Derived Algebraic Geometry
Shane Kelly, UTokyo
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Last week we defined the category of affine schemes as the opposite of the category of rings.

$$\mathcal{A}\text{ff} = \mathcal{R}\text{ing}^{\text{op}}.$$

In order to glue affine schemes together along open immersions we put them in a larger category, where such colimits exist. One choice is the category of locally ringed spaces,

$$\mathcal{A}\text{ff} \subseteq \{ \text{ locally ringed spaces } \}.$$

This is featured in texts such as EGA, [GD71], and Hartshorne's book, [Har77]. In the ∞-category setting (or even the 2-category setting, i.e.,, when using groupoids instead of sets), defining the higher version of the category of locally ringed spaces is annoying.

Alternatively, we can also put the category of affine schemes into the category of sheaves

$$\mathcal{A}\text{ff} \subseteq \text{Shv}(\mathcal{A}\text{ff}).$$

This approach is featured in texts such as SGA3, [ABD+66], and Jantzen's book, [Jan87], and is particularly well suited to moduli questions, which are often formulated in terms of a functor which sends a ring R to some set/groupoid/space of objects over R.

Remark 1. As a general principle, it's difficult to define an ∞-category or a functor between ∞-categories because one must specify not just objects and morphisms, but also homotopies, homotopies between homotopies, homotopies between homotopies between homotopies, etc and check an infinite list of compatibilities. However, once we have and ∞-category C, it is easy to define full sub-∞-categories $D \subseteq C$, because this is just a class of objects.

So as much as possible one tries to use ∞-categories and ∞-functors that have already been defined, or exist naturally. For any "good" theory of ∞-categories, there should be:

1. an ∞-category $\mathcal{S}$ freely generated under colimits by a single object $*$,
2. for any two ∞-categories C, D, there should be an ∞-category $\text{Fun}(C, D)$ of functors between them, and
3. there should be a fully faithful Yoneda embedding $j : C \to \text{Fun}(C^{\text{op}}, \mathcal{S})$.

So defining schemes in terms of sheaves is technically low-level in the sense that it is heuristically close to the hypothetical axioms of the theory of ∞-categories.

The main theorems in this lecture are:

1. sheafification, Thm.22,
2. subcanonicity of the Zariski topology, Thm.19.

References:
- [Lurie, Higher topos theory]
- [SGA41 Exposé.II]
- [Maclane, Moerdijk, Sheaves in geometry and logic]
- [Jantzen, Representations of algebraic groups]

1 Presheaves

Definition 2. Let C be a category. A *presheaf* is a functor $F : C^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$, a morphism of presheaves is a natural transformation.

Example 3.

1. Let $\mathcal{B}\mathrm{all}_d \subseteq \mathcal{M}\mathrm{an}_d$ be the full subcategory of the category of d-dimensional manifolds whose objects are diffeomorphic to $\mathbb{R}^d$. Every manifold M defines a presheaf $U \mapsto \hom_{\mathcal{M}\mathrm{an}}(U, M)$ on $\mathcal{B}\mathrm{all}_d$.
2. Any object $X \in C$ defines a presheaf
$$jX : Y \mapsto \hom(Y, X).$$
Presheaves of this form are called *representable*. A morphism $f : X \to X'$ in C induces a morphism of presheaves $\hom(-, X) \to \hom(-, X')$. In this way we obtain a functor
$$j : C \to \mathrm{PSh}(C).$$
3. Limits and colimits in $\mathrm{PSh}(C)$ exist and are computed object wise. That is, if $\Lambda \to \mathrm{PSh}(C)$; $\lambda \mapsto F_\lambda$ is a diagram of presheaves then we have
$$(\lim F_\lambda)(Y) = \lim(F_\lambda(Y))$$
and similarly for colimits.

