Derived Algebraic Geometry
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Lecture 4: Quasi-coherent modules

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In this lecture our main goal is to describe the category of quasi-coherent modules on a presheaf F sufficiently well that we can give some examples:

1. the structure module $\mathcal{O}$,
2. the invertible module $\mathcal{O}(1)$ on $\mathbb{P}^n$, and
3. the module of Kähler differentials Ω_X on a scheme X.

For affine schemes we set

$$\mathcal{Q}\mathrm{Coh}(\mathrm{Spec}(R)) = R\text{-mod}.$$

More generally, any presheaf F can be written as a colimit of affines $F = \mathrm{colim}_{\mathcal{A}\mathrm{ff}_{/F}} j\,\mathrm{Spec}(R)$. So giving a module over F, should be the same as giving a module over each $j\,\mathrm{Spec}(R) \to F$ such that these modules are compatible with the way F is glued out of the $\mathrm{Spec}(R)$. That is, we want to define

$$\mathcal{Q}\mathrm{Coh}(F) :\overset{?}{=} \lim_{j\,\mathrm{Spec}(R)\to F} R\text{-mod}.$$

Since (strict) limits do not preserve equivalences of categories, the above does not work well. We begin with the notion of 2-limit, which is a version of limit that takes equivalences into account. In fact, 2-limits are a special case of the homotopy limits which we will see later in the ∞-categorical setting.

1 2-functors

We would like to consider sheaves of categories. For this, we need to consider limits of categories. In general limits of categories do not preserve equivalence.

Example 1. The square on the left and right are both cartesian in the 1-category of categories.

$$\begin{array}{ccc} \varnothing & \longrightarrow & \{0\} \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \{0 \cong 1\} \end{array} \qquad\qquad \begin{array}{ccc} * & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

Note that the lower right corners are equivalent, but the pullbacks are not.

So we need to incorporate this "unique up to isomorphism" into the framework.

Definition 2 (2-functor)**.** A *2-functor* $F : I \to \mathcal{C}\mathrm{at}$ associates to every:

object	i	a category	F_i
morphism	$i \xrightarrow{u} j$	a functor	$F_i \xrightarrow{F_u} F_j$
triangle	$\xrightarrow{u}\xrightarrow{v}$	a natural isomorphism	$F_{(u,v)} : F_v F_u \xrightarrow{\sim} F_{vu}$
tetrahdron	$\xrightarrow{u}\xrightarrow{v}\xrightarrow{w}$	a commutative square	$\begin{array}{ccc} F_w F_v F_u & \xrightarrow{\sim} & F_w F_{vu} \\ \downarrow \sim & & \downarrow \sim \\ F_{wv} F_u & \xrightarrow[\sim]{} & F_{wvu}. \end{array}$

To avoid spending our short lives checking coherence diagrams, we also make the harmless requirements $F_{\mathrm{id}_i} = \mathrm{id}_{F_i}$, $F_{\mathrm{id}_i,u} = \mathrm{id}_{F_u}$, and , $F_{u,\mathrm{id}_j} = \mathrm{id}_{F_u}$.[1]

Remark 3. In this setting, the tetrahedron line is a condition, not a datum, but from the ∞-category point of view it is the datum of a homotopy between homotopies. Since we are working with 2-categories at the moment, all 3-cells are identities. So the choice of a 3-cell is the same as the affirmation that (the unique one) exists.

Example 4.

1. Any normal functor
$$F : I \to \mathcal{S}\mathrm{et}$$
can be considered as a 2-functor, by thinking of each set $F(i)$ as a category whose objects are elements of $F(i)$, and all morphisms are identity morphisms.
2. The association
$$R \mapsto R\text{-mod}$$
with the functors $R\text{-mod} \to S\text{-mod} : M \mapsto S \otimes_R M$ is naturally a 2-functor. Similarly, for $R \mapsto R\text{-alg}$.
3. The assignment $\mathrm{Spec}(R) \mapsto \mathrm{Sch}_{/j\,\mathrm{Spec}(R)}$ has a natural structure of 2-functor.
4. If X is a topological space and $I = \mathcal{O}\mathrm{pen}(X)^{\mathrm{op}}$ the corresponding category then the assignment
$$U \mapsto \mathrm{Shv}(U)$$
together with the restriction functor $\mathrm{Shv}(U) \to \mathrm{Shv}(V)$ for inclusions $V \subseteq U$ has a structure of 2-functor.

