\}_{\alpha \in A}$ is a collection of simplicial sets and $p : \sqcup K_\alpha \to C$ is a diagram, then $\lim p = \prod \lim p|_{K_\alpha}$, assuming all those limits exist.

Proposition 30 ([HTT, Cor.5.1.2.3] Limits of presheaves are calculated object wise)**.** *Let $K, S \in \mathcal{S}\mathrm{et}_\Delta$ and suppose $C \in \mathcal{Q}\mathrm{Cat}$ admits K-indexed limits. Then*

1. *The quasi-category $\mathrm{Fun}(S, C)$ admits K-indexed limits.*
2. *A map $K^{\triangleleft} \to \mathrm{Fun}(S, C)$ is a limit diagram if and only if for each vertex $s \in S$, the induced map $K^{\triangleleft} \to C$ is a limit diagram.*

That is, for $F : K \to \mathrm{Fun}(S, C)$ and $s \in S_0$ we have

$$(\lim_K F_k)(s) = \lim_K (F_k(s)).$$

[6]Recall a simplex $\sigma \in K_n$ is non-degenerate if it is not in the image of any $K_{n-1} \to K_n$.

[7]A cardinal κ is *regular* if $I \in \mathcal{S}\mathrm{et}_{<\kappa}$ and $\{K_i\}_{i \in I} \subseteq \mathcal{S}\mathrm{et}_{<\kappa}$ implies $\mathrm{colim}_{i \in I} K_i \in \mathcal{S}\mathrm{et}_{<\kappa}$ where $\mathcal{S}\mathrm{et}_{<\kappa}$ is the category of sets of size $< \kappa$.

Proposition 31 ([HTT, 5.1.3.2], Yoneda preserves limits)**.** *Let $C \in \mathcal{Q}\mathrm{Cat}$ be a small quasi-category and $j : C \to \mathrm{Fun}(C^{op}, N\mathcal{K}\mathrm{an})$ the Yoneda embedding (that we will define in the next lecture). Then j preserves all small limits which exists in C,*

$$j \lim = \lim j.$$

Remark 32. It follows from the previous two propositions that (if the limits exist), for any object $X \in C$ and $p : I \to C$ we have

$$\mathrm{Map}(X, \lim_{i\in I} p(i)) \cong \lim_{i\in I} \mathrm{Map}(X, p(i)).$$

Indeed, composing $p : I \to C$ wth Yoneda $j : C \to \mathrm{PSh}(C)$ we have a diagram $jp : I \to \mathrm{PSh}(C)$ which we can compose with the evaluation-at-X functor $ev_X : \mathrm{PSh}(C) \to N\mathcal{K}\mathrm{an}$. Then we have equivalences

$$\begin{aligned} \mathrm{Map}(X, \lim p(i)) &\cong ev_X j \lim_{i\in I} p \\ &\overset{Prop.31}{\cong} ev_X \lim_{i\in I} jp \\ &\overset{Prop.30}{\cong} \lim_{i\in I} ev_X jp \cong \lim_{i\in I} \mathrm{Map}(X, p(i)) \end{aligned}$$

Proposition 33 ([HTT, Lem.5.1.5.3], Every presheaf is the colimit of its sections)**.** *Suppose $C \in \mathcal{S}\mathrm{et}_\Delta$, let $j : C \to \mathrm{PSh}(C) = \mathrm{Fun}(C^{op}, N\mathcal{K}\mathrm{an})$ denote the Yoneda embedding, and take $F \in \mathrm{PSh}(C)$. Consider the slice category $C_{/F} = C \times_{\mathrm{PSh}(C)} (\mathrm{PSh}(C)_{/F})$ whose objects are the morphisms $j(c) \to F$ for $c \in C$. The canonical cocone $C^{\triangleright}_{/F} \to \mathrm{PSh}(C)$ exhibits F as a colimit over $C_{/F}$:*

$$F = \mathrm{colim}_{c\in C_{/F}} j(c).$$

Definition 34. A *adjunction* between two quasi-categories is a pair of functors $L : C \rightleftarrows D : R$ and a natural transformation $\mathrm{id} \to RL$ such that for all $d \in D$, $c \in C$, the canonical map

$$\mathrm{Map}(Lc, d) \to \mathrm{Map}(RLc, Rd) \to \mathrm{Map}(c, Rd)$$

is an equivalence.