Exercise 4 (Yoneda's Lemma)**.** Suppose C is a category, X is an object, and $F \in \mathrm{PSh}(C)$ is a presheaf. Given $s \in F(X)$ define a natural transformation $\phi_s : jX \to F$ which sends $f \in jX(Y) = \hom(Y, X)$ to the image of s under $F(f) : F(X) \to (Y)$. Show that this defines a bijection
$$F(X) \xrightarrow{\sim} \hom_{\mathrm{PSh}}(jX, F).$$
Deduce that for any other object X' of C we get a bijection
$$\hom_C(X, X') \xrightarrow{\sim} \hom_{\mathrm{PSh}}(jX, jX').$$
Check that this induces a fully faithful functor
$$j : C \to \mathrm{PSh}(C).$$

Exercise 5 (Harder)**.** Show that every presheaf $F \in \mathrm{PSh}(C)$ is a colimit of representables. That is, $F \cong \mathrm{colim}_\Lambda\, jX_\lambda$ for some appropriately chosen diagram $\Lambda \to C$.

Remark 6. In fact, $j : C \to \mathrm{PSh}(C)$ is the free cocompletion of C in the sense that if $C \to D$ is any morphism to a category D admitting all colimits, there is unique (up to natural isomorphism) factorisation through a colimit preserving functor $\mathrm{PSh}(C) \to D$.

2 Topologies

Definition 7. Let C be a category. A *sieve* is a subpresheaf $R \subseteq jX$ of a representable presheaf. The collection of sieves on X is denoted $\mathrm{Sub}(jX)$.

Remark 8. Sometimes it can be useful to think of a sieve as the full subcategory

$$C_{/R} \subseteq C_{/X}$$

of the comma category consisting of those objects $f : Y \to X$ such that $f \in R(Y)$.

Remark 9. Sometimes it can be useful to think of a sieve as the full subcategory $C_{/R}$ of the comma category $C_{/X}$ consisting of those objects $f : Y \to X$ such that $f \in R(Y)$. For another way of thinking about a sieve see Exercise 11, Eq.(1).

Example 10. For any family of morphisms $\mathcal{U} = \{U_\lambda \to X\}_{\lambda \in \Lambda}$, we have an associated sieve $R_\mathcal{U} = \bigcup_{i \in \Lambda} \mathrm{im}(j(U_\lambda) \to jX)$. That is for $Y \in C$, we have

$$R_\mathcal{U}(Y) = \left\{ f : Y \to X \,\middle|\, \begin{array}{c} \text{there exists a factorisation} \\ Y \to U_\lambda \to X \text{ for some } \lambda \in \Lambda \end{array} \right\}.$$

Exercise 11.

1. Suppose $A \xrightarrow{f} B \in \mathcal{S}\mathrm{et}$ is a morphism of sets. Show that

$$A \xrightarrow{\pi} \mathrm{coeq}(A \times_B A \rightrightarrows A) \xrightarrow{\iota} B$$

   is the canonical factorisation of f into an epimorphism π followed by a monomorphism ι, or in other words, show that $\mathrm{coeq}(A \times_B A \rightrightarrows A)$ is canonically isomorphic to the image of f.
2. The same question, but now $A \to B \in \mathrm{PSh}(C)$ is a morphism of presheaves on some category C.
3. Deduce that if $R_\mathcal{U} \subseteq jX$ is the sieve associated to a family $\mathcal{U} = \{U_\lambda \to X\}_{\lambda \in \Lambda}$, of morphisms in C and we set $U := \sqcup_{\lambda \in \Lambda} j(U_\lambda)$, then the canonical morphism

$$\mathrm{coeq}(U \times_{jX} U \rightrightarrows U) \to R_\mathcal{U} \tag{1}$$

   is an isomorphism.
4. Show furthermore, that (if C is small) then *every* sieve R is the sieve associated to some family $\mathcal{U}$ of morphisms in C.

Definition 12. A *topology* on a category C is the data of: for each object X a collection $J(X)$ of sieves called *covering sieves*. These covering sieves are required to satisfy the following axioms.

