Derived Algebraic Geometry
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Lecture 7: Infinity categories

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In this lecture we introduce two models for infinity categories—*quasi-categories* and *simplicial categories*—and the adjunction between them. We finish with the definition of the quasi-category of derived rings, or equivalently, the quasi-category of affine derived schemes.

1 Quasi-categories

Just as a small category is a directed graph with composable edges, a quasi-category is a kind of simplicial set.

Definition 1 (Boardman, Vogt, 1973)**.** A *quasi-category* is a simplicial set K such that for every $0 < i < n$ and each diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

there exists a (not necessarily unique) dashed arrow making a commutative triangle.

A *functor* between quasi-categories is a morphism of simplicial sets. That is, the category of quasi-categories is a full subcategory of the category of simplicial sets

$$\mathcal{Q}\mathrm{Cat} \subset \mathcal{S}\mathrm{et}_\Delta.$$

Elements of K_0 are called *objects* and elements of K_1 are called 1-morphisms, or often just *morphisms*. Given two morphisms $f, g \in K_1$ such that $d_0 f = d_1 g$ (equivalently, a morphism of simplicial sets $\Lambda^2_1 \to K$), for any factorisation $\Lambda^2_1 \to \Delta^2 \overset{\sigma}{\dashrightarrow} K$, the morphism $d_1\sigma \in K_1$ will be called a *composition* of g and f. For any object $X \in K_0$, the morphism $s_0 X \in K_1$ is called the *identity morphism* of X, and written id_X.

Example 2. Let C be a small category. Considering the ordered sets $[n]$ as categories[1] the assignment

$$N : [n] \mapsto \mathrm{Fun}([n], C)$$

sending $[n]$ to the set of functors $[n] \to C$ defines a simplicial set. This is called the *nerve* of C.

Explicitly,

[1] So, for $0 \leq i, j \leq n$ there is exactly one morphism $i \to j$ if $i \leq j$, and no morphisms otherwise.

1. $N(C)_0$ is the set of objects of C,
2. $N(C)_1$ is the set of (all) morphisms in C,
3. The two morphisms $N(C)_1 \rightrightarrows N(C)_0$ induced by the two morphisms $[0] \rightrightarrows [1]$ send morphisms in $N(C)_1$ to their source and target.

$$(X \xrightarrow{f} Y) \qquad \mapsto \qquad X, Y$$

4. The morphism $N(C)_0 \to N(C)_1$ induced by $[1] \to [0]$ sends each object to its identity morphism.

$$X \qquad \mapsto \qquad (X \xrightarrow{\mathrm{id}_X} X)$$

5. $N(C)_2$ is the set of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.
6. The three maps $d_0, d_1, d_2 : N(C)_2 \overrightarrow{\rightrightarrows} N(C)_1$ induced by the three monomorphisms $[1] \overrightarrow{\rightrightarrows} [2]$ send $\xrightarrow{f}\xrightarrow{g}$ to g, $g \circ f$, and f respectively.

$$\begin{array}{ccccc} & & Y & & \\ & {}^{f}\nearrow & & \searrow^{g} & \\ X & & \xrightarrow[g\circ f]{} & & Z \end{array} \qquad \mapsto \qquad (Y \xrightarrow{g} Z), \quad (X \xrightarrow{g\circ f} Y), \quad (X \xrightarrow{f} Y)$$

7. More generally, $N(C)_n$ is the set of sequences of n composable morphisms $\xrightarrow{f_1} \cdots \xrightarrow{f_n}$ and the various maps $N(C)_n \to N(C)_m$ come from various combinations of composition and inserting identities.

Note that we can completely recover C from $N(C)$. In fact we have a lot of degenerate information.

Exercise 3. Suppose that C is a simplicial set such that:
1. Each $\Lambda^2_1 \to C$ extends to a unique $\Delta^2 \to C$, and
2. Each $\Lambda^3_1 \to C$ extends to some $\Delta^3 \to C$.

Show that C canonically determines a category whose set of objects is C_0 and set of morphisms is C_1.

Exercise 4 (HTT, Proposition 1.1.2.2)**.** (Difficult) Show that a simplicial set K is of the form $N(C)$ if and only if for every $0 < i < n$ and each diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

there exists a *unique* dotted arrow making a commutative triangle.

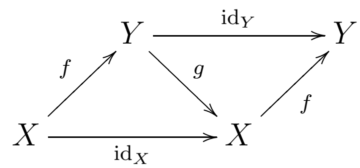
Example 5. Any Kan complex is an quasi-category. That is, we have fully faithful inclusions

$$\mathcal{S}\mathrm{et}_\Delta \supset \mathcal{Q}\mathrm{Cat} \supset \mathcal{K}\mathrm{an}.$$

In particular, for any topological space X, the simplicial set $\operatorname{Sing} X$ is a quasi-category.

Exercise 6.

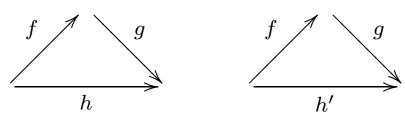
1. Show that every Kan complex is a quasi-category.
2. Show that if K is a Kan complex, then every morphism in K is invertible up to homotopy in the sense that:
   - For every $X \xrightarrow{f} Y$ in K_1 we can find two 2-cells in K_2 fitting into a diagram of the form



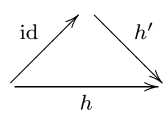
3. (Harder) Show that if K is a quasi-category satisfying the above property, then K is a Kan complex. Hint.[2]

Note that in general, for a topological space X, composition in $\operatorname{Sing} X$ is not unique, but any two choices of composition are homotopic. This is a general feature of ∞-categories.