Definition 5 (2-natural transformation)**.** Suppose that $F, G : I \to \mathcal{C}\mathrm{at}$ are two 2-functors. A *2-natural transformation* $f : F \to G$ of 2-functors associates to every

object	i	a functor	$F_i \xrightarrow{f_i} G_i$
morphism	$i \xrightarrow{u} j$	a natural *isomorphism*	$f_u : G_u f_i \xrightarrow{\sim} f_j F_u$
triangle	$i \xrightarrow{u} j \xrightarrow{v} k$	a commutative pentagon	$\begin{array}{ccccc} G_v G_u f_i & & \xrightarrow{\sim} & & G_{vu} f_i \\ \downarrow \sim & & \circlearrowleft & & \downarrow \sim \\ G_v f_j F_u & \xrightarrow{\sim} & f_k F_v F_u & \xrightarrow{\sim} & f_k F_{vu} \end{array}$

[1] Some definitions only require that $F(\mathrm{id}_i)$ be equipped with an isomorphism to $\mathrm{id}_{F(i)}$ compatible with the $\phi_{u,v}$ in an appropriate way.

Remark 6.

1. The natural isomorphisms in the pentagon come from $f_{vu}, G_{(u,v)}, F_{(u,v)}, f_v, f_u$.
2. The pentagon lives in the category of functors $\mathrm{Fun}(F_i, G_k)$ which is a 1-category: all 2-cells are identities. So again, the triangle line is a condition, not a datum.

Example 7.

1. Any morphism of functors $F, G : I \to \mathcal{S}\mathrm{et}$ induces a morphism of the associated 2-functors via the canonical embedding $\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at}$ sending a set to the associated discrete category.
2. If F is a discrete 2-functor, (e.g., the 2-functor associated to a sieve), and $f : F \to G$ is a morphism towards some other 2-functor, then the pentagon in part three becomes a square because all $F_{(u,v)} : F_v F_u \to F_{vu}$ are identities

$$\begin{array}{ccccc} G_v G_u f_i & & \xrightarrow{\sim} & & G_{vu} f_i \\ \downarrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\sim} \\ G_v f_j F_u & \xrightarrow{\sim} & f_k F_v F_u & = & f_k F_{vu} \end{array} \tag{1}$$

3. Suppose F is a constant 2-functor. So all F_u and $F_{(u,v)}$ are identities. Then the pentagon becomes even simpler.

$$\begin{array}{ccc} G_v G_u f_i & \xrightarrow{\sim} & G_{vu} f_i \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ G_v f_j & \xrightarrow{\sim} & f_k \end{array}$$

Definition 8 (2-modification)**.** Suppose $f, g : F \rightrightarrows G$ are two 2-morphisms between 2-functors. A *2-modification* $\phi : f \to g$ associates to every

object	i	a natural transformation	$f_i \xrightarrow{\phi_i} g_i$
morphism	$i \xrightarrow{u} j$	a commutative square	$g_u \phi_i = \phi_j f_u$

We will write

$$\mathcal{H}\mathrm{om}(F, G)$$

for the category whose objects are 2-natural transformations and morphisms are 2-modifications. The *2-limit* of a 2-functor $F : I \to \mathcal{C}\mathrm{at}$ is the category

$$2\text{-}\lim_I F := \mathcal{H}\mathrm{om}(*, F)$$

where $*$ is the 2-functor that sends all $i \in I$ to the terminal category $*$.

Example 9. Let $I = \{0 \xrightarrow{u} 1 \xleftarrow{v} 2\}$ and suppose $F : I \to \mathcal{C}\mathrm{at}$ is a 2-functor. Then an object of the category $2\text{-}\lim_I F$ is a quintuple

$$(M_0, M_1, M_2, M_u, M_v)$$

where M_i are objects in F_i $(i = 0, 1, 2)$ and $M_u : F_u M_0 \xrightarrow{\sim} M_1$, $M_v : F_v M_2 \xrightarrow{\sim} M_1$ are isomorphisms in M_1. A morphism $\phi : M \to N$ is a triple (ϕ_0, ϕ_1, ϕ_2) where $\phi_i : M_i \to N_i$ is a morphism in F_i and the two squares

$$\begin{array}{ccccc} F_u M_0 & \longrightarrow & M_1 & \longleftarrow & F_v M_2 \\ \downarrow & & \downarrow & & \downarrow \\ F_u N_0 & \longrightarrow & N_1 & \longleftarrow & F_v N_2 \end{array}$$

are commutative.

