Derived Algebraic Geometry
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Lecture 3: Schemes

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We define the category of schemes. The main theorem today is that Sch admits all finite limits, and the canonical inclusion $\mathrm{Sch} \subseteq \mathrm{Shv}(\mathcal{A}\mathrm{ff})$ preserves them.

1 Epimorphisms

Example 1. Consider the morphism $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^2$. This induces a morphism of sheaves $\hom_{holo.}(-, \mathbb{C}^*) \to \hom_{holo.}(-, \mathbb{C}^*)$ of invertible holomorphic functions on $\mathcal{O}\mathrm{pen}(\mathbb{C}^*)$. The map $\hom_{holo.}(\mathbb{C}^*, \mathbb{C}^*) \to \hom_{holo.}(\mathbb{C}^*, \mathbb{C}^*)$ is not surjective (because z^2 doesn't have a global inverse), but $(-)^2 : \hom_{holo.}(B, \mathbb{C}^*) \to \hom_{holo.}(B, \mathbb{C}^*)$ *is* surjective on any open ball $B \subseteq \mathbb{C}^*$.

Exercise 2. Let C be a category equipped with a topology and $F \to G$ a morphism in $\mathrm{Shv}(C)$. Show that the following are equivalent.

1. For every sheaf H the map
$$\hom(G, H) \to \hom(F, H)$$
is injective.
2. The canonical morphism
$$\mathrm{coeq}(F \times_G F \rightrightarrows F) \to G$$
is an isomorphism of sheaves.
3. For every $s \in \hom(jX, G)$ there exists a family $\mathcal{U} = \{U_\lambda \to X\}_{\lambda \in \Lambda}$ and commutative squares
$$\begin{array}{ccc} jU_\lambda & \longrightarrow & jX \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$
such that $\mathcal{R}_\mathcal{U} \to jX$ is a covering sieve.

Definition 3. A morphism satisfying the equivalent conditions of Exercise 2 is called an *epimorphism*.

Exercise 4.

1. Show that if $F \twoheadrightarrow G$, $G \twoheadrightarrow H$ are two epimorphisms of sheaves, then the composition $F \to H$ is an epimorphism.

2. Show that if $F \twoheadrightarrow G$ is an epimorphism of sheaves and $H \to G$ is any morphism of sheaves, then $F \times_G H \to H$ is an epimorphism of sheaves.
3. Show that if a composition $F \to G \to H$ of morphisms of sheaves is an epimorphism then so is $G \to H$.

Exercise 5. Show that the canonical morphism

$$\coprod_{\lambda \in \Lambda} j \operatorname{Spec}(A_\lambda) \to j \coprod_{\lambda \in \Lambda} \operatorname{Spec}(A_\lambda)$$

is not an epimorphism in the category of presheaves unless all but one A_λ are the zero ring. Show that it becomes an epimorphism in the category of sheaves. Deduce that it is an isomorphism in the category of sheaves.

2 Open immersions

Definition 6. A subsheaf $U \subseteq j \operatorname{Spec}(A)$ is an *open immersion* if there exists an epimorphism *of sheaves* of the form

$$\coprod_{\lambda \in \Lambda} j \operatorname{Spec}(A[f_\lambda^{-1}]) \to U.$$

In other words, if it is the sheafification of a union of opens $\operatorname{Spec}(A[f_\lambda^{-1}]) \subseteq \operatorname{Spec}(A)$.

Example 7. The subsheaf

$$\{(f, g) \in R \times R \mid af + bg = 1,\ \exists\, a, b\} \subseteq \mathbb{A}^2(R)\}$$

is the sheafification of the image of

$$j \operatorname{Spec}(\mathbb{Z}[x, y, y^{-1}]) \sqcup j \operatorname{Spec}(\mathbb{Z}[x, x^{-1}, y]) \to j \operatorname{Spec}(\mathbb{Z}[x, y])$$

Note that this is larger than the image presheaf

$$im = \{(f, g) \in R \times R \mid f \in R^* \text{ or } g \in R^*\}$$

Exercise 8. Show that if $U \subseteq j \operatorname{Spec}(A)$ is an open immersion and $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ any morphism then $U \times_{j \operatorname{Spec}(A)} j \operatorname{Spec}(B) \subseteq j \operatorname{Spec}(B)$ is an open immersion.