Exercise 7. Show that in a quasi-category C, any two compositions are "homotopic" in the sense that if there exist two 2-cells in C_2 of the form

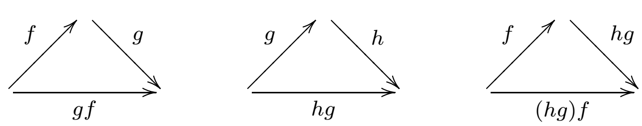


then there exists a 2-cell of the form

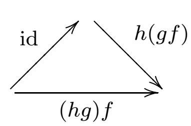


Similarly, in $\operatorname{Sing} X$ composition is not associative on the nose, but only up to homotopy.

Exercise 8. Show that composition in a quasi-category C is associative "up to homotopy" in the sense that if we have 2-cells in C_2 of the form



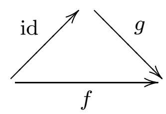
Then $(hg)f$ is a composition of gf and h. In particular, by Exercise 7, if $h(gf)$ is any other choice of composition of gf and h, then there is a 2-cell of the form:



[2]Start with the case $\Lambda^2_0 \to C$ and work up to Λ^n_0 by induction. Use opposite categories to deduce Λ^n_n from Λ^n_0.

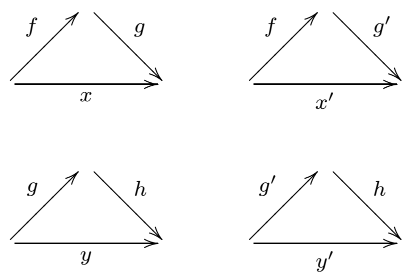
Exercise 9. Recall the nerve functor from Example 2. We will show that the nerve functor admits a left adjoint.

1. Let C be a quasi-category. Define a relation on 1-morphisms in C by saying $f \sim g$ if f is a composition of g and id. That is, if there exists a 2-cell in C_2 of the form



   Show that this is an equivalence relation.

2. Show that the above equivalence relation preserves composition. That is, suppose that $g \in C_1$ is equivalent to $g' \in C_1$, and suppose we have 2-cells of the following form.



   Show that $x \sim x$ and $y \sim y'$. (Use Exercise 7 if necessary).

3. Define hC to be the category whose objects are vertices C_0, morphisms are edges C_1 modulo the above equivalence relation, and composition is induced by composition in C. Show that this is actually a category. That is, show that it satisfies the identity and associativity axioms. (Use Exercise 8 for associativity).

4. Show that

$$h : \mathcal{Q}\mathrm{Cat} \to \mathcal{C}\mathrm{at}$$

   defines a functor which is left adjoint to N. Hint.[3]

Definition 10. The category hC defined above is called the *homotopy category* of C. A morphism $X \xrightarrow{f} Y \in C_1$ in a quasi-category is said to be an *equivalence* if it becomes an isomorphism in hC. If such an equivalence exists, we say X and Y are equivalent.

2 Mapping spaces

We wanted to replace sets with homotopy types, so for any two objects $x, y \in C_0$ in a quasi-category, we should have a homotopy type $\mathrm{Map}_C(x, y)$ of morphisms. Here are two models for this homotopy type.

[3] It suffices to show that $hN = \mathrm{id}$ and to give a natural transformation $\eta : \mathrm{id} \to Nh$ such that $h(\eta)$ is the identity natural transformation.

Definition 11. Let C be a quasi-category, and $x, y \in C_0$ objects. Define

$$\hom^R_C(x,y)_J = \{z : \Delta^{J \sqcup [0]} \to C \mid z|_{\Delta^J} = x \text{ and } z|_{\Delta^0} = y\}$$

where $J \sqcup [0] = \{j_0 < \cdots < j_n\} \sqcup \{0\} = \{j_0 < j_1 < \cdots < j_n < 0\}$ and we use x for the constant morphism $\Delta^J \to \Delta^0 \xrightarrow{x} C$. Similarly, define

$$\hom^L_C(x,y)_J = \{z : \Delta^{[0] \sqcup J} \to C \mid z|_{\Delta^0} = x \text{ and } z|_{\Delta^j} = y\}$$

where $[0] \sqcup J = \{0\} \sqcup \{j_0 < \cdots < j_n\} = \{0 < j_0 < j_1 < \cdots < j_n\}$.

Exercise 12. Suppose C is a quasi-category and $x, y \in C_0$ are objects. Show that $\hom^R_C(x,y)$ and $\hom^L_C(x,y)$ are Kan complexes.

Exercise 13.

