Derived Algebraic Geometry Shane Kelly, UTokyo Autumn Semester 2023-2024

# Lecture 5: Blowups

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The goal of this course will be derived blowups. In this lecture we discuss the 1-categorical version. At the end we paint a picture of the deformation to the normal cone construction.

#### 1 Blowups of affine schemes

**Definition 1.** Suppose that R is a ring and  $I \subseteq R$  an ideal. Similar to  $\mathbb{P}^n$ , we define a presheaf on  $\mathcal{A}\mathrm{ff}_{/R}$  by sending a ring homomorphism  $R \to A$  to the set

$$Bl_RI(A) := \{A \otimes_R I \twoheadrightarrow L\} / \sim$$

of equivalence classes of surjections  $A \otimes_R I \longrightarrow L$  towards an invertible A-module L assuch that: For each d the morphism induced by  $I \longrightarrow A \otimes_R I \longrightarrow L$  factors as<sup>1</sup>

$$\operatorname{Sym}_{R}^{d}(I) \xrightarrow{} \operatorname{Sym}_{A}^{d}(L)$$

As with  $\mathbb{P}^n$ , one declares two surjections to be equivalent if there exists an isomorphism  $L \xrightarrow{\sim} L'$  of A-modules making a commutative triangle

$$A \otimes_R I \underbrace{\bigvee}_{L'}^L$$

**Proposition 2.** For any set of generators  $I = \langle r_{\lambda} \rangle_{\lambda \in \Lambda}$  there is a closed immersion  $Bl_R I \hookrightarrow \mathbb{P}^{\Lambda}_R$ . Consequently, the presheaf  $Bl_R I$  is a scheme.

*Proof.* Let  $R^{\oplus \Lambda} \twoheadrightarrow I$  be the surjection of R-modules induced by the generators  $r_{\lambda}$ . Then for each  $R \to A$ , the pullback  $A^{\oplus \Lambda} \to A \otimes_R I$  is surjective, and sending  $[A \otimes_R I \twoheadrightarrow L]$  to  $[A^{\oplus \Lambda} \twoheadrightarrow A \otimes_R I \twoheadrightarrow L]$  defines a morphism of presheaves

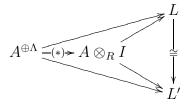
$$Bl_R I \to \mathbb{P}_R^{\Lambda}.$$

$$\operatorname{Sym}_{A}^{d} M := \operatorname{colim}_{\sigma \in \Sigma_{d}} \underbrace{M \otimes_{A} \cdots \otimes_{A} M}_{d \text{ factors}}$$

where the colimit is over elements in the symmetric group  $\Sigma_d$ , which acts by permuting the factors of  $M^{\otimes_A d}$ .

<sup>&</sup>lt;sup>1</sup> Given an A-module M, one defines

We claim that this is a closed immersion. First note that it is a monomorphism, since the outer triangle below is commutative if and only if the inner one is by surjectivity of (\*).



Let  $j \operatorname{Spec}(A) \to \mathbb{P}^{\Lambda}$  be any morphism from an affine, with corresponding epimorphism  $[A^{\oplus \Lambda} \twoheadrightarrow L]$ . We wish to find an isomorphism  $\operatorname{Spec}(A/J) \xrightarrow{\sim} Bl_R I \times_{\mathbb{P}^{\Lambda}} \operatorname{Spec}(A)$  for some ideal J. Since L is an invertible module, it is projective, so there exists an isomorphism  $L \oplus L' \cong A^{\oplus M}$  for some A-module L' and set M. On the other hand, choose a surjection from a free module to the kernel of  $R^{\oplus \Lambda} \twoheadrightarrow I$  so we get an exact sequence

$$R^{\oplus \Lambda'} \to R^{\oplus \Lambda} \to I \to 0.$$

Finally, consider the composition  $\Phi : A^{\oplus \Lambda'} \to A^{\oplus \Lambda} \twoheadrightarrow L \hookrightarrow A^{\oplus M}$ . The A-module morphism  $\Phi$  is represented by a matrix  $[\Phi]$ . Let  $J \subseteq A$  be the ideal generated by the coefficients of  $[\Phi]$ . Since  $(A/J) \otimes_A \Phi = 0$  (by definition of J) we get a factorisation