Example 10. Let $I = \mathcal{A}\text{ff}^{\text{op}}$ and consider the functor $\mathcal{M} : \text{Spec}(R) \mapsto R\text{-mod}$. An object of $2\text{-}\lim_{\mathcal{A}\text{ff}^{\text{op}}} \mathcal{M}$ is an assignment of:

an R-module	M_R	for each ring R,
a homomorphism	$S \otimes_R M_R \xrightarrow{\sim} M_S$	for each homomorphism $R \to S$
a commutative square	$\begin{array}{ccc} T \otimes_S S \otimes_R M_R & \longrightarrow & T \otimes_R M_R \\ \downarrow & \circlearrowleft & \downarrow \\ T \otimes_S M_S & \longrightarrow & M_T \end{array}$	for every triangle $R \to S \to T$.

A morphism $M \to N$ in $2\text{-}\lim_{\mathcal{A}\text{ff}^{\text{op}}} \mathcal{M}$ is an assignment:

a homomorphism	$M_R \to N_R$	for each ring R
a commutative square	$\begin{array}{ccc} S \otimes_R M_R & \longrightarrow & S \otimes_R N_R \\ \downarrow & \circlearrowleft & \downarrow \\ M_S & \longrightarrow & N_S \end{array}$	for every homomorphism $R \to S$.

Exercise 11. Show that $2\text{-}\lim_{\mathcal{A}\text{ff}^{\text{op}}} \mathcal{M}$ is equivalent to the category of abelian groups. More generally, show that if I has an initial object $\varnothing$, then for any 2-functor F there is an equivalence of categories $F_\varnothing \xrightarrow{\sim} 2\text{-}\lim_I F$.

Exercise 12. Suppose that $F : I \to \mathcal{S}\text{et}$ is a functor of sets and $G : I \to \mathcal{C}\text{at}$ is a 2-functor. Show that we have

$$2\text{-}\lim_{I_{/F}} (G \circ \pi) = \mathcal{H}\text{om}(F, G)$$

where $\pi : I_{/F} \to I$ is the canonical forgetful functor.

Exercise 13. Show that for any category C and 2-functor $F : I \to \mathcal{C}\text{at}$ we have

$$\text{Fun}(C, 2\text{-}\lim_I F) = \mathcal{H}\text{om}(\gamma C, F)$$

where $\gamma C : I \to \mathcal{C}\text{at}$ is the constant 2-functor with value C.

2 Descent conditions

Definition 14. Let $F : \mathcal{A}\text{ff}^{\text{op}} \to \mathcal{C}\text{at}$ be a 2-functor. Consider the following three conditions that F might satisfy.

(Desc.) For every covering sieve $R \to jX$ the functor

$$F(X) \to \mathcal{H}\text{om}(R, F)$$

is an equivalence of categories.

(Loc.) For any morphism of presheaves of sets $G \to H$ in $\text{PSh}(\mathcal{A}\text{ff})$ such that $LG \to LH$ is an isomorphism of sheaves in $\text{Shv}(\mathcal{A}\text{ff})$, the functor

$$\mathcal{H}\text{om}(H, F) \to \mathcal{H}\text{om}(G, F)$$

is an equivalence of categories.

(MV) For every covering in $\mathcal{A}\text{ff}$ of the form $\{U \to X, V \to X\}$ the morphism

$$F(X) \to F(U) \overset{2}{\times}_{F(W)} F(V)$$

is an equivalence of categories, where $\overset{2}{\times}$ means the 2-limit and $W = U \times_X V$.

Remark 15. We will see eventually that these conditions are actually equivalent.

Exercise 16. Show that the assignment $R \mapsto R\text{-mod}$ satisfies excision. That is, show that for every $f, g \in R$ such that $1 = af + bg$ form some $a, b \in R$, the canonical functor

$$R\text{-mod} \to R[f^{-1}]\text{-mod} \overset{2}{\times}_{R[(fg)^{-1}]\text{-mod}} R[g^{-1}]\text{-mod}.$$

is an equivalence of categories.