1. Let C be a small category. Show that $\hom^R_{NC}(x,y)_J = \hom_C(x,y)$ for all J.
2. Let X be a topological space and $x, y \in X$ two points. Let PX denote the set $\hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ equipped with the compact-open topology[4] and $PX(x,y) \subseteq \hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ the subspace of maps $\gamma : \Delta^1_{\mathrm{top}} \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Define an isomorphism of simplicial sets

$$\hom^R_{\mathrm{Sing}\,X}(x,y) \cong \mathrm{Sing}\, PX(x,y).$$

Definition 14. A morphism $C \to D$ of quasi-categories is:

1. *fully faithful* if for every pair of objects $X, Y \in C_0$ the induced morphism $\hom^R_C(X,Y) \to \hom^R_D(FX, FY)$ is an equivalence of Kan complexes,
2. *essentially surjective* if $hC \to hD$ is essentially surjective,
3. a *categorical equivalence* if it is essentially surjective and fully faithful.

Exercise 15. Let $F : C \to C'$ be a functor between small categories. Show that F is an equivalence of categories if and only if $F : NC \to NC'$ is an equivalence of quasi-categories.

2.1 Simplicial categories

References:

[1982 Max Kelly, Basic Concepts of Enriched Category Theory]

[2003 Hirschorn, Model categories and their localisations, Def.9.1.2]

[2012 Lurie, Higher Topos Theory]

Quasi-categories are good for some things but not so good for other things. For example, proving the Yoneda lemma purely in the context of quasi-categories is particularly uncomfortable (cf. Cisinski's book). For such things (i.e., Yoneda) simplicial categories are much nicer.

Definition 16 ([HTT, Def.1.1.4.1])**.** A *simplicial category* C is a category enriched over $\mathcal{S}\mathrm{et}_\Delta$. Explicitly, it is the data of:

[4] Or indeed, any topology such that $\hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, \hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)) = \hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}} \times \Delta^1_{\mathrm{top}}, X)$.

1. A collection of objects $Ob\ C$.
2. For every pair of objects $X, Y \in Ob\ C$, a simplicial set $\mathrm{Map}_C(X, Y)$.
3. For every triple of objects $W, X, Y \in Ob\ C$ a morphism of simplicial sets

$$- \circ - : \mathrm{Map}_C(W, X) \times \mathrm{Map}_C(X, Y) \to \mathrm{Map}_C(W, Y).$$

These data are required to satisfy:

(Id.) Every object has an identity morphism. That is, for every $X \in Ob\ C$ there is a vertex $\mathrm{id}_X \in \mathrm{Map}(X, X)_0$ such that

$$\Delta^0 \times \mathrm{Map}(X, Y) \xrightarrow{\{\mathrm{id}_X\} \times \mathrm{id}_{\mathrm{Map}(X,Y)}} \mathrm{Map}(X, X) \times \mathrm{Map}(X, Y) \xrightarrow{\circ} \mathrm{Map}(X, Y)$$

is the canonical identification $\Delta^0 \times \mathrm{Map}(X, Y) \cong \mathrm{Map}(X, Y)$, and similarly for $\mathrm{Map}(W, X) \times \mathrm{Map}(X, X) \to \mathrm{Map}(W, X)$.

(Assoc.) The composition is associative. That is the following diagram of simplicial sets commutes for any objects W, X, Y, Z.

$$\begin{array}{ccc} \mathrm{Map}_C(W, X) \times \mathrm{Map}_C(X, Y) \times \mathrm{Map}_C(Y, Z) & \longrightarrow & \mathrm{Map}_C(W, Y) \times \mathrm{Map}_C(Y, Z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_C(W, X) \times \mathrm{Map}_C(X, Z) & \longrightarrow & \mathrm{Map}_C(W, Z) \end{array}$$

A simplicial category is called *fibrant* if all $\mathrm{Map}_C(X, Y)$ are Kan complexes.

Example 17. The simplicial category of simplicial sets is defined as follows. Objects are simplicial sets. Given two simplicial sets K, L the mapping space is defined by

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, L)_n = \hom_{\mathcal{S}\mathrm{et}_\Delta}(K \times \Delta^n, L).$$

The simplicial set structure comes from functoriality in $[n] \in \Delta$. Composition is defined using the diagonal maps $\Delta^n \to \Delta^n \times \Delta^n$. Explicitly, the composition of two n-cells $f : K \times \Delta^n \to L$ and $g : L \times \Delta^n \to M$ is

$$K \times \Delta^n \xrightarrow{diag.} K \times \Delta^n \times \Delta^n \xrightarrow{f \times \mathrm{id}_{\Delta^n}} L \times \Delta^n \xrightarrow{g} M.$$

Exercise 18. Show that composition in the simplicial category $\mathcal{S}\mathrm{et}_\Delta$ satisfies the identity and associativity axioms.

Exercise 19 ([HTT, Prop.1.2.7.3], [Gabriel-Zisman, 3.1.3])**.** Let C be a quasi-category (resp. Kan complex). It turns out [HTT, Cor.2.3.2.4],[5] [Gabriel-Zisman, Prop.2.2] that C satisfies the stronger property:

(*) For every simplicial set K, every $0 < i < n$ (resp. $0 \leq i \leq n$), and every morphism $\Lambda^n_i \times K \to C$ there exists a factorisation

$$\begin{array}{ccc} \Lambda^n_i \times K & \longrightarrow & C \\ \downarrow & \nearrow & \\ \Delta^n \times K & & \end{array}$$

[5] This is a result of Joyal.