The surjection  $(A/J) \otimes_A I \longrightarrow (A/J) \otimes_A L$  corresponds to a morphism  $\operatorname{Spec}(A/J) \longrightarrow Bl_R I$ . Moreover, the fact that the composition  $(A/J)^{\oplus \Lambda} \longrightarrow (A/J) \otimes_A I \longrightarrow (A/J) \otimes_A L$  is the pullback of the original  $A^{\oplus \Lambda} \longrightarrow L$  corresponds to the following square being commutative.

So we have found a morphism

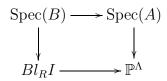
$$\operatorname{Spec}(A/J) \to Bl_R I \times_{\mathbb{P}^{\Lambda}_R} \operatorname{Spec}(A).$$

We claim it is an isomorphism. It is clearly injective since  $\text{Spec}(A/J) \to \text{Spec}(A)$  is injective, so it suffices to show that it is surjective. But this is almost by design. Suppose

$$s: \operatorname{Spec}(B) \to Bl_R I \times_{\mathbb{P}^\Lambda} \operatorname{Spec}(A)$$

is any morphism. Since  $\operatorname{Spec}(A/J) \to Bl_R I \times_{\mathbb{P}^A} \operatorname{Spec}(A) \to \operatorname{Spec}(A)$  are both monomorphisms, to show that s factors through  $\operatorname{Spec}(A/J)$ , it suffices to show that

 $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$  factors through  $\operatorname{Spec}(A/J)$ . I.e., that the corresponding  $A \to B$  sends all elements of J to zero. The original s corresponds to a commutative square



The existence of such a square implies there is a factorisation  $B^{\oplus \Lambda} \longrightarrow B \otimes_R I \longrightarrow B \otimes_A L$ . This implies  $B^{\oplus \Lambda'} \longrightarrow B^{\oplus M}$  is zero (cf. the diagram Eq.(1)), which implies all coefficients of  $[\Phi]$  are zero, which implies all elements of J are sent to zero in B.

#### Example 3.

1. Suppose that the canonical morphism  $A \otimes_R I \to A$  is an isomorphism (e.g.,  $A = R[r^{-1}]$ ) for some  $r \in I$ ). In this case we are considering surjections  $A \to L$ . But any surjection of invertible modules is an isomorphism<sup>2</sup> so in this case  $Bl_R I(A)$  consists of a single element  $[A \otimes_R I \xrightarrow{\sim} A]$ . In particular, every  $r \in I$ , induces an isomorphism

$$U \times_X Bl_R I \xrightarrow{\sim} U$$

where  $U \to X$  is the open immersion  $j \operatorname{Spec}(R[r^{-1}]) \to j \operatorname{Spec}(R)$ . In other words,  $Bl_R I \to j \operatorname{Spec}(R)$  is an isomorphism outside of the closed subscheme  $j \operatorname{Spec}(R/I) \subseteq j \operatorname{Spec}(R)$ .

2. Next, consider the case A = R/I. In this case  $A \otimes_R I \cong I/I^2$ . If furthermore there are  $t_1, \ldots, t_c \in I$  that induce an isomorphism  $I/I^2 \cong (R/I)^{\oplus c}$ , then  $Bl_R I(R/I)$  is the set of equivalence classes of quotients of  $A \otimes_R I \cong I/I^2 \cong (R/I)^{\oplus c}$ . That is,

$$Bl_R I(R/I) \cong \mathbb{P}^{c-1}(R/I).$$

It follows that, in this case, there is an isomorphism

$$Z \times_X Bl_R I \xrightarrow{\sim} Z \times \mathbb{P}^{c-1}$$

where  $Z = j \operatorname{Spec}(R/I)$ .

The geometric interpretation is as follows. The R/I-module  $I/I^2$  is the module of functions on  $\operatorname{Spec}(R)$  which vanish on R/I, module the relation:  $f \sim g$  if their if their "linear terms" agree. This is called the *conormal* module. Then a point x of  $Bl_R I$  over a point z in Z is a "normal direction" to Z at z up to scalar.

