Derived Algebraic Geometry
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Lecture 5: Blowups

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The goal of this course will be derived blowups. In this lecture we discuss the 1-categorical version. At the end we paint a picture of the deformation to the normal cone construction.

1 Blowups of affine schemes

Definition 1. Suppose that R is a ring and $I \subseteq R$ an ideal. Similar to $\mathbb{P}^n$, we define a presheaf on $\mathcal{A}\mathrm{ff}_{/R}$ by sending a ring homomorphism $R \to A$ to the set

$$Bl_R I(A) := \{A \otimes_R I \twoheadrightarrow L\}/\sim$$

of equivalence classes of surjections $A \otimes_R I \twoheadrightarrow L$ towards an invertible A-module L assuch that: For each d the morphism induced by $I \to A \otimes_R I \to L$ factors as[1]

$$\begin{array}{ccc} \mathrm{Sym}^d_R(I) & \twoheadrightarrow & \mathrm{Sym}^d_A(L) \\ & \searrow\!\!\!\!\searrow \quad I^d \quad \dashrightarrow & \end{array}$$

As with $\mathbb{P}^n$, one declares two surjections to be equivalent if there exists an isomorphism $L \xrightarrow{\sim} L'$ of A-modules making a commutative triangle

$$\begin{array}{ccc} & \nearrow & L \\ A \otimes_R I & & \downarrow \cong \\ & \searrow & L' \end{array}$$

Proposition 2. *For any set of generators $I = \langle r_\lambda \rangle_{\lambda \in \Lambda}$ there is a closed immersion $Bl_R I \hookrightarrow \mathbb{P}^\Lambda_R$. Consequently, the presheaf $Bl_R I$ is a scheme.*

Proof. Let $R^{\oplus\Lambda} \twoheadrightarrow I$ be the surjection of R-modules induced by the generators r_λ. Then for each $R \to A$, the pullback $A^{\oplus\Lambda} \to A \otimes_R I$ is surjective, and sending $[A \otimes_R I \twoheadrightarrow L]$ to $[A^{\oplus\Lambda} \twoheadrightarrow A \otimes_R I \twoheadrightarrow L]$ defines a morphism of presheaves

$$Bl_R I \to \mathbb{P}^\Lambda_R.$$

[1] Given an A-module M, one defines

$$\mathrm{Sym}^d_A M := \operatorname*{colim}_{\sigma \in \Sigma_d} \underbrace{M \otimes_A \cdots \otimes_A M}_{d \text{ factors}}$$

where the colimit is over elements in the symmetric group Σ_d, which acts by permuting the factors of $M^{\otimes_A d}$.

We claim that this is a closed immersion. First note that it is a monomorphism, since the outer triangle below is commutative if and only if the inner one is by surjectivity of $(*)$.

$$\begin{tikzcd} & & L \arrow[dd, "\cong"] \\ A^{\oplus\Lambda} \arrow[r, two heads, "(*)"] \arrow[urr] \arrow[drr] & A \otimes_R I \arrow[ur] \arrow[dr] & \\ & & L' \end{tikzcd}$$

Let $j\,\mathrm{Spec}(A) \to \mathbb{P}^\Lambda$ be any morphism from an affine, with corresponding epimorphism $[A^{\oplus\Lambda} \twoheadrightarrow L]$. We wish to find an isomorphism $\mathrm{Spec}(A/J) \xrightarrow{\sim} Bl_R I \times_{\mathbb{P}^\Lambda} \mathrm{Spec}(A)$ for some ideal J. Since L is an invertible module, it is projective, so there exists an isomorphism $L \oplus L' \cong A^{\oplus M}$ for some A-module L' and set M. On the other hand, choose a surjection from a free module to the kernel of $R^{\oplus\Lambda} \twoheadrightarrow I$ so we get an exact sequence

$$R^{\oplus\Lambda'} \to R^{\oplus\Lambda} \to I \to 0.$$

Finally, consider the composition $\Phi : A^{\oplus\Lambda'} \to A^{\oplus\Lambda} \twoheadrightarrow L \hookrightarrow A^{\oplus M}$. The A-module morphism Φ is represented by a matrix $[\Phi]$. Let $J \subseteq A$ be the ideal generated by the coefficients of $[\Phi]$. Since $(A/J) \otimes_A \Phi = 0$ (by definition of J) we get a factorisation

$$\begin{tikzcd} (A/J)^{\oplus\Lambda'} \arrow[r] \arrow[drrr, "(A/J)\otimes_A\Phi"] & (A/J)^{\oplus\Lambda} \arrow[r, two heads] \arrow[d, equal] & (A/J) \otimes_A I \arrow[r] \arrow[d, dashed] & 0 \\ & (A/J)^{\oplus\Lambda} \arrow[r, two heads] & (A/J) \otimes_A L \arrow[r, "\subseteq"] & (A/J)^{\oplus M} \cong (A/J) \otimes_A (L \oplus L') \end{tikzcd} \tag{1}$$