Using this property, show that for any $K \in \mathcal{S}\mathrm{et}_\Delta$, the simplicial set $\mathrm{Map}(K, C)$ is an quasi-category (resp. Kan complex).

Deduce that the simplicial category of Kan complexes is fibrant.

Exercise 20. Give an example of $C, C' \in \mathcal{Q}\mathrm{Cat}$ such that $\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(C, C')$ is not a Kan complex.

Like quasi-categories, simplicial categories also have associated categories.

Exercise 21.

1. Let C be a simplicial category. For $X, Y \in Ob\ C$ define $\hom_C(X, Y) = \mathrm{Map}_C(X, Y)_0$. Show that this defines a category. This category is sometimes denoted C_0. Be careful not to confuse this with the set of 0-simplicies of a simplicial set.
2. (Harder) If K, L are simplicial sets, define a map $\pi_0|K| \times \pi_0|L| \to \pi_0|K \times L|$. Hint.[6]
3. Let C be a fibrant simplicial category. For $X, Y \in Ob\ C$ define $\hom_{hC}(X, Y) = \pi_0|\mathrm{Map}_C(X, Y)|$. Show that this defines a category.

Definition 22. A *morphism* $F : C \to D$ between two simplicial categories is defined in the obvious way. We have a map $Ob\ C \to Ob\ D$, for every pair $X, Y \in Ob\ C$ we have a morphism of simpicial sets $\mathrm{Map}_C(X, Y) \to \mathrm{Map}_D(FX, FY)$, and these morphisms are required to be compatible with composition and send identity morphisms to identity morphisms. The category of simplicial categories is denoted $\mathcal{C}\mathrm{at}_\Delta$.

Definition 23 ([HTT, Def.1.1.4.4])**.** A morphism $F : C \to C'$ of simplicial categories is an *equivalence* if

1. it is *fully faithful* in the sense that for every $X, Y \in Ob\,C$ the map $\mathrm{Map}_C(X, Y) \to \mathrm{Map}_{C'}(FX, FY)$ is a weak equivalence of simplicial sets, and
2. it is *essentially surjective* in the sense that $hC \to hC'$ is essentially surjective.

3 Comparing quasi-categories and simplicial categories

In this section we construct the adjunction

$$\mathfrak{C} : \mathcal{Q}\mathrm{Cat} \rightleftarrows \mathcal{C}\mathrm{at}_\Delta : N.$$

As with geometric realisation $|-| : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathrm{Top} : \mathrm{Sing}$, the strategy is to define $\mathfrak{C}[\Delta^n]$ for the quasi-categories Δ^n, take the hom out of this functor to define N, and then observe that N admits a left adjoint, determined by its values on Δ^n and the requirement that it preserve colimits.

[6] Note that for diagrams $X, Y : \mathbb{N} \rightrightarrows \mathrm{Top}$ such that for each n, the maps $X(n) \to X(n+1)$, $Y(n) \to Y(n+1)$ are inclusions of closed subspaces, we have $\mathrm{colim}_{\mathbb{N}} X(n) \times \mathrm{colim}_{\mathbb{N}} Y(m) \cong \mathrm{colim}_{\mathbb{N}\times\mathbb{N}} X(n) \times Y(m)$, and $\hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, \mathrm{colim}_{n\in\mathbb{N}} X_n) = \mathrm{colim}_{n\in\mathbb{N}} \hom(\Delta^1_{\mathrm{top}}, X(n))$.

Definition 24 (Cordier 1982, [HTT, §1.1.5])**.** Define $\mathfrak{C}[\Delta^n]$ to be the simplicial category whose objects are elements of $[n] = \{0 < \cdots < n\}$. For $0 \leq i, j \leq n$ the mapping space is the nerve of the partially ordered set

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = N\Big\{\{i,j\} \subseteq J \subseteq \{i, i{+}1, \ldots, j\}\Big\}$$

of subsets J containing i, j and contained in $\{i, i{+}1, \ldots, j\}$. Composition

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) \times \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(j,k) \to \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,k)$$

is induced by union.

Exercise 25. Show that $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = N[1]^{j-i-1}$ where $[1]^m$ is the poset

$$\underbrace{[1] \times \cdots \times [1]}_{m \text{ times}} = \{(\varepsilon_1, \ldots, \varepsilon_m) \mid \varepsilon_k \in \{0,1\}\}.$$

That is, show that $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) = \Delta^1 \times \cdots \times \Delta^1$ is the $(j{-}i{-}1)$-dimensional simplicial cube.

Remark 26. The 0-simplices of $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j)$ can be interpreted as all of the different ways of writing the morphism $i \to j$ in $N[n]$ as a composition

$$i = k_0 \to k_1 \to \cdots \to k_m \to k_{m+1} = j,$$

with $k_\ell \neq k_{\ell+1}$ (unless $i = j$). The higher simplicies can be interpreted as homotopies between these various compositions. See Remark 32 for more details.

Note that $\mathfrak{C}[\Delta^n]$ is functorial in n, cf.[HTT, Def.1.1.5.3], so we obtain a functor

$$\mathfrak{C}[\Delta^-] : \Delta \to \mathcal{C}\mathrm{at}_\Delta$$

Definition 27. The *nerve* of a simplicial category C is the simplicial set, [HTT, Def.1.1.5.5],

$$NC : [n] \mapsto \hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[\Delta^n], C).$$

Here is the main comparison theorem.