<sup>&</sup>lt;sup>2</sup>A homomorphism  $A \to L$  is an isomorphism if  $A[f^{-1}] \to L[f^{-1}]$  is an isomorphism for all elements  $A[f^{-1}]$  of a covering family. Choose one which trivialises L. Then we are reduced to showing that every epimorphism of A-modules  $A \twoheadrightarrow A$  is an isomorphism. Morphisms of Amodules  $\phi : A \to A$  are all of the form  $a \mapsto ab$  for some b, namely,  $b = \phi(1)$ . Such a  $\phi$  is surjective if and only if 1 is in the image, i.e., if and only if ab = 1 for some a, i.e., if and only if b is a unit. But in this case  $\phi$  is invertible with inverse  $a \mapsto ab^{-1}$ .

3. Suppose that  $I = \langle 0 \rangle$  is the zero ideal, but R is not the zero ring. Then there are no surjections  $A \otimes_R I \longrightarrow L$  unless A is the zero ring. So in this case  $Bl_R I \cong \emptyset$  is the empty sheaf.

#### 2 Strict transform

**Definition 4.** Let  $R \to R'$  be a ring homomorphism,  $I \subseteq R$  an ideal and set I' := IR'. Then there is a canonical morphism

 $Bl_{R'}I' \to Bl_RI$ 

which sends a given  $R' \to A$  and  $[A \otimes_{R'} I' \twoheadrightarrow L]$  to the compositions  $R \to R' \to A$  and  $[A \otimes_R I \cong A \otimes'_R R' \otimes_R I \twoheadrightarrow A \otimes_{R'} I' \twoheadrightarrow L]$ .

Exercise 5. Check that this is well defined and functorial. That is, check that:

1. if  $\operatorname{Sym}_{R'}^n I' \to L$  factors through  $(I')^n$ , then  $\operatorname{Sym}_R^n I \to L$  factors through  $I^n$ ,

2. given  $R' \to A \to B$ , the square

$$\begin{array}{c} Bl_{R'}I'(A) \longrightarrow Bl_{R}I(A) \\ \downarrow & \downarrow \\ Bl_{R'}I'(B) \longrightarrow Bl_{R}I(B) \end{array}$$

commutes.

**Exercise 6.** Suppose that  $R \to R'$  is a flat ring homomorphism. I.e., the functor  $M \mapsto R' \otimes_R M$  preserves monomorphisms. Let  $I \subseteq R$  be an ideal, and set I' := IR'. Show that the canonical morphism

$$Bl_{R'}I' \to \operatorname{Spec}(R') \times_{\operatorname{Spec}(R)} Bl_RI$$

is an isomorphism.

**Example 7.** The morphism

$$Bl_{R'}I' \to \operatorname{Spec}(R') \times_{\operatorname{Spec}(R)} Bl_R I$$

is not always an isomorphism. For example, if  $R = \mathbb{Z}[x, y]$ ,  $I = \langle x, y \rangle$ , and  $R' = \mathbb{Z}$ (with *R*-algebra structure  $x, y \mapsto 0$ ) then  $I' = \langle 0 \rangle$  so  $Bl_{R'}I' = \emptyset$ . However, since  $I/I^2 \cong \mathbb{Z} \oplus \mathbb{Z}$  is free, the pullback along  $\operatorname{Spec}(\mathbb{Z}) \to \operatorname{Spec}(\mathbb{Z}[x, y])$  is  $\mathbb{P}^1$ , as discussed at the beginning of the lecture.

$$\varnothing \cong Bl_{R'}I' \xrightarrow{\varphi} \mathbb{P}^1 \cong \operatorname{Spec}(R') \times_{\operatorname{Spec}(R)} Bl_RI \longrightarrow Bl_RI$$

## **3** Descent for schemes

We want to define blowups along closed immersions of schemes. We will do this by a descent argument. We will use the following easy fact.