Hint. To begin with show that this functor admits a right adjoint. Then show that this right adjoint is actually a left and right inverse.

Example 17. We will see next week that $R \mapsto \text{Sch}_R = \text{Sch}_{/j\,\text{Spec}(R)}$ satisfies excision.

3 Quasi-coherent modules

Definition 18. Let $F \in \text{PSh}(\mathcal{A}\text{ff})$ be a presheaf. The category of *quasi-coherent modules on* F is the category

$$\mathcal{Q}\text{Coh}(F) := 2\text{-lim}_{\text{Spec}(R) \in \mathcal{A}\text{ff}_{/F}} R\text{-mod}.$$

Example 19. The assignment $\mathcal{O}_F$ sending $\text{Spec}(R)$ to R assembles to give a quasi-coherent module on F. This is called the *structure module*.

Exercise 20. Recall that

$$\hom(j\operatorname{Spec}(R), \mathbb{P}^n) \cong \mathbb{P}^n(R) = \{R^{\oplus n+1} \twoheadrightarrow L\}/\sim$$

For each $s : j\operatorname{Spec}(R) \to \mathbb{P}^n$ choose an L_s in the corresponding isomorphism class. Show that $s \mapsto L_s \in R\text{-mod}$ has a structure of quasi-coherent module over $\mathbb{P}^n$. This is denoted $\mathcal{O}_{\mathbb{P}^n}(1)$ or sometimes just $\mathcal{O}(1)$. Show that each vector $v \in \mathbb{Z}^{\oplus n+1}$ canonically induces a morphism of line bundles $\mathcal{O} \to \mathcal{O}(1)$.

Exercise 21. Show that for any presheaf F the category $\mathcal{Q}\mathrm{Coh}(F)$ is an *additive category*. That is,

1. $\mathcal{Q}\mathrm{Coh}(F)$ has finite products.
2. $\mathcal{Q}\mathrm{Coh}(F)$ has finite coproducts.
3. The canonical comparison morphisms $\sqcup_{i=1}^n M_i \to \sqcap_{i=1}^n M_i$ are isomorphisms.

Show furthermore that $\mathcal{Q}\mathrm{Coh}(F)$ admits all small colimits.

4 Kähler differentials

Definition 22. Let R be a ring and M an R-module. We construct a new ring $R \oplus M$, whose underlying abelian group is $R \oplus M$ and multiplication is defined via

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1).$$

This is called the *trivial square zero extension of R by M.*

Exercise 23. Show that if $A \to B$ is a morphism of rings and $M \in B\text{-mod}$ a B-module then $A \oplus \omega M \cong (B \oplus M) \times_B A$ in the category of A-algebras where we write $\omega : B\text{-mod} \to A\text{-mod}$ for the forgetful functor.

Definition 24. (If it exists) the module of *Kähler differentials* Ω_A of a ring A is the module corepresenting the functor $M \mapsto \hom_A(A, A \oplus -)$. That is, the module Ω_A equipped with isomorphisms

$$\hom_{A\text{-mod}}(\Omega_A, M) \cong \hom_A(A, A \oplus M)$$

which are natural in M.

Exercise 25. In this exercise we show that Ω_A always exists. This is essentially the usual generators and relations proof, just written in a categorical way that will generalise more easily to the derived setting.

1. Show that $\Omega_{\mathbb{Z}[x_1,\ldots,x_n]}$ exists and there is an isomorphism

$$\Omega_{\mathbb{Z}[x_1,\ldots,x_n]} \cong \mathbb{Z}[x_1, \ldots, x_n]^{\oplus n}.$$

2. Suppose that $A_- : \Lambda \to \mathcal{R}\mathrm{ing}$ is a diagram of rings. Using the definitions, adjunctions, and Exercise 23, show that $\Omega_{\operatorname{colim} A_\lambda}$ exists and there is an isomorphism

$$\Omega_{\operatorname{colim} A_\lambda} \cong \operatorname{colim}((\operatorname{colim} A_\lambda) \otimes_{A_\lambda} \Omega_{A_\lambda}).$$