Theorem 28 ([HTT, §2.2], [HTT, Prop.1.1.5.10, Thm.2.2.5.1])**.**
1. *The nerve functor admits a left adjoint*

$$\mathfrak{C} : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{C}\mathrm{at}_\Delta : N.$$

2. *The functor N sends fibrant simplicial categories*[7] *to quasi-categories.*
3. *Both $\mathfrak{C}$ and N both preserve and reflect categorical equivalences.*[8]

[7] Recall, a simplicial category if *fibrant* if all Map are Kan complexes.

[8] That is, a morphism f in $\mathcal{C}\mathrm{at}_\infty$ (resp. $\mathcal{C}\mathrm{at}_\Delta$) is a categorical equivalence if and only if $\mathfrak{C}(f)$ (resp. $N(f)$) is a categorical equivalence.

4. *Given $C \in \mathcal{Q}\mathrm{Cat}$ and $X, Y \in C_0$ there exist homotopy equivalences of Kan complexes*

$$\hom^L_C(X,Y) \cong \operatorname{Sing}|\operatorname{Map}_{\mathfrak{C}[C]}(X,Y)| \cong \operatorname{Map}^R_C(X,Y).$$

Remark 29.

1. Since the functor $\mathfrak{C}$ is a left adjoint and we know its values on the representables Δ^n, its value on a general simplicial set K is a kind of geometric realisation $\mathfrak{C}[K] = \operatorname{colim}_{([n],f)\in\Delta_{/K}} \mathfrak{C}[\Delta^n]$.[9] This description is usually useless since colimits (for example coequalisers) in $\mathcal{C}\mathrm{at}_\Delta$ are difficult to describe in general. Only in some simple cases (e.g. $\partial\Delta^n$, Λ^n_i) something can be said.
2. In [HTT, Thm.2.2.5.1] categorical equivalences of simplicial sets are *defined* as those morphisms sent to equivalences under $\mathfrak{C}[-]$. So this part of the above theorem is empty in some sense. However, as we saw above, for quasi-categories C, the mapping spaces in $\mathfrak{C}[C]$ can also be computed via other more accessible models.

Definition 30. The *quasi-category of spaces* is the nerve of the simplicial category of Kan complexes.

$$\mathcal{S} := N(\mathcal{K}\mathrm{an}).$$

Remark 31 ([HTT, §1.2.15])**.** Here we run into Russell's paradox, the set of all sets cannot be a set. There are various ways to resolve this. One way is to choose a Grothendieck universe, or equivalently, a strongly inaccessible cardinal κ. This is a cardinal such that the category $\mathcal{S}\mathrm{et}_\kappa$ of sets of cardinality $< \kappa$ satisfies: if $f : X \to Y$ is a morphism of sets such that $Y \in \mathcal{S}\mathrm{et}_\kappa$ and all $f^{-1}(y) \in \mathcal{S}\mathrm{et}_\kappa$ then $X \in \mathcal{S}\mathrm{et}_\kappa$ and $\{Z \subseteq Y\} \in \mathcal{S}\mathrm{et}_\kappa$. Then we define $\mathcal{S}\mathrm{et}_\Delta$ to be the category of simplicial sets in $\mathcal{S}\mathrm{et}_\kappa$, i.e., $(\mathcal{S}\mathrm{et}_\kappa)_\Delta$. In this way it's not a member of itself.

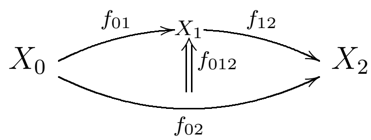
Remark 32.

1. Elements of $\mathcal{S}_0$ are Kan complexes.
2. Elements of $\mathcal{S}_1$ are morphisms between Kan complexes.
3. Elements of $\mathcal{S}_2$ are tuples

$$\begin{gathered}(X_0, X_1, X_2,\\ X_0 \xrightarrow{f_{01}} X_1, X_1 \xrightarrow{f_{12}} X_2, X_0 \xrightarrow{f_{02}} X_2,\\ X_0 \times \Delta^1 \xrightarrow{f_{012}} X_1)\end{gathered}$$

such that such X_0, X_1, X_2 are Kan complexes and f_{012} is a simplicial homotopy from f_{02} to $f_{12}\circ f_{01}$, in the sense that $f_{012}|_{X_0\times\{0\}} = f_{02}$ and $f_{012}|_{X_0\times\{1\}} = f_{12}\circ f_{01}$.

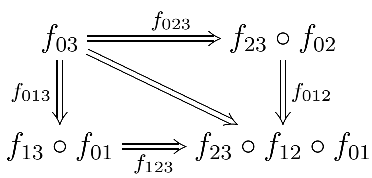
[9] For this, we also need to know that $\mathcal{C}\mathrm{at}_\Delta$ admits colimits. This follows from abstract nonsense because it sits in a monadic adjunction $\mathcal{G}\mathrm{r}_\Delta \rightleftarrows \mathcal{C}\mathrm{at}_\Delta$ with the category $\mathcal{G}\mathrm{r}_\Delta$ of simplicial graphs, i.e., graph objects $E \rightrightarrows V$ in $\mathcal{S}\mathrm{et}_\Delta$ such that V is a constant simplicial set. Cf. the Barr-Beck Theorem.