The surjection $(A/J) \otimes_A I \twoheadrightarrow (A/J) \otimes_A L$ corresponds to a morphism $\mathrm{Spec}(A/J) \to Bl_R I$. Moreover, the fact that the composition $(A/J)^{\oplus\Lambda} \to (A/J) \otimes_A I \twoheadrightarrow (A/J) \otimes_A L$ is the pullback of the original $A^{\oplus\Lambda} \twoheadrightarrow L$ corresponds to the following square being commutative.

$$\begin{tikzcd} \mathrm{Spec}(A/J) \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d] \\ Bl_R I \arrow[r] & \mathbb{P}^\Lambda_R \end{tikzcd}$$

So we have found a morphism

$$\mathrm{Spec}(A/J) \to Bl_R I \times_{\mathbb{P}^\Lambda_R} \mathrm{Spec}(A).$$

We claim it is an isomorphism. It is clearly injective since $\mathrm{Spec}(A/J) \to \mathrm{Spec}(A)$ is injective, so it suffices to show that it is surjective. But this is almost by design. Suppose

$$s : \mathrm{Spec}(B) \to Bl_R I \times_{\mathbb{P}^\Lambda} \mathrm{Spec}(A)$$

is any morphism. Since $\mathrm{Spec}(A/J) \to Bl_R I \times_{\mathbb{P}^\Lambda} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ are both monomorphisms, to show that s factors through $\mathrm{Spec}(A/J)$, it suffices to show that

$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ factors through $\mathrm{Spec}(A/J)$. I.e., that the corresponding $A \to B$ sends all elements of J to zero. The original s corresponds to a commutative square

$$\begin{array}{ccc} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ Bl_R I & \longrightarrow & \mathbb{P}^\Lambda \end{array}$$

The existence of such a square implies there is a factorisation $B^{\oplus\Lambda} \twoheadrightarrow B \otimes_R I \dashrightarrow B \otimes_A L$. This implies $B^{\oplus\Lambda'} \to B^{\oplus M}$ is zero (cf. the diagram Eq.(1)), which implies all coefficients of $[\Phi]$ are zero, which implies all elements of J are sent to zero in B. $\square$

Example 3.

1. Suppose that the canonical morphism $A \otimes_R I \to A$ is an isomorphism (e.g., $A = R[r^{-1}]$) for some $r \in I$). In this case we are considering surjections $A \twoheadrightarrow L$. But any surjection of invertible modules is an isomorphism[2] so in this case $Bl_R I(A)$ consists of a single element $[A \otimes_R I \xrightarrow{\sim} A]$. In particular, every $r \in I$, induces an isomorphism
$$U \times_X Bl_R I \xrightarrow{\sim} U$$
where $U \to X$ is the open immersion $j\,\mathrm{Spec}(R[r^{-1}]) \to j\,\mathrm{Spec}(R)$. In other words, $Bl_R I \to j\,\mathrm{Spec}(R)$ is an isomorphism outside of the closed subscheme $j\,\mathrm{Spec}(R/I) \subseteq j\,\mathrm{Spec}(R)$.
2. Next, consider the case $A = R/I$. In this case $A \otimes_R I \cong I/I^2$. If furthermore there are $t_1, \ldots, t_c \in I$ that induce an isomorphism $I/I^2 \cong (R/I)^{\oplus c}$, then $Bl_R I(R/I)$ is the set of equivalence classes of quotients of $A \otimes_R I \cong I/I^2 \cong (R/I)^{\oplus c}$. That is,
$$Bl_R I(R/I) \cong \mathbb{P}^{c-1}(R/I).$$
It follows that, in this case, there is an isomorphism
$$Z \times_X Bl_R I \xrightarrow{\sim} Z \times \mathbb{P}^{c-1}$$
where $Z = j\,\mathrm{Spec}(R/I)$.
The geometric interpretation is as follows. The R/I-module I/I^2 is the module of functions on $\mathrm{Spec}(R)$ which vanish on R/I, module the relation: $f \sim g$ if their if their "linear terms" agree. This is called the *conormal* module. Then a point x of $Bl_R I$ over a point z in Z is a "normal direction" to Z at z up to scalar.