3. Show that every ring A can be written as a (not necessarily filtered) colimit of polynomial rings in finitely many variables. That is, there are finite sets I_λ indexed by the objects of some small category Λ, and a diagram of the form $\lambda \mapsto \mathbb{Z}[x_i : i \in I_\lambda]$ such that

$$A \cong \operatorname{colim}_{\lambda \in \Lambda} \mathbb{Z}[x_i : i \in I_\lambda],$$

Note: the transition morphisms $\mathbb{Z}[x_i : i \in I_\lambda] \to \mathbb{Z}[x_i : i \in I_\mu]$ *don't* necessarily come from a morphism of finite sets $I_\lambda \to I_\mu$.

4. Deduce that for any ring R, the module Ω_R exists.

Exercise 26.

1. Suppose A is a ring and $f \in A$ an element. Let $\omega : A[f^{-1}]\text{-mod} \to A\text{-mod}$ denote the forgetful functor. Show that the two functors $A[f^{-1}]\text{-mod} \to \mathcal{S}\mathrm{et}$

$$M \mapsto \hom_A(A, A \oplus \omega M)$$

$$M \mapsto \hom_{A[f^{-1}]}(A[f^{-1}], A[f^{-1}] \oplus M)$$

are isomorphic.

2. Deduce that we have isomorphisms

$$\Omega_A[f^{-1}] \cong \Omega_{A[f^{-1}]}$$

such that for every $f, g \in A$ the square

$$\begin{array}{ccc}
\Omega_A[f^{-1}][g^{-1}] & \longrightarrow & \Omega_{A[f^{-1}]}[g^{-1}] \\
\downarrow & & \downarrow \\
\Omega_A[(fg)^{-1}] & \longrightarrow & \Omega_{A[(fg)^{-1}]} \cong \Omega_{A[f^{-1}][g^{-1}]}
\end{array}$$

commutes. Consequently, for any scheme X there is a unique quasi-coherent module Ω_X such that for every open affine immersion $\mathrm{Spec}(A) \to X$ the value of Ω_X at A is Ω_A.

5 Excision (omitted from the lecture)

Proposition 27. *The following conditions are equivalent for $F \in \mathrm{PSh}(\mathcal{A}\mathrm{ff})$ a presheaf.*

1. *F is a sheaf. That is, for every covering sieve $R \subseteq jX$ we have*

$$\hom_{\mathrm{PSh}}(jX, F) \xrightarrow{\sim} \hom_{\mathrm{PSh}}(R, F).$$

2. *For every morphism $G \to H$ in $\mathrm{PSh}(C)$ such that $LG \xrightarrow{\sim} LH$ is an isomorphism of sheaves, we have*

$$\hom_{\mathrm{PSh}}(H, F) \xrightarrow{\sim} \hom_{\mathrm{PSh}}(G, F)$$

3. *(Mayer-Vietoris) For every ring A and elements f, g generating the unit ideal $\langle f, g\rangle = R$, the square*

$$\begin{array}{ccc} F(\operatorname{Spec} A) & \longrightarrow & F(\operatorname{Spec} A[f^{-1}]) \\ \downarrow & & \downarrow \\ F(\operatorname{Spec} A[g^{-1}]) & \longrightarrow & F(\operatorname{Spec} A[(fg)^{-1}]) \end{array}$$

is cartesian and $F(\varnothing) = *$.

We will use the following exercise from two weeks ago in the proof.

Exercise 28. Let $R_0 \subseteq R_1 \subseteq jX$ be two sieves. Suppose that F is a presheaf such that:
1. $\hom(jX, F) \to \hom(R_0, F)$ is bijective.
2. $\hom(jY, F) \to \hom(jY \times_{jX} R_0, F)$ is injective for every $jY \to R_1$.

Show that $\hom(jX, F) \to \hom(R_1, F)$ is bijective.