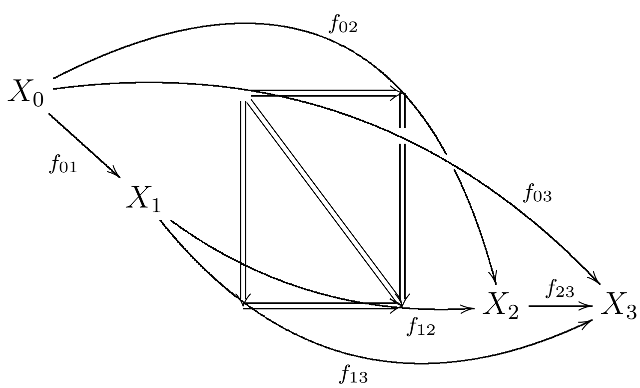


4. Elements of $\mathcal{S}_3$ are tuples

$$((X_i : 0 \leq 1 \leq 3),$$
$$(X_i \overset{f_{ij}}{\to} X_j : 0 \leq i < j \leq 3),$$
$$(X_i \times \Delta^1 \overset{f_{ijk}}{\to} X_j : 0 \leq i < j < k \leq 3)$$
$$(X_0 \times \Delta^1 \times \Delta^1 \overset{f_{0123}}{\to} X_3)$$

   such X_0, X_1, X_2, X_3 are Kan complexes, each of the four f_{ijk} satisfies the property analogous to f_{012} above, and f_{0123} restricted to the four edges $\Delta^1 \times \{\epsilon\} \subset \Delta^1 \times \Delta^1$ and $\{\epsilon\} \times \Delta^1 \subset \Delta^1 \times \Delta^1$ for $\epsilon = 0, 1$ correspond to the four f_{ijk}.





4 The quasi-category of derived rings

Definition 33. The category $\mathcal{R}\mathrm{ing}_\Delta$ of simplicial rings is the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{R}\mathrm{ing})$ of functors from Δ^{op} into the category of rings.

Example 34. Every simplicial set determines a simplicial ring via the canonical free/forgetful adjunction

$$\mathbb{Z}[-] : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{R}\mathrm{ing}_\Delta : U.$$

So $\mathbb{Z}[K]_n$ is the polynomial ring with one variable x_k for each element $k \in K_n$.

Example 35.

1. The simplicial ring $\mathbb{Z}[\Delta^0]$ has $\mathbb{Z}[\Delta^0]_n = \mathbb{Z}[x]$ for all n.
2. The simplicial ring $\mathbb{Z}[\partial\Delta^1]$ has $\mathbb{Z}[\partial\Delta^1]_n = \mathbb{Z}[x_{\delta_1}, x_{\delta_0}]$ for all n.
3. The simplicial ring $\mathbb{Z}[\Delta^1]$ has

$$\mathbb{Z}[\Delta^1]_n \cong \mathbb{Z}[x_{\delta_1}, x_{\sigma_1}, \dots, \dots, x_{\sigma_n}, x_{\delta_0}],$$

where $\sigma_i : [n] \to [1]$ is the unique surjection with $\sigma_i(i-1) \neq \sigma_i(i)$ and $\delta_\varepsilon : [n] \to [1]$ is the constant map $i \mapsto 1-\varepsilon$. For $p : [m] \to [n]$ the induced map $\mathbb{Z}[\Delta^1]_n \to \mathbb{Z}[\Delta^1]_m$ sends x_q to $x_{q \circ p}$.

Example 36. Suppose that $L : C \rightleftarrows D : R$ is an adjunction. So we have natural transformations $\eta : LR \to \mathrm{id}_D$ and $\varepsilon : \mathrm{id}_C \to RL$. Define $\Phi := LR : D \to D$. Inserting the counit $\eta : \Phi = LR \to \mathrm{id}_D$ in the ith place defines a natural transformation

$$d_i : \Phi^{\circ i} \circ \Phi \circ \Phi^{\circ n-i} \to \Phi^{\circ i} \circ \mathrm{id}_D \circ \Phi^{\circ n-i}.$$

for $i = 0, \dots, n$. On the other hand the unit ε defines a natural transformation

$$\Phi = L\,\mathrm{id}_C\,R \to L(RL)R = \Phi \circ \Phi.$$

Inserting this in the ith place defines natural transformations

$$s_i : \Phi^{\circ i} \circ \Phi \circ \Phi^{\circ n-i} \to \Phi^{\circ i} \circ (\Phi \circ \Phi) \circ \Phi^{\circ n-i}.$$

for $i = 0, \dots, n$. Since the two compositions $R \to (RL)R = R(LR) \to R$ and $L \to L(RL) = (LR)L \to L$ are the identity natural transformation, and every morphism $p : [n] \to [m]$ of Δ can be written as a composition of δ_i's and σ_i's, unique up to the simplicial identities,[10] we get a functor

$$\Delta^{\mathrm{op}} \to \mathrm{End}(D); \qquad [n] \mapsto \Phi^{\circ n+1}$$

into the category of endofunctors of D. Doing this in the case of the adjunction

$$\mathbb{Z} : \mathcal{S}\mathrm{et} \rightleftarrows \mathcal{R}\mathrm{ing} : U$$

gives a functorial procedure to associate a simplicial ring $B \in \mathcal{R}\mathrm{ing}_\Delta$ to any ring $A \in \mathcal{R}\mathrm{ing}$. So

$$B_0 = \mathbb{Z}[UA], \qquad B_1 = \mathbb{Z}[U\mathbb{Z}[UA]], \qquad B_2 = \mathbb{Z}[U\mathbb{Z}[U\mathbb{Z}[UA]]], \qquad \dots.$$