Definition 9. A morphism $F \to G$ of sheaves is an *open immersion* if for every $j \operatorname{Spec}(A) \to G$ the pullback $j \operatorname{Spec}(A) \times_G F \to j \operatorname{Spec}(A)$ is an open immersion.

Exercise 10.
1. Suppose $F \hookrightarrow G$ is an open immersion and $H \to G$ any morphism. Show that $F \times_G H \to H$ is an open immersion.
2. Suppose that $F \hookrightarrow G$ and $G \hookrightarrow H$ are open immersions. Show that the composition $F \to H$ is an open immersion.

3. Suppose that $F \to G$ is a morphism, $H \twoheadrightarrow G$ is an epimorphism, and that $F \times_G H \hookrightarrow H$ is an open immersion. Show that $F \to G$ is an open immersion.

Exercise 11. Let F be a sheaf. Show that the collection $\mathcal{O}\mathrm{pen}(F)$ of open immersions $U \hookrightarrow F$ with its canonical partial order satisfy the axioms of a topological space. That is:

1. $\varnothing \to F$ and $\mathrm{id}_F : F \to F$ are open immersions.
2. If $U \hookrightarrow F$ and $V \hookrightarrow F$ are open immersions then $U \times_F V \to F$ is an open immersion.
3. If $\{U_\lambda \hookrightarrow F\}_{\lambda \in \Lambda}$ is a family of open immersions, then $V := \mathrm{im}(\sqcup U_\lambda \to F) \to F$ is an open immersion which satisfies: For every open immersion $W \hookrightarrow F$ admitting factorisations $U_\lambda \to W \to F$ for each λ there is a factorisation $V \to W \to F$.

Deduce that for any field K, the set $F(\mathrm{Spec}(K))$ has an induced structure of topological space, and for every field extension L/K the map $F(\mathrm{Spec}(L)) \to F(\mathrm{Spec}(K))$ is continuous in this topology. The colimit $\mathrm{colim}_K\, F(K)$ over all fields of these topological spaces is called the *underlying topological space of* F.

Exercise 12. Recall that an R-module L is *invertible* if there exists a covering family $\{\mathrm{Spec}\, R[f_\lambda^{-1}] \to \mathrm{Spec}\, R\}_{\lambda \in \Lambda}$ and for each λ isomorphisms $L[f_\lambda^{-1}] \cong R[f_\lambda^{-1}]$. Let L be an invertible R-module and $s \in L$ an element. Say that s is *nowhere vanishing* if the induced R-module homomorphism $R \to L;\ r \mapsto rs$ is an isomorphism. Define

$$U_{L,s}(A) := \left\{ R \to A \;\middle|\; 1 \otimes s \in A \otimes_R L \quad \text{is nowhere vanishing} \right\} \subseteq j\,\mathrm{Spec}(R)$$

Show that $U_{L,s} \to j\,\mathrm{Spec}(R)$ is an open immersion.

3 Projective space

Example 13 (Projective space)**.** In this example we fix a natural number $n \geq 0$ and consider equivalence classes of surjections of R-modules $R^{\oplus n+1} \twoheadrightarrow L$ towards an invertible module L. Here, two surjections $R^{\oplus n+1+1} \twoheadrightarrow L$, $R^{\oplus n} \twoheadrightarrow L'$ are equivalent if there exists an isomorphism of R-modules forming a commutative triangle.

$$\begin{array}{ccc} & \nearrow & L \\ R^{\oplus n+1} & & \downarrow\cong \\ & \searrow & L' \end{array}$$

The set of such equivalence classes of surjections is denoted

$$\mathbb{P}^n(R) := \{R^{\oplus n+1} \twoheadrightarrow L\}_{/\sim}$$

The transition morphisms $\mathbb{P}^n(R) \to \mathbb{P}^n(R')$ are induced by $- \otimes_R R'$.[1] For every $i = 0, \ldots, n$ we can consider the basis vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^{\oplus n+1}$.

[1] Note that $- \otimes_R R'$ does not preserve monomorphisms in general. This is why we work with equivalence classes of surjections instead of the set of invertible submodules $L \subseteq R^{\oplus n+1}$. Of course, $\hom(-, R)$ sets up a bijection between equivalence classes of surjections towards invertible modules, and the set of submodules which are invertible.