(T1) If $R \subseteq jX$ is a covering sieve and $Y \to X$ is any morphism in C then $jY \times_{jX} R$ is a covering sieve of Y.

$$\begin{array}{ccc} jY \times_{jX} R & \longrightarrow & R \\ \cap| & & \cap| \\ jY & \longrightarrow & jX. \end{array}$$

(T2) If $R \subseteq jX$ is a covering sieve and $R' \subseteq jX$ is any other sieve satisfying: for every morphism $jY \to R \to jX$ in $R(Y)$, the pullback $jY \times_{jX} R' \subseteq jY$ is a covering sieve, then R' is also a covering sieve.

$$\begin{array}{ccccc} jY \times_{jX} R' & \longrightarrow & R \cap R' & \longrightarrow & R' \\ \cap\!| & & \cap\!| & & \cap\!| \\ jY & \longrightarrow & R & \longrightarrow & jX \end{array}$$

(T3) For every object X, the maximal sieve

$$R = jX \subseteq jX$$

is a covering sieve.

Example 13.

1. Let X be a topological space, and consider the poset $\mathcal{O}\mathrm{pen}(X)$ of open subsets of X considered as a category. A sieve $R \subseteq j(U)$ is a covering sieve if there is a covering family $\mathcal{U} = \{V_\lambda \subseteq U\}_{\lambda\in\Lambda}$ such that $V_\lambda \to U$ is in $R(V_\lambda)$ for all λ.
2. A sieve $R \subseteq j(\mathrm{Spec}(A))$ is called a *Zariski covering sieve* if there exists a covering $\mathcal{U} = \{\mathrm{Spec}(A[a_\lambda^{-1}]) \to \mathrm{Spec}(A)\}_{\lambda\in\Lambda}$ such that $\mathrm{Spec}(A[a_\lambda^{-1}]) \to \mathrm{Spec}(A)$ is in $R(\mathrm{Spec}(A))$ for all λ, or equivalently, such that $R_\mathcal{U} \subseteq R$.
3. We can extend the previous example to comma categories. Let $F \in \mathrm{PSh}(\mathcal{A}\mathrm{ff})$ be a presheaf and consider the category $\mathcal{A}\mathrm{ff}_{/F}$ whose objects are morphisms of the form $j(\mathrm{Spec}(A)) \to F$, and morphisms of $\mathcal{A}\mathrm{ff}_{/F}$ are commutative triangles in $\mathrm{PSh}(\mathcal{A}\mathrm{ff})$. This is equipped with a canonical forgetful functor $\pi : \mathcal{A}\mathrm{ff}_{/F} \to \mathcal{A}\mathrm{ff}$. Note that sieves on $j(\mathrm{Spec}(A)) \to F$ are in bijection with sieves on $\mathrm{Spec}(A)$. We say a sieve on $j(\mathrm{Spec}(A)) \to F$ is a Zariski covering sieve if its corresponding sieve on $\mathrm{Spec}(A)$ is a Zariski covering sieve.

Exercise 14. Show that any of the examples above satisfy the axioms for a topology.

3 Sheaves

Definition 15. Let C be a category equipped with a Grothendieck topology. A presheaf $F \in \mathrm{Shv}(C)$ is called a *sheaf*, resp. *separated* presheaf, if for every covering sieve $R \subseteq jX$, the associated morphism

$$\hom(jX, F) \to \hom(R, F) \tag{$*$}$$

is a bijection, resp. injection.

Exercise 16. Suppose that $R = R_\mathcal{U}$ is the sieve associated to a family of morphisms $\mathcal{U} = \{U_\lambda \to X\}_{\lambda\in\Lambda}$ and each fibre product $U_\lambda \times_X U_\mu$ exists in C. Show that

$$\hom(jX, F) \to \hom(R_\mathcal{U}, F)$$

is a bijection if and only if

$$F(X) \to \mathrm{eq}\left(\prod_\lambda F(U_\lambda) \rightrightarrows \prod_{\lambda,\mu} F(U_\lambda \times_X U_\mu)\right)$$

is a bijection. Hint. Cf.Exercise 11.

Exercise 17.