[2] A homomorphism $A \to L$ is an isomorphism if $A[f^{-1}] \to L[f^{-1}]$ is an isomorphism for all elements $A[f^{-1}]$ of a covering family. Choose one which trivialises L. Then we are reduced to showing that every epimorphism of A-modules $A \twoheadrightarrow A$ is an isomorphism. Morphisms of A-modules $\phi : A \to A$ are all of the form $a \mapsto ab$ for some b, namely, $b = \phi(1)$. Such a ϕ is surjective if and only if 1 is in the image, i.e., if and only if $ab = 1$ for some a, i.e., if and only if b is a unit. But in this case ϕ is invertible with inverse $a \mapsto ab^{-1}$.

3. Suppose that $I = \langle 0 \rangle$ is the zero ideal, but R is not the zero ring. Then there are no surjections $A \otimes_R I \twoheadrightarrow L$ unless A is the zero ring. So in this case $Bl_R I \cong \varnothing$ is the empty sheaf.

2 Strict transform

Definition 4. Let $R \to R'$ be a ring homomorphism, $I \subseteq R$ an ideal and set $I' := IR'$. Then there is a canonical morphism

$$Bl_{R'} I' \to Bl_R I$$

which sends a given $R' \to A$ and $[A \otimes_{R'} I' \twoheadrightarrow L]$ to the compositions $R \to R' \to A$ and $[A \otimes_R I \cong A \otimes_R' R' \otimes_R I \twoheadrightarrow A \otimes_{R'} I' \twoheadrightarrow L]$.

Exercise 5. Check that this is well defined and functorial. That is, check that:

1. if $\mathrm{Sym}^n_{R'} I' \to L$ factors through $(I')^n$, then $\mathrm{Sym}^n_R I \to L$ factors through I^n,
2. given $R' \to A \to B$, the square

$$\begin{array}{ccc} Bl_{R'} I'(A) & \longrightarrow & Bl_R I(A) \\ \downarrow & & \downarrow \\ Bl_{R'} I'(B) & \longrightarrow & Bl_R I(B) \end{array}$$

commutes.

Exercise 6. Suppose that $R \to R'$ is a flat ring homomorphism. I.e., the functor $M \mapsto R' \otimes_R M$ preserves monomorphisms. Let $I \subseteq R$ be an ideal, and set $I' := IR'$. Show that the canonical morphism

$$Bl_{R'} I' \to \mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} Bl_R I$$

is an isomorphism.

Example 7. The morphism

$$Bl_{R'} I' \to \mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} Bl_R I$$

is not always an isomorphism. For example, if $R = \mathbb{Z}[x, y]$, $I = \langle x, y \rangle$, and $R' = \mathbb{Z}$ (with R-algebra structure $x, y \mapsto 0$) then $I' = \langle 0 \rangle$ so $Bl_{R'} I' = \varnothing$. However, since $I/I^2 \cong \mathbb{Z} \oplus \mathbb{Z}$ is free, the pullback along $\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}[x, y])$ is $\mathbb{P}^1$, as discussed at the beginning of the lecture.

$$\begin{array}{ccccc} \varnothing \cong Bl_{R'} I' & \xrightarrow{\not\cong} & \mathbb{P}^1 \cong \mathrm{Spec}(R') \times_{\mathrm{Spec}(R)} Bl_R I & \longrightarrow & Bl_R I \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}(\mathbb{Z}) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[x, y]) \end{array}$$

3 Descent for schemes

We want to define blowups along closed immersions of schemes. We will do this by a descent argument. We will use the following easy fact.

Exercise 8. Suppose that X is a sheaf and $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a family of open immersions such that each U_λ is a scheme and $\sqcup U_\lambda \to X$ is an epimorphism of sheaves. Show that X is a scheme.