[10] Recall that $\delta_i : [n] \to [n+1]$ is the unique injection which misses i, and $\sigma_i : [n+1] \to [n]$ is the unique surjection which hits i twice. The simplicial identities are:

$$\begin{aligned} \delta_j\delta_i &= \delta_i\delta_{j-1} && i < j \\ \sigma_j\delta_i &= \delta_i\sigma_{j-1} && i < j \\ \sigma_j\delta_i &= \mathrm{id} && i = j, j+1 \\ \sigma_j\delta_i &= \delta_{i-1}\sigma_j && j < i-1 \\ \sigma_j\sigma_i &= \sigma_i\sigma_{j+1} && i \le j \end{aligned}$$

Notice that each B_n is a polynomial ring over $\mathbb{Z}$, and the degeneracy morphisms $s_i : B_n \to B_{n+1}$ are all of the form $\mathbb{Z}[-]$ applied to some morphism of sets.

Example 37. Colimits in functor categories are computed objectwise. In the category of rings, pushout is degreewise tensor product. So if $A \leftarrow B \to C$ are morphisms of simplicial rings the colimit is the simplicial ring with $(A \otimes_B C)_n = A_n \otimes_{B_n} C_n$.

Example 38. Let $f \in \mathbb{Z}$ be any non-zero integer. Consider the morphism $\mathbb{Z}[\partial\Delta^1] \to \mathbb{Z}$ which sends δ_1, resp. δ_0, to f, resp. 0. Define

$$\mathbb{Z}/\!/f := \mathbb{Z}[\Delta^1] \underset{\mathbb{Z}[\partial\Delta^0]}{\otimes} \mathbb{Z}.$$

So we have

$$(\mathbb{Z}/\!/f)_n \cong \frac{\mathbb{Z}[\Delta^1]_n}{\langle x_{\delta_1} - f, x_{\delta_0} \rangle} \cong \mathbb{Z}[x_{\sigma_1}, \dots, x_{\sigma_n}]$$

with $d_0 : (\mathbb{Z}/\!/f)_n \to (\mathbb{Z}/\!/f)_{n-1}$ sending x_{σ_1} to 0 and $d_n : (\mathbb{Z}/\!/f)_n \to (\mathbb{Z}/\!/f)_{n-1}$ sending x_{σ_n} to f.

Definition 39. The normalised chain complex associated to a simplicial ring A is $NA_n = \cap_{i=1}^n \ker(d_i)$ equipped with differential $d_0 : NA_n \to NA_{n-1}$. The homotopy groups of A are

$$\pi_n A = \frac{\ker(NA_n \xrightarrow{d_0} NA_{n-1})}{\mathrm{im}(NA_{n+1} \xrightarrow{d_0} NA_n)}.$$

where we set $NA_{-1} = 0$.

A morphism of simplicial rings is a *weak equivalence* if it induces isomorphisms on all homotopy groups.

Exercise 40. Show that $\mathbb{Z}[\Delta^n] \to \mathbb{Z}[\Delta^0]$ are quasi-isomorphisms for all n.

Exercise 41. Let $f \in \mathbb{Z}$ be a nonzero integer.

1. Show that $\pi_0(\mathbb{Z}/\!/f) \cong \mathbb{Z}/f$.
2. Show that $\pi_n(\mathbb{Z}/\!/f) = 0$ for $n > 0$.
3. Show that the canonical morphism

$$\mathbb{Z}/\!/f \to \mathbb{Z}/f$$

is a weak equivalence but there are no morphisms from $\mathbb{Z}/f$ to $\mathbb{Z}/\!/f$. Here $\mathbb{Z}/f$ means the constant functor $\Delta^{\mathrm{op}} \to \mathcal{R}\mathrm{ing}$ with value $\mathbb{Z}/f$.

Exercise 42. Consider the simplicial replacement $A \mapsto B$ from Example 36. Hint.[11]

1. Show that $\pi_0 B \cong A$.
2. Show that $\pi_n B \cong 0$ for $n > 0$.

The category of simplicial rings admits a structure of simplicial category.

[11]Use the degeneracy morphism $s_0 : B_n \to B_{n+1}$ and the canonical morphism of sets $A \to \mathbb{Z}[UA]$.

Definition 43. For any simplicial ring A and simplicial set K define a functor

$$\begin{aligned}\mathcal{R}\text{ing}_\Delta \times \mathcal{S}\text{et}_\Delta &\to \mathcal{R}\text{ing}_\Delta \\ (A, K) &\mapsto A \otimes K\end{aligned}$$

by $(A \otimes K)_n = \bigotimes_{k \in K_n} A_n$. Note that $A \otimes (K \times L) = (A \otimes K) \otimes L$. Setting

$$\text{Map}(A, B)_n = \hom(A \otimes \Delta^n, B).$$

endows $\mathcal{R}\text{ing}_\Delta$ with a structure of simplicial category. Composition is analogous to Example 17.