**Exercise 8.** Suppose that X is a sheaf and  $\{U_{\lambda} \to X\}_{\lambda \in \Lambda}$  is a family of open immersions such that each  $U_{\lambda}$  is a scheme and  $\sqcup U_{\lambda} \to X$  is an epimorphism of sheaves. Show that X is a scheme.

**Theorem 9.** Write  $\operatorname{Sch}_R = \operatorname{Sch}_{/\operatorname{Spec}(R)}$  for the comma category. The assignment  $R \mapsto \operatorname{Sch}_R$  is a 2-functor satisfying excision. That is, for every Zariski covering in Aff of the form  $\{U \to X, V \to X\}$ , the canonical functor

$$\Phi: \operatorname{Sch}_{/X} \to 2\operatorname{-lim}\left(\operatorname{Sch}_{/U} \times \operatorname{Sch}_{/V} \rightrightarrows \operatorname{Sch}_{/W}\right)$$
(2)

is an equivalence of categories. Here  $W = U \times_X V$ .

*Proof.* To begin with we construct a left adjoint to (2). Suppose that we have morphisms of schemes  $Y_U \to U$ ,  $Y_V \to V$ ,  $Y_W \to W$  and an isomorphisms of Wschemes  $Y_U \times_U W \stackrel{a}{\cong} Y_W \stackrel{b}{\cong} Y_V \times_V W$ . Define Y via the pushout of *sheaves* 

$$\Psi(Y_V, Y_W, Y_U, a, b) := Y := Y_V \sqcup_{Y_W} Y_U.$$

This is clearly functorial in  $Y_{\bullet}$ . Since  $X = V \sqcup_W V$  it is equipped with a canonical morphism  $Y \to X$ . We show that Y is a scheme.

Since  $Y_V$  and  $Y_U$  are schemes, it suffices to show that  $Y_V, Y_U \to Y$  are open immersions and  $Y_V \sqcup Y_U \to Y$  is an epimorphism, see Exercise 8. Since  $Y = Y_V \sqcup_{Y_W}$  $Y_U$ , the morphism  $Y_V \sqcup Y_U \to Y$  is automatically an epimorphism. For the open immersions, it suffices to show that the two squares

are cartesian. We have

$$V \times_X Y_W \cong V \times_X W \times_W Y_W \cong W \times_W Y_W \cong Y_W$$
(3)

and similarly,  $V \times_X Y_U \cong Y_W$  and  $V \times_X Y_V \cong Y_V$  so

$$V \times_X (Y_V \sqcup_{Y_W} Y_U) \cong (V \times_X Y_V \sqcup_{V \times_X Y_W} V \times_X Y_U) \cong (Y_V \sqcup_{Y_W} Y_W) \cong Y_V$$

So we have a functor  $\Psi$ . To conclude it suffices to show that the canonical natural transformations  $\Psi \Phi \to \text{id}$  and  $\text{id} \to \Phi \Psi$  are isomorphisms. That is, we want to show that for  $(Y_V, Y_U, Y_W, a, b)$  as above, and  $T \to X$  in  $\text{Sch}_{/X}$  the canonical morphisms

$$T \times_X U \sqcup_{T \times_X W} T \times_X V \to T$$

and

$$Y_U \to Y \times_X U$$
$$Y_V \to Y \times_X V$$
$$Y_W \to Y \times_X W$$

are isomorphisms. The first one is true because colimits are universal, and the second set of three is the same as Eq.(3). 

#### 4 Blowups of schemes

**Corollary 10.** Let  $Z \to X$  be a closed immersion of schemes. Then there exists an X-scheme

$$Bl_XZ$$

unique up to unique isomorphism, such that for open immersion  $U := \operatorname{Spec}(A) \to X$ we have an isomorphism

$$U \times_X Bl_X Z \cong Bl_A I$$

where  $\operatorname{Spec}(A) \times_X Z = \operatorname{Spec}(A/I)$ , and for every open immersion  $V = \operatorname{Spec}(B) \to$ Spec(A) = U we have a commutative square

where (\*) is the canonical morphism from Definition 4.