Let $U_i \subseteq \mathbb{P}^n$ be the subfunctor of those $\pi : R^{\oplus n+1} \twoheadrightarrow L$ such that $\pi(e_i)$ is nowhere vanishing, cf. Exercise 12.

$$U_i(R) := \{[R^{\oplus n+1} \overset{\pi}{\twoheadrightarrow} L] \mid \pi(e_i) \text{ is nowhere vanishing }\}.$$

Exercise 14 (Harder.)**.**

1. Show that $\mathbb{P}^n$ is a sheaf.
2. Show that the $U_i \to \mathbb{P}^n$ are open immersions. Hint. Use Exercise 12.
3. Let $\iota_i : \mathbb{A}^n \to \mathbb{P}^n$ be the functor corresponding to the surjection $P^{\oplus n+1} \to P$

$$e_j \mapsto \begin{cases} x_{j+1} & j < i \\ 1 & j = i \\ x_j & j > i \end{cases}$$

   where $P := \mathbb{Z}[x_1, \dots, x_n]$ so $\mathbb{A}^n = j\,\mathrm{Spec}(P)$. Show that this map induces an isomorphism

$$\mathbb{A}^n(R) \xrightarrow{\sim} U_i(R).$$

4. Show that

$$\coprod_{i=0}^{n} U_i \to \mathbb{P}^n$$

   is an epimorphism of sheaves.
5. Describe the subpresheaf $U_i \cap U_j \subseteq U_i \cong \mathbb{A}^n$ as a subpresheaf of $\mathbb{A}^n$.

4 Schemes and finite limits

Definition 15. A *scheme* is a sheaf $X \in \mathrm{Shv}(\mathcal{A}\mathrm{ff})$ which admits an epimorphism of sheaves of the form

$$\coprod_{\lambda \in \Lambda} j\,\mathrm{Spec}(A_\lambda) \to X$$

such that each $j\,\mathrm{Spec}(A_\lambda) \to X$ is an open immersion. Such a morphism is called an *open affine covering*. The category of schemes is the full subcategory

$$\mathrm{Sch} \subseteq \mathrm{Shv}(\mathcal{A}\mathrm{ff})$$

whose objects are schemes.

Example 16. In Exercise 14 we showed that $\mathbb{P}^n$ is a scheme.

Exercise 17. Suppose $U \subseteq j\,\mathrm{Spec}(A)$ is an open immersion. Show that U is a scheme.

Exercise 18. Suppose that $X \to Y$ is a morphism of schemes and $\{V_\mu \to Y\}_{\mu \in M}$ an open affine covering of Y. Show that there exists an open affine covering $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ of X such that for each λ there is some μ_λ admitting a commutative square

$$\begin{array}{ccc} U_\lambda & \longrightarrow & V_{\mu_\lambda} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Theorem 19. *The category of schemes admits finite limits and the canonical inclusion* $\mathrm{Sch} \subseteq \mathrm{PSh}(\mathcal{A}\mathrm{ff})$ *preserves those limits.*

Proof. It suffices to discuss fibre products and terminal objects.

Final object. Clearly, $j\,\mathrm{Spec}(\mathbb{Z})$ is the terminal presheaf $*$.

Fibre products. Suppose that $X \to Y \leftarrow Z$ are morphisms of schemes. By Exercise 18 we can find open affine coverings $\{V_\mu \to Y\}_{\mu\in M}$, $\{U_{\mu\lambda} \to X\}_{\mu\in M,\lambda\in\Lambda_\mu}$, $\{W_{\mu\nu} \to Z\}_{\mu\in M,\nu\in N_\mu}$ such that each $U_{\mu\lambda} \to Y$ resp. $W_{\mu\nu} \to Y$ factors through $V_\mu \to Y$. We claim that $\{U_{\mu\lambda} \times_{V_\mu} W_{\mu\nu} \to X \times_Y Z\}_{\mu\in M,\lambda\in\Lambda_\mu,\nu\in N_\mu}$ is an open affine covering.

Affineness. Certainly, each $U_{\mu\lambda} \times_{V_\mu} W_{\mu\nu}$ is affine, since $j\,\mathrm{Spec}(A) \times_{j\,\mathrm{Spec}(B)} j\,\mathrm{Spec}(C) = j\,\mathrm{Spec}(A \otimes_B C)$.