Proof.
($1 \Rightarrow 2$) This is just the adjunction $L : \mathrm{PSh} \rightleftarrows \mathrm{Shv}$.
($2 \Rightarrow 3$) Since $F(-) = \hom(j(-), F)$, it suffices to show that the square

$$\begin{array}{ccc} j\operatorname{Spec}(A[(fg)^{-1}]) & \longrightarrow & j\operatorname{Spec}(A[f^{-1}]) \\ \downarrow & & \downarrow \\ j\operatorname{Spec}(A[g^{-1}]) & \longrightarrow & j\operatorname{Spec}(A) \end{array}$$

is cocartesian in the category of sheaves. Since f, g generate the unit ideal, we have $A = \operatorname{eq}(A[f^{-1}] \times A[g^{-1}] \rightrightarrows A[(fg)^{-1}])$ in the category of rings. So it suffices to show that

$$j\operatorname{Spec}(A[g^{-1}]) \sqcup j\operatorname{Spec}(A[f^{-1}]) \to j\operatorname{Spec}(A)$$

is an epimorphism of sheaves. This holds because $\{\operatorname{Spec} A[f^{-1}], \operatorname{Spec} A[g^{-1}]\}$ is a covering of $\operatorname{Spec} A$.
($3 \Rightarrow 1$) We want to show every covering sieve induces an isomorphism $\hom(jX, F) \cong \hom(R, F)$. By Exercise 28 it suffices to consider sieves generated by a Zariski covering family $\{\operatorname{Spec}(A[f_\lambda^{-1}]) \to \operatorname{Spec}(A)\}$. Since every such family contains a finite subfamily which is also covering, applying Exercise 28 again, it suffices to consider finite families $\{\operatorname{Spec}(A[f_i^{-1}]) \to \operatorname{Spec}(A)\}_{i=1}^n$. Now we use induction on the size of the covering. The case $n = 0$ is only valid for the zero ring $A = 0$ which holds by (3). The case $n = 1$ is an isomorphism, which is obvious. The case $n = 2$ is, essentially,[2] also (3).

[2]The case of a covering of the form $\{U \to X, V \to X\}$ is actually $F(X) \cong \operatorname{eq}(F(U) \times F(V) \rightrightarrows F(U) \times F(U \times_X V) \times F(V \times_X U) \times F(V))$ but this equaliser is easily seen to be isomorphic to $F(U) \times_{F(U \times_X V)} F(V)$.

So consider a Zariski covering

$$\mathcal{U} := \{U_i \to X\}_{i=1}^n \tag{2}$$

where $n > 2$ and $U_i = \mathrm{Spec}(A[f_i^{-1}])$, $X = \mathrm{Spec}(A)$ and assume that:

$(\mathrm{Desc})_{<n}$ $\hom(jX', F) \cong \hom(R', F)$ for every covering sieve $R' \subseteq jX'$ that is generated by a Zariski covering of size $< n$.

Since $\mathcal{U}$ above is a covering, $1 = \sum_{i=1}^n f_i g_i$ for some $g_i \in A$. Set $h = \sum_{i=1}^{n-1} f_i g_i$ and consider the additional open immersion $V := \mathrm{Spec}(A[h^{-1}]) \to X$. Since h, f_n generate the unit ideal $\{V \to X, U_n \to X\}$ is a covering. Let $R \subseteq jX$ be the sieve associated to our covering $\mathcal{U}$ from Eq.(2) above, and consider the cartesian squares

$$\begin{array}{ccccc}
jY|_{U_n\times_X V} & \rightrightarrows & jY|_{U_n} \sqcup jY|_V & \longrightarrow & jY \\
\downarrow & & \downarrow & & \downarrow \\
R|_{U_n\times_X V} & \rightrightarrows & R|_{U_n} \sqcup R|_V & \longrightarrow & R \\
\downarrow{\scriptstyle (a)} & & \downarrow{\scriptstyle (b)} & & \downarrow{\scriptstyle (c)} \\
j(U_n\times_X V) & \rightrightarrows & jU_n \sqcup jV & \longrightarrow & jX
\end{array}$$

where $jY \to jX$ is any morphism scheme in R and $(-)|_W$ means $(-) \times_{jX} jW$. Now we make the following observations. Since each $\{V \times_X Y \to Y,\ U_n \times_X Y \to Y\}$ is a covering of size two, applying $\hom(-, F)$ to the upper row produces an equaliser diagram. Moreover, since $R = \mathrm{colim}_{jY\to R}\, jY$, and colimits are universal, the middle row is the colimit of the upper rows as Y ranges over all objects in $\mathcal{A}\mathrm{ff}_{/R}$. So applying $\hom(-, F)$ to the centre row also produces an equaliser diagram. So to show that (c) is sent to an isomorphism by $\hom(-, F)$, it suffices to show that (a) and (b) are sent to isomorphisms by $\hom(-, F)$.