At this point, one could try and use the quasi-category associated to the simplicial category $\mathcal{R}\text{ing}_\Delta$, however this does not do the "correct" thing because there are quasi-isomorphisms in $\mathcal{R}\text{ing}_\Delta$ which do not become isomorphisms in the homotopy category of the simplicial category $\mathcal{R}\text{ing}_\Delta$, cf. Example 41.

We have seen this before. Recall that in the category of topological spaces, weak equivalence also didn't necessarily imply homotopy equivalence. However there was a nice subcategory—the category of CW complexes— where these notions did align. A retract of a CW complex is called a *cofibrant* topological space. We do the same thing for simplicial rings.

$$\begin{array}{ccccc}\text{Top}^{\text{cof}} & \subseteq & \text{CW} & \subseteq & \text{Top} \\ \mathcal{R}\text{ing}_\Delta^{\text{cof}} & \subseteq & \mathcal{R}\text{ing}_\Delta^{\text{cell}} & \subseteq & \mathcal{R}\text{ing}_\Delta\end{array}$$

Definition 44. A simplicial ring is *celluar* if it is a colimit of the form

$$A(-1) \to A(0) \to A(1) \to A(2) \to \dots$$

where $A(-1) = \mathbb{Z}$ and for $n \geq 0$ the morphism $A(n-1) \to A(n)$ is a pushout of the form

$$\begin{array}{ccc}\bigotimes_{I_n} \mathbb{Z}[\partial\Delta^n] & \xrightarrow{g_n} & A(n-1) \\ \downarrow & & \downarrow \\ \bigotimes_{I_n} \mathbb{Z}[\Delta^n] & \longrightarrow & A(n)\end{array}$$

for some set I_n and morphism g_n. Retracts of cellular simplicial rings are called *cofibrant*. The full subcategory of cofibrant simplicial rings is written $\mathcal{R}\text{ing}_\Delta^{\text{cof}}$.

Note that since pushouts of rings are tensor products, in the above squares we have $A(n)_i = (\otimes_{I_n} \mathbb{Z}[\Delta^n])_i \bigotimes_{(\otimes_{I_n} \mathbb{Z}[\partial\Delta^n])_i} A(n-1)_i$.

Exercise 45 (Harder)**.** Suppose A is a cellular simplicial ring. Show that

$$A_n \cong \mathbb{Z}[\sqcup_{[n] \twoheadrightarrow [k]} I_k]$$

where the disjoint union is over all surjections $[n] \twoheadrightarrow [k]$ in Δ. Hint.[12]

Show furthermore that for any surjection $p : [m] \twoheadrightarrow [n]$ the corresponding morphism $A_p : A_n \to A_m$ is the morphism associated to $\sqcup_{[n]\twoheadrightarrow[k]} I_k \to \sqcup_{[m]\twoheadrightarrow[k]} I_k$ which sends the $([n] \twoheadrightarrow [k])$th copy of I_k to the $([m] \twoheadrightarrow [n] \twoheadrightarrow [k])$th copy.

Exercise 46.

1. Suppose that A is a simplicial ring. Let $\mathbb{Z}[\partial\Delta^n] \to A$ be the morphism which sends all variables in $\mathbb{Z}[\partial\Delta^n]$ to zero. Define $B := \mathbb{Z}[\Delta^n] \underset{\mathbb{Z}[\partial\Delta^n]}{\otimes} A$. Show that the canonical map $\pi_i A \to \pi_i B$ is an isomorphism for $i < n$, and for $i = n$ it is injective, admits a retraction, and $x_{\mathrm{id}_{[n]}}$ defines a nonzero element of $\pi_n B$ not in the image of $\pi_n A$.
2. Suppose that A is a simplicial ring and $\overline{\alpha} \in \pi_n A$ any element with lift $\alpha \in A_n$. Let $\mathbb{Z}[\partial\Delta^{n+1}] \to A$ be the map which sends $x_{\delta_i} \in \mathbb{Z}[\partial\Delta^{n+1}]_n$ to zero for $i < n+1$ and $x_{\delta_{n+1}}$ to α and define $B = \mathbb{Z}[\Delta^{n+1}] \underset{\mathbb{Z}[\partial\Delta^{n+1}]}{\otimes} A$. Show that the canonical morphism $\pi_i A \to \pi_i B$ is an isomorphism for $i < n$, and for $i = n$ is a surjection sending $\overline{\alpha}$ to zero.

Exercise 47. Suppose A is a cofibrant simplicial ring and $X \to Y$ any morphism of simplicial rings such that each $X_n \to Y_n$ is surjective. Show that for any morphism $A \to Y$ there exists a dashed morphism making a commutative triangle

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow \\ A & \longrightarrow & Y \end{array}$$

Definition 48. The quasi-category of derived rings is the nerve of the simplicial category of cofibrant simplicial rings $N(\mathcal{R}\mathrm{ing}_{\Delta}^{\mathrm{cof}})$.

[12] Describe $(\partial\Delta^k)_n \subseteq (\Delta^k)_n$ for $n < k$ and $n \leq k$. Using this describe $A(k)_n$ for $n < k$ and $k \leq n$.