Openness. To see that they are open, first notice that since $V_\mu \to Y$ is a monomorphism we have $U_{\mu\lambda} \times_{V_\mu} W_{\mu\nu} = U_{\mu\lambda} \times_Y W_{\mu\nu}$. Now consider the cartesian squares

$$\begin{array}{ccccc} U \times_Y W & \longrightarrow & X \times_Y W & \longrightarrow & W \\ \downarrow & \searrow & \downarrow & & \downarrow \\ U \times_Y Z & \longrightarrow & X \times_Y Z & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ U & \longrightarrow & X & \longrightarrow & Y \end{array} \tag{1}$$

where we omit the indices, as they are clear form context. Since pullbacks and compositions of open immersions are open immersions, the diagonal map is an open immersion.

Epimorphicity. Again, since each $V_\mu \to Y$ is a monomorphism, we have $\coprod(U_{\mu\lambda} \times_{V_\mu} W_{\mu\nu}) = \coprod(U_{\mu\lambda} \times_Y W_{\mu\nu})$. Moreover, we have $\coprod(U_{\mu\lambda} \times_Y W_{\mu\nu}) = (\coprod U_{\mu\lambda}) \times_Y (\coprod W_{\mu\nu})$.

The analogous diagram to (1) shows that $(\coprod U_{\mu\lambda}) \times_Y (\coprod W_{\mu\nu}) \to X \times_Y Z$ is an epimorphism since epimorphisms are also preserved by pullback and composition. □

5 Affine morphisms

Definition 20. A morphism of sheaves $F \to G$ is *affine* if for every $j\,\mathrm{Spec}(A) \to G$ the pullback $F \times_G j\,\mathrm{Spec}(A)$ is representable. That is, $F \times_G j\,\mathrm{Spec}(A) \cong j\,\mathrm{Spec}(B)$ for some B.

Example 21.

1. Show that any morphism of the form $j\,\mathrm{Spec}(B) \to j\,\mathrm{Spec}(A)$ is affine.
2. Show that the composition of two affine morphisms is affine.
3. Show that the pullback of an affine morphism is affine.

Proposition 22. *Suppose* $\Lambda \to \mathrm{Sch}$; $\lambda \mapsto X_\lambda$ *is a small filtered system of schemes. That is,* Λ *is a small category admitting all finite cones. Suppose furthermore that for every* $\lambda \to \mu$*, the morphism* $X_\lambda \to X_\mu$ *is affine. Then the limit* $\lim X_\lambda$ *in* $\mathrm{PSh}(\mathcal{A}\mathrm{ff})$ *is a scheme.*

Proof. The inclusion Shv $\to$ PSh preserves limits, so $\lim X_\lambda$ is a sheaf. Choose any $\lambda_0 \in \Lambda$ and an open affine covering $\{U_i \to X_0\}_{i\in I}$. We claim that $\{U_i \times_{X_0} \lim X_\lambda \to \lim X_\lambda\}_{i\in I}$ is an open affine covering. Pullbacks of opens and epimorphisms are such, so it suffices to check that each $U_i \times_{X_0} \lim X_\lambda$ is affine. Since each $X_\mu \to X_\lambda$ is affine, each $X \to X_0$ is affine, so each $X_\mu \times_{X_0} U_i$ is an affine scheme. Then we have $U_i \times_{X_0} \lim X_\lambda = \lim U_i \times_{X_0} X_\lambda$ since limits commute with limits. So it suffices to show that a limit of affine schemes is an affine scheme. But this is clear since $\mathcal{A}$ff admits limits ($\mathcal{R}$ing admits colimits) and Yoneda and sheafification commute with them. ☐

Definition 23. A morphism of of affine schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is a *closed immersion* if $B \to A$ is surjective.

A morphism of presheaves $F \to G$ in PSh($\mathcal{A}$ff) is a *closed immersion* if it is affine and for every $\mathrm{Spec}(A) \to G$, the pullback $F \times_G j\,\mathrm{Spec}(A) \to j\,\mathrm{Spec}(A)$ is a closed immersion.

Exercise 24. Show the following.

1. A composition of closed immersions is a closed immersion.
2. The pullback of a closed immersion is a closed immersion.

Exercise 25. Suppose that X is a scheme and $Y \to X$ is a closed immersion.

1. Show that $Y \to X$ is a monomorphism of presheaves.
2. Show that Y is a sheaf.
3. Show that Y is a scheme.