Since $U_n \to X$ and $U_n \times_X V \to X$ are in R, we have isomorphisms

$$jU_n \cong R|_{U_n}, \qquad \text{and} \qquad j(U_n \times_X V) \cong R|_{U_n\times_X V}. \tag{3}$$

So to conclude it suffices to show that $\hom(jV, F) \to \hom(R|_V, F)$ is an isomorphism.

For this note that $f_1, \dots, f_{n-1}$ generate the unit ideal in $A[h^{-1}]$ since $1 = hh^{-1} = \sum_{i=1}^{n-1} f_i g_i h^{-1}$. So

$$\mathcal{V} = \{V \times_X U_i \to V\}_{i=1}^{n-1}$$

is a covering and more generally, for any $W \to V$ the pullback $\mathcal{V}|_W = \{W \times_X U_i \to W\}_{i=1}^{n-1}$ is a covering. Since these have size $n-1$, by the induction hypothesis $(\mathrm{Desc})_{<n}$ we have $F(W) \cong \hom(\mathcal{R}_{\mathcal{V}}|_W, F)$ for every $W \to V$ in $\mathcal{A}\mathrm{ff}$. Combining this with the factorisation $R_{\mathcal{V}} \to R|_V \to jV$, using Exercise 28, we deduce that

$$\hom(jV, F) \to \hom(R|_V, F). \tag{4}$$

is an isomorphism, as desired.

□

In the following I write $L_{\mathrm{Zar}} : \mathrm{PSh}(\mathcal{A}\mathrm{ff}) \to \mathrm{Shv}(\mathcal{A}\mathrm{ff})$ to avoid confusion with derived colimits.

Lemma 29. *Suppose X is a scheme and $\mathcal{A}\mathrm{ff}^{open}_{/X} \subseteq \mathcal{A}\mathrm{ff}_{/X}$ the full subcategory of morphisms $j\,\mathrm{Spec}(R) \to X$ which are open immersions. Then*

$$L_{\mathrm{Zar}} \operatorname{colim}_{U \in \mathcal{A}\mathrm{ff}^{open}_{/X}} U \to X$$

is an isomorphism of sheaves.

Proof. Suppose that $\sqcup V_\lambda \to X$ is an open affine covering. In particular it is a epimorphism. Since epimorphisms detect isomorphisms it suffices to show that for every λ the pullback $(L_{\mathrm{Zar}} \operatorname{colim}_{U \in \mathcal{A}\mathrm{ff}^{\mathrm{open}}_{/X}} U) \times_X V_\lambda = L_{\mathrm{Zar}} \operatorname{colim}_{U \in \mathcal{A}\mathrm{ff}^{\mathrm{open}}_{/X}} (U \times_X V_\lambda) \to V_\lambda$ is an isomorphism. We will show that actually $\operatorname{colim}_{U \in \mathcal{A}\mathrm{ff}^{\mathrm{open}}_{/X}} (U \times_X V_\lambda) \to V_\lambda$ is an isomorphism of presheaves.

Since each $U \times_X V_\lambda \to V_\lambda$ is an open immersion, it is a monomorphism of presheaves. So the colimit is the union of the images (since the analogous statement is true in $\mathcal{S}\mathrm{et}$). The union $\bigcup_{U \in \mathcal{A}\mathrm{ff}^{\mathrm{open}}_{/X}} U \times_X V_\lambda \subseteq V_\lambda$ clearly contains the union $\bigcup_{U \in \mathcal{A}\mathrm{ff}^{\mathrm{open}}_{/V_\lambda}} U \subseteq V_\lambda$, but this contains $V_\lambda \subseteq V_\lambda$ since V_λ is affine. □

Corollary 30. *For any presheaf of sets $F \in \mathrm{PSh}(\mathcal{A}\mathrm{ff})$ which satisfies excision, and scheme X, we have*

$$\hom(X, F) = \lim_{U \in \mathcal{A}\mathrm{ff}^{open}_{/X}} F(U).$$