Theorem 9. *Write* $\mathrm{Sch}_R = \mathrm{Sch}_{/\,\mathrm{Spec}(R)}$ *for the comma category. The assignment* $R \mapsto \mathrm{Sch}_R$ *is a 2-functor satisfying excision. That is, for every Zariski covering in* $\mathcal{A}\mathrm{ff}$ *of the form* $\{U \to X, V \to X\}$*, the canonical functor*

$$\Phi : \mathrm{Sch}_{/X} \to 2\text{-}\lim\Big(\mathrm{Sch}_{/U} \times \mathrm{Sch}_{/V} \rightrightarrows \mathrm{Sch}_{/W}\Big) \tag{2}$$

is an equivalence of categories. Here $W = U \times_X V$*.*

Proof. To begin with we construct a left adjoint to (2). Suppose that we have morphisms of schemes $Y_U \to U$, $Y_V \to V$, $Y_W \to W$ and an isomorphisms of W-schemes $Y_U \times_U W \overset{a}{\cong} Y_W \overset{b}{\cong} Y_V \times_V W$. Define Y via the pushout of *sheaves*

$$\Psi(Y_V, Y_W, Y_U, a, b) := Y := Y_V \sqcup_{Y_W} Y_U.$$

This is clearly functorial in $Y_\bullet$. Since $X = V \sqcup_W V$ it is equipped with a canonical morphism $Y \to X$. We show that Y is a scheme.

Since Y_V and Y_U are schemes, it suffices to show that $Y_V, Y_U \to Y$ are open immersions and $Y_V \sqcup Y_U \to Y$ is an epimorphism, see Exercise 8. Since $Y = Y_V \sqcup_{Y_W} Y_U$, the morphism $Y_V \sqcup Y_U \to Y$ is automatically an epimorphism. For the open immersions, it suffices to show that the two squares

$$\begin{array}{ccc} Y_V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array} \qquad\qquad \begin{array}{ccc} Y_U & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

are cartesian. We have

$$V \times_X Y_W \cong V \times_X W \times_W Y_W \cong W \times_W Y_W \cong Y_W \tag{3}$$

and similarly, $V \times_X Y_U \cong Y_W$ and $V \times_X Y_V \cong Y_V$ so

$$V \times_X (Y_V \sqcup_{Y_W} Y_U) \cong (V \times_X Y_V \sqcup_{V \times_X Y_W} V \times_X Y_U) \cong (Y_V \sqcup_{Y_W} Y_W) \cong Y_V$$

So we have a functor Ψ. To conclude it suffices to show that the canonical natural transformations $\Psi\Phi \to \mathrm{id}$ and $\mathrm{id} \to \Phi\Psi$ are ismorphisms. That is, we want to show that for (Y_V, Y_U, Y_W, a, b) as above, and $T \to X$ in $\mathrm{Sch}_{/X}$ the canonical morphisms

$$T \times_X U \sqcup_{T \times_X W} T \times_X V \to T$$

and

$$\begin{aligned} Y_U &\to Y \times_X U \\ Y_V &\to Y \times_X V \\ Y_W &\to Y \times_X W \end{aligned}$$

are isomorphisms. The first one is true because colimits are universal, and the second set of three is the same as Eq.(3). $\square$

4 Blowups of schemes

Corollary 10. *Let $Z \to X$ be a closed immersion of schemes. Then there exists an X-scheme*

$$Bl_X Z,$$

unique up to unique isomorphism, such that for open immersion $U := \mathrm{Spec}(A) \to X$ we have an isomorphism

$$U \times_X Bl_X Z \cong Bl_A I$$

where $\mathrm{Spec}(A) \times_X Z = \mathrm{Spec}(A/I)$, and for every open immersion $V = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = U$ we have a commutative square

$$\begin{array}{ccc} V \times_X Bl_X Z & \longrightarrow & Bl_B IB \\ \downarrow & & \downarrow (*) \\ V \times_U U \times_X Bl_X Z & \longrightarrow & V \times_U Bl_A I \end{array}$$

where $(*)$ is the canonical morphism from Definition 4.

Proof. Since $X = \mathrm{colim}_{\mathcal{A}\mathrm{ff}^{\mathrm{open}}_{/X}} j\,\mathrm{Spec}(A)$, by Theorem 9 (and the 2-category version of the excision theorem from last week) we have

$$\mathrm{Sch}_X \xrightarrow{\sim} \underset{\substack{\mathrm{Spec}(A)\to X \\ \mathrm{open}}}{2\text{-}\lim} \mathrm{Sch}_A.$$

Above we have constructed:

1. For every A an A-scheme $Bl_A I$,
2. For every open immersion $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ an isomorphism $Bl_B IB \xrightarrow{\sim} \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} Bl_A I$.