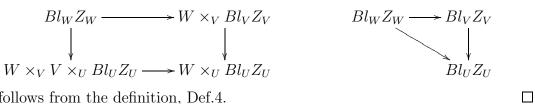
*Proof.* Since  $X = \operatorname{colim}_{\mathcal{A}\mathrm{ff}_{/X}^{\operatorname{open}}} j\operatorname{Spec}(A)$ , by Theorem 9 (and the 2-category version of the excision theorem from last week) we have

$$\operatorname{Sch}_X \xrightarrow{\sim} 2\text{-lim}_{\operatorname{Spec}(A) \to X} \operatorname{Sch}_A.$$

Above we have constructed:

- 1. For every A an A-scheme  $Bl_AI$ ,
- 2. For every open immersion  $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$  an isomorphism  $Bl_B IB \xrightarrow{\sim}$  $\operatorname{Spec}(B) \times_{\operatorname{Spec}(A)} Bl_A I.$

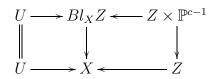
So to finish defining an object of 2-lim  $Sch_A$  we have to show that for open immersions  $W \to V \to U \to X$  with W, V, U affine, the square (below) of isomorphisms is commutative. Here we write  $Z_U, Z_V, Z_W$  for the respective pullbacks of  $Z \to X$  to U, V, W. For this it suffices to check that the triangle on the right is commutative.



This follows from the definition, Def.4.

### 5 Deformation to the normal cone

Suppose that  $\operatorname{Spec}(B) = \operatorname{Spec}(A/I) = Z \to X = \operatorname{Spec}(A)$  is a closed immersion of affine schemes. Suppose furthermore, that  $I/I^2 \cong B^{\oplus c}$  for some c. if  $U = im(\bigsqcup_{a \in I} \operatorname{Spec}(A[a^{-1}]) \to \operatorname{Spec}(A)$  is the open complement to Z, as discussed in Example 3, we obtain the following cartesian squares



Now we consider  $X \times \mathbb{A}^1 = \operatorname{Spec}(A[t])$  with closed subscheme  $Z \times \{0\} = \operatorname{Spec}(A[t]/J)$ where J = IA[t] + tA[t]. One can check that we have  $J/J^2 \cong (I/I^2) \oplus B$ . So this leads to the cartesian square

Moreover, we have the zero section  $s_0: X \to X \times \mathbb{A}^1$  and it's induced square

This square is *not* cartesian, but the horizontal morphisms are none-the-less closed immersions. We want to consider the open complement

$$\mathscr{D}_{X,Z} := Bl_{X \times \mathbb{A}^1} Z \times \{0\} \setminus Bl_X Z$$

with it's canonical morphism

$$\pi: \mathscr{D}_{X,Z} \to \mathbb{A}^1.$$

More precisely,  $\mathscr{D}_{X,Z}$  is the union of all open immersions  $V \subseteq Bl_{X \times \mathbb{A}^1} Z \times \{0\}$  whose intersection with  $Bl_X Z$  is empty.

This comes equipped with the following closed immersion  $Z \times \mathbb{A}^1 \to \mathscr{D}_{X,Z}$ . Since the ideal  $tB[t] \subseteq B[t]$  of  $Z \times \{0\} \to Z \times \mathbb{A}^1$  is generated by a single non-zero divisor, there is an isomorphism  $tB[t] \cong B[t]$ , so  $Bl_{Z \times \mathbb{A}^1}Z \times \{0\} \cong Z \times \mathbb{A}^1$  and we get a canonical morphism

$$Z \times \mathbb{A}^1 \cong Bl_{Z \times \mathbb{A}^1} Z \times \{0\} \to Bl_{X \times \mathbb{A}^1} Z \times \{0\}.$$

One can check that this doesn't intersection  $Z \times \mathbb{P}^{c-1} \subseteq Z \times \mathbb{P}^c$  so it factors as

$$\iota: Z \times \mathbb{A}^1 \to \mathscr{D}_{X,Z}.$$

By construction, away from  $\{0\} \subseteq \mathbb{A}^1$  the maps  $\mathscr{D}_{X,Z} \to Bl_{X \times \mathbb{A}^1}Z \times \{0\} \to X$  are isomorphisms. That is, if we pullback  $\iota$  along  $\mathbb{A}^1 \setminus \{0\}$  we get the closed immersion  $Z \times (\mathbb{A}^1 \setminus \{0\}) \to X \times (\mathbb{A}^1 \setminus \{0\})$ . On the other hand, by construction, over  $\{0\}$  we have a closed immersion of the form  $Z \to Z \times \mathbb{A}^c$  and one can check that this is the zero section. So we have the following cartesian squares.