Remark 35. The meaning of the word *composition* in the above definition needs some comment. A morphism $X \to Y$ in a quasi-category, is a functor $p : \Delta^1 \to C$ and one can consider the over category $C_{/p}$ together with its two projections $\pi_0 : C_{/p} \to C_{/p(0)}$ and $\pi_1 : C_{/p} \to C_{/p(1)}$. One can show that the first projection $C_{/p} \to C_{/p(0)}$ is a categorical equivalence, and therefore admits a section $\sigma : C_{/p(0)} \to C_{/p}$, which is unique up to homotopy (more accurately, the space $\mathrm{Map}(C_{/p(0)}, C_{/p}) \times_{\mathrm{Map}(C_{/p(0)}, C_{/p(0)})} \{\mathrm{id}\}$ of sections is contractible). Composing with any choice of section defines a functor

$$C_{/p(0)} \to C_{/p} \to C_{/p(1)}$$

which can be thought of as a composition with $X \to Y$.

Now note that for any $W \in C$, the right mapping space is a (not full) subcategory of the overcategory $\mathrm{Map}_C^R(W,X) \subseteq C_{/X}$. It then follows from the fact that σ is a section to π that our functor $\pi_1 \circ \sigma$ carries $\mathrm{Map}_C^R(W,X)$ into the subcategory $\mathrm{Map}_C^R(W,Y)$. Indeed, a simplex $\alpha \in C_n$ is in $\mathrm{Map}_C^R(W,X)$ if and only if its restriction to $\Delta^{\{0,\dots,n-1\}}$ is W and to $\Delta^{\{n\}}$ is X. If $\sigma\alpha \in C_{n+1}$ is the image of α under the section, then we have $\sigma\alpha|_{\Delta^{\{0,\dots,n-1\}}} = W$ and $\sigma\alpha|_{\Delta^{\{n,n+1\}}} = X \to Y$. The projection π_1 is induced by the inclusion loses the nth vertex, and as such we have $\pi_1\sigma\alpha|_{\Delta^{\{0,\dots,n-1\}}} = W$ and $\sigma\alpha|_{\Delta^{\{n\}}} = Y$.

Proposition 36 ([HTT, Prop.5.2.3.5])**.** *Let $L : C \to D \in \mathcal{Q}\mathrm{Cat}$ be a functor which admits a right adjoint $R : D \to C$. Then L preserves all colimits which exist in C and R preserves all limits which exists in D.*

Proposition 37 ([HTT, Prop.5.3.3.3] Filtered colimits commute with finite limits)**.** *Suppose that I is a quasi-category. Then the following are equivalent.*

1. *K is* cofiltered. *That is, every finite diagram $D \to K$ admits a (not necessarily limit) cone $D^{\triangleleft} \to K$.*
2. *The limit functor* $\lim : \mathrm{Fun}(K, N\mathcal{K}\mathrm{an}) \to N\mathcal{K}\mathrm{an}$ *preserves finite colimits.*

$$\mathrm{colim}_D \lim_K p = \lim_K \mathrm{colim}_D\, p.$$

Proposition 38 ([HTT, Lem.5.5.2.3] Limits commute with limits)**.** *Let K, L be simplicial sets, let $p : (K^{\triangleleft}) \times (L^{\triangleleft}) \to C$ be a diagram. Suppose that:*

1. *For every vertex $k \in K^{\triangleleft}$, the associated map $p_k : L^{\triangleleft} \to C$ is a limit diagram.*
2. *For every vertex $l \in L$, the associated map $p_l : K^{\triangleleft} \to C$ is a limit diagram.*

Then the restriction $p_0 : K^{\triangleleft} \to C$ is a limit diagram, where $0 \in K^{\triangleleft}$ is the cone point. That is,

$$\lim_{k\in K} \lim_{l \in L} p(k,l) = \lim_{l\in L} \lim_{k\in K} p(k,l).$$

Proposition 39 ([HTT, Def.6.1.1.2, Lem.6.1.3.14], Colimits are universal in $\mathcal{S}$)**.** *For any morphism $X \to Y$ in $N\mathcal{K}\mathrm{an}$ the associated pullback functor $N\mathcal{K}\mathrm{an}^{/Y} \to N\mathcal{K}\mathrm{an}^{/X}$ preserves (small) colimits. That is, for any diagram $p : K \to N\mathcal{K}\mathrm{an}^{/Y}$, we have*

$$X \times_Y (\mathrm{colim}_{k\in K}\, p(k)) = \mathrm{colim}_{k \in K}\, (X \times_Y p(k))$$

where the colimits are taken in $N\mathcal{K}\mathrm{an}$.

Remark 40. Combining Prop.39 with Prop.30 we see that for any $\mathcal{C} \in \mathcal{Q}\mathrm{Cat}$ Prop.39 also holds in $\mathrm{PSh}(\mathcal{C}, N\mathcal{K}\mathrm{an})$. Moreover, if $\mathrm{PSh}(\mathcal{C}, N\mathcal{K}\mathrm{an}) \to \mathcal{T}$ is any finite limit preserving functor admitting a fully faithful right adjoint, then Prop.39 also holds in $\mathcal{T}$. In fact, Prop.39 is one of the fundamental characterising properties of higher topoi.