So to finish defining an object of 2-lim Sch_A we have to show that for open immersions $W \to V \to U \to X$ with W, V, U affine, the square (below) of isomorphisms is commutative. Here we write Z_U, Z_V, Z_W for the respective pullbacks of $Z \to X$ to U, V, W. For this it suffices to check that the triangle on the right is commutative.

$$\begin{array}{ccc} Bl_W Z_W & \longrightarrow & W \times_V Bl_V Z_V \\ \downarrow & & \downarrow \\ W \times_V V \times_U Bl_U Z_U & \longrightarrow & W \times_U Bl_U Z_U \end{array} \qquad \begin{array}{ccc} Bl_W Z_W & \longrightarrow & Bl_V Z_V \\ & \searrow & \downarrow \\ & & Bl_U Z_U \end{array}$$

This follows from the definition, Def.4. $\square$

5 Deformation to the normal cone

Suppose that $\mathrm{Spec}(B) = \mathrm{Spec}(A/I) = Z \to X = \mathrm{Spec}(A)$ is a closed immersion of affine schemes. Suppose furthermore, that $I/I^2 \cong B^{\oplus c}$ for some c. if $U = im(\sqcup_{a\in I} \mathrm{Spec}(A[a^{-1}]) \to \mathrm{Spec}(A)$ is the open complement to Z, as discussed in Example 3, we obtain the following cartesian squares

$$\begin{array}{ccccc} U & \longrightarrow & Bl_X Z & \longleftarrow & Z \times \mathbb{P}^{c-1} \\ \| & & \downarrow & & \downarrow \\ U & \longrightarrow & X & \longleftarrow & Z \end{array}$$

Now we consider $X \times \mathbb{A}^1 = \mathrm{Spec}(A[t])$ with closed subscheme $Z \times \{0\} = \mathrm{Spec}(A[t]/J)$ where $J = IA[t] + tA[t]$. One can check that we have $J/J^2 \cong (I/I^2) \oplus B$. So this leads to the cartesian square

$$\begin{array}{ccc} Bl_{X\times\mathbb{A}^1} Z\times\{0\} & \longleftarrow & Z \times \mathbb{P}^c \\ \downarrow & & \downarrow \\ X \times \mathbb{A}^1 & \longleftarrow & Z \times \{0\} \end{array}$$

Moreover, we have the zero section $s_0 : X \to X \times \mathbb{A}^1$ and it's induced square

$$\begin{array}{ccc} Bl_X Z & \longrightarrow & Bl_{X\times\mathbb{A}^1} Z\times\{0\} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times \mathbb{A}^1 \end{array}$$

This square is *not* cartesian, but the horizontal morphisms are none-the-less closed immersions. We want to consider the open complement

$$\mathscr{D}_{X,Z} := Bl_{X\times\mathbb{A}^1} Z\times\{0\} \setminus Bl_X Z$$

with it's canonical morphism

$$\pi : \mathscr{D}_{X,Z} \to \mathbb{A}^1.$$

More precisely, $\mathscr{D}_{X,Z}$ is the union of all open immersions $V \subseteq Bl_{X\times\mathbb{A}^1} Z\times\{0\}$ whose intersection with $Bl_X Z$ is empty.

This comes equipped with the following closed immersion $Z \times \mathbb{A}^1 \to \mathscr{D}_{X,Z}$. Since the ideal $tB[t] \subseteq B[t]$ of $Z \times \{0\} \to Z \times \mathbb{A}^1$ is generated by a single non-zero divisor, there is an isomorphism $tB[t] \cong B[t]$, so $Bl_{Z\times\mathbb{A}^1} Z \times \{0\} \cong Z \times \mathbb{A}^1$ and we get a canonical morphism

$$Z \times \mathbb{A}^1 \cong Bl_{Z\times\mathbb{A}^1} Z \times \{0\} \to Bl_{X\times\mathbb{A}^1} Z\times\{0\}.$$

One can check that this doesn't intersection $Z \times \mathbb{P}^{c-1} \subseteq Z \times \mathbb{P}^c$ so it factors as

$$\iota : Z \times \mathbb{A}^1 \to \mathscr{D}_{X,Z}.$$

By construction, away from $\{0\} \subseteq \mathbb{A}^1$ the maps $\mathscr{D}_{X,Z} \to Bl_{X\times\mathbb{A}^1} Z\times\{0\} \to X$ are isomorphisms. That is, if we pullback ι along $\mathbb{A}^1 \setminus \{0\}$ we get the closed immersion $Z \times (\mathbb{A}^1 \setminus \{0\}) \to X \times (\mathbb{A}^1 \setminus \{0\})$. On the other hand, by construction, over $\{0\}$ we have a closed immersion of the form $Z \to Z \times \mathbb{A}^c$ and one can check that this is the zero section. So we have the following cartesian squares.