That is, we have "deformed" the closed immersion  $Z \to X$  into the zero section  $Z \to Z \times \mathbb{A}^c$  of a vector bundle. In fact, all of this works with the weaker assumption that  $I/I^2$  is finite rank projective, but not necessarily free, in which case we get the vector bundle  $\operatorname{Spec}(\oplus \operatorname{Sym}_B^d I/I^2)$  over  $Z = \operatorname{Spec}(B)$ .

#### 6 Further exercises

**Exercise 11** ((Harder) The standard open affine covering). Consider the closed immersion  $Bl_R I \to \mathbb{P}_R^{\Lambda}$  from Proposition 2. Recall from the Schemes lecture that for each  $\lambda \in \Lambda$  we get an open immersion  $\mathbb{A}_R^{\Lambda \setminus \{\lambda\}} \to \mathbb{P}_R^{\Lambda}$ , and these form an open affine covering of  $\mathbb{P}_R^{\Lambda}$ . This induces an open affine covering

$$\sqcup_{\lambda \in \Lambda} Bl_R I \times_{\mathbb{P}^{\Lambda}_R} \mathbb{A}_R^{\Lambda \setminus \{\lambda\}} \to Bl_R I.$$

Show that  $Bl_R I \times_{\mathbb{P}^{\Lambda}_R} \mathbb{A}^{\Lambda \setminus \{\lambda\}}_R$  is the affine scheme associated to the ring

$$S_{r_{\lambda}} := \operatorname{colim}(R \xrightarrow{r_{\lambda}} I \xrightarrow{r_{\lambda}} I^2 \xrightarrow{r_{\lambda}} \dots)$$

where the transition morphisms are multiplication by  $r_{\lambda}$ . Note that  $S_{r_{\lambda}}$  can be considered as the subring

$$\{\frac{m}{r_{\lambda}^n} \in R[r_{\lambda}^{-1}] \mid m \in I^n\}$$

by assembling the morphisms  $I^n \to R[r_{\lambda}^{-1}]; m \mapsto \frac{m}{r_{\lambda}^n}$ .

**Definition 12.** As with  $\mathbb{P}^n$ , the assignment

$$\mathcal{O}(1): [A \otimes_R I \twoheadrightarrow L] \mapsto L$$

(where we make a choice in each equivalence class) defines an invertible module on the presheaf  $Bl_R I$ .

**Exercise 13** (The invertible module  $\mathcal{O}(1)$ ). Consider the assignment

$$\mathcal{O}(1): [A \otimes_R I \longrightarrow L] \mapsto L$$

(where we make a choice in each equivalence class).

Show that  $\mathcal{O}(1)$  actually is a quasi-coherent module. That is, show that one can choose an isomorphism  $\phi_{s,f} : B \otimes_A \mathcal{O}(1)(s) \xrightarrow{\sim} \mathcal{O}(1)(sf)$  for every pair of morphisms  $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$  and  $s \in Bl_RI(A) \cong \operatorname{hom}(\operatorname{Spec}(A), Bl_RI)$ , such that given a third morphism  $g : \operatorname{Spec}(C) \to \operatorname{Spec}(B)$  the square

commutes.

**Example 14.** Show that  $Bl_{\mathbb{Z}[x_1,\ldots,x_n]}\langle x_1,\ldots,x_n\rangle$  admits an open affine covering by the schemes  $\operatorname{Spec} \mathbb{Z}[\frac{x_1}{x_i},\ldots,\frac{x_{i-1}}{x_i},x_i,\frac{x_{i+1}}{x_i},\ldots,\frac{x_n}{x_i}] \cong \mathbb{A}^n$ .