$$\begin{array}{ccccc}
Z \times (\mathbb{A}^1 \setminus 0) & \longrightarrow & Z \times \mathbb{A}^1 & \longleftarrow & Z \times \{0\} \\
\downarrow & & \downarrow & & \downarrow \\
X \times (\mathbb{A}^1 \setminus 0) & \longrightarrow & \mathscr{D}_{X,Z} & \longleftarrow & Z \times \mathbb{A}^c \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{A}^1 \setminus 0 & \longrightarrow & \mathbb{A}^1 & \longleftarrow & \{0\}
\end{array}$$

That is, we have "deformed" the closed immersion $Z \to X$ into the zero section $Z \to Z \times \mathbb{A}^c$ of a vector bundle. In fact, all of this works with the weaker assumption that I/I^2 is finite rank projective, but not necessarily free, in which case we get the vector bundle $\mathrm{Spec}(\oplus \mathrm{Sym}^d_B I/I^2)$ over $Z = \mathrm{Spec}(B)$.

6 Further exercises

Exercise 11 ((Harder) The standard open affine covering)**.** Consider the closed immersion $Bl_R I \to \mathbb{P}^\Lambda_R$ from Proposition 2. Recall from the Schemes lecture that for each $\lambda \in \Lambda$ we get an open immersion $\mathbb{A}_R^{\Lambda\setminus\{\lambda\}} \to \mathbb{P}^\Lambda_R$, and these form an open affine covering of $\mathbb{P}^\Lambda_R$. This induces an open affine covering

$$\sqcup_{\lambda\in\Lambda} Bl_R I \times_{\mathbb{P}^\Lambda_R} \mathbb{A}_R^{\Lambda\setminus\{\lambda\}} \to Bl_R I.$$

Show that $Bl_R I \times_{\mathbb{P}^\Lambda_R} \mathbb{A}_R^{\Lambda\setminus\{\lambda\}}$ is the affine scheme associated to the ring

$$S_{r_\lambda} := \mathrm{colim}(R \xrightarrow{r_\lambda} I \xrightarrow{r_\lambda} I^2 \xrightarrow{r_\lambda} \dots)$$

where the transition morphisms are multiplication by r_λ. Note that S_{r_λ} can be considered as the subring

$$\{\tfrac{m}{r_\lambda^n} \in R[r_\lambda^{-1}] \mid m \in I^n\}$$

by assembling the morphisms $I^n \to R[r_\lambda^{-1}];\ m \mapsto \frac{m}{r_\lambda^n}$.

Definition 12. As with $\mathbb{P}^n$, the assignment

$$\mathcal{O}(1) : [A \otimes_R I \twoheadrightarrow L] \mapsto L$$

(where we make a choice in each equivalence class) defines an invertible module on the presheaf $Bl_R I$.

Exercise 13 (The invertible module $\mathcal{O}(1)$)**.** Consider the assignment

$$\mathcal{O}(1) : [A \otimes_R I \twoheadrightarrow L] \mapsto L$$

(where we make a choice in each equivalence class).

Show that $\mathcal{O}(1)$ actually is a quasi-coherent module. That is, show that one can choose an isomorphism $\phi_{s,f} : B \otimes_A \mathcal{O}(1)(s) \xrightarrow{\sim} \mathcal{O}(1)(sf)$ for every pair of morphisms $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $s \in Bl_R I(A) \cong \hom(\mathrm{Spec}(A), Bl_R I)$, such that given a third morphism $g : \mathrm{Spec}(C) \to \mathrm{Spec}(B)$ the square

$$\begin{array}{ccc} C \otimes_B B \otimes_A \mathcal{O}(1)(s) & \longrightarrow & C \otimes_A \mathcal{O}(1)(s) \\ \downarrow{\scriptstyle C\otimes_B \phi_{s,f}} & & \downarrow{\scriptstyle \phi_{s,fg}} \\ C \otimes_B \mathcal{O}(1)(sf) & \xrightarrow[\phi_{sf,g}]{} & \mathcal{O}(1)(sfg) \end{array}$$

commutes.

Example 14. Show that $Bl_{\mathbb{Z}[x_1,\dots,x_n]}\langle x_1, \dots, x_n\rangle$ admits an open affine covering by the schemes $\mathrm{Spec}\,\mathbb{Z}[\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}] \cong \mathbb{A}^n$.