1. Suppose that $f, g : G \rightrightarrows F$ are two morphisms of presheaves such that for every $s : jY \to G$ we have $f \circ s = g \circ s$. Show that $f = g$.
2. Let $R_0 \subseteq R_1 \subseteq jX$ be two sieves. Suppose that F is a presheaf such that:
   (a) $\hom(jX, F) \to \hom(R_0, F)$ is bijective.
   (b) $\hom(jY, F) \to \hom(jY \times_{jX} R_0, F)$ is injective for every $jY \to R_1$.
   Show that $\hom(jX, F) \to \hom(R_1, F)$ is bijective.

Example 18.

1. If Y, X are topological spaces, then $U \mapsto \hom_{cont.}(U, Y)$ defines a sheaf on $\mathcal{O}\mathrm{pen}(X)$.
2. Sending an open $U \subseteq \mathbb{C}$ to the set $\hom_{holo.}(U, \mathbb{C})$ of holomorphic functions defines a sheaf on $\mathcal{O}\mathrm{pen}(\mathbb{C})$.

Theorem 19.

1. *For any R-module M, the assignment*

$$\widetilde{M} : \mathrm{Spec}(S) \mapsto S \otimes_R M$$

   defines a sheaf $\widetilde{M}$ on $\mathcal{A}\mathrm{ff}_R$.
2. *For any R-algebra A, the representable presheaf $j\,\mathrm{Spec}(A)$ is a sheaf on $\mathcal{A}\mathrm{ff}_R$.*

Proof. We give the proof for the second one, and leave the first as an exercise (the same strategy works). We want to show that $\hom(-, j\,\mathrm{Spec}(A))$ sends covering sieves to isomorphisms. By Exercise 17 it suffices to consider sieves generated by a Zariski covering family $\{\mathrm{Spec}(B[f_\lambda^{-1}]) \to \mathrm{Spec}(B)\}$. Since every such family contains a finite subfamily which is also covering, applying Exercise 17 again, it suffices to consider finite families. In this case, by Exercise 16 and the definition $\mathcal{A}\mathrm{ff}^{\mathrm{op}} = \mathcal{R}\mathrm{ing}$ we are asking if

$$\hom(A, B) \to \mathrm{eq}\left(\prod_{i=1}^n \hom(A, B[f_i^{-1}]) \rightrightarrows \prod_{i,j=1}^n \hom(A, B[(f_i f_j)^{-1}])\right)$$

is an isomorphism. We can bring the products and equalisers inside since $\lim \hom(-,-) = \hom(-, \lim -)$. Since finite products commute with tensor product, setting $C := \prod_{i=1}^n B[f_i^{-1}]$ we have $C \otimes_B C = \prod_{i,j=1}^n B[(f_i f_j)^{-1}]$, and so now we are asking if

$$B \to \mathrm{eq}(C \rightrightarrows C \otimes_B C) \tag{2}$$

is an isomorphism. A morphism of rings ϕ is an isomorphism if and only if $\phi[g^{-1}]$ is an isomorphism for every element in a covering family. So it suffices to show that each

$$B[f_i^{-1}] \to \mathrm{eq}(C[f_i^{-1}] \rightrightarrows C[f_i^{-1}] \otimes_{B[f_i^{-1}]} C[f_i^{-1}])$$

is an isomorphism. Since $C = \prod_{i=1}^{n} B[f_i^{-1}]$, the morphism $B[f_i^{-1}] \to C[f_i^{-1}]$ admits a retraction $C[f_i^{-1}] \to B[f_i^{-1}]$. Now it suffices to check that in any category with pushouts, for any morphism $B \to C$ admitting a retraction

$$B \overset{\curvearrowleft}{\to} C$$

the morphism (2) is an isomorphism. An easy diagram chase using the following two cocartesian squares shows that any morphism $g : D \to B$ satisfying $inc_2 g = inc_1 g$ factors uniquely as $D \stackrel{sg}{\to} B \stackrel{f}{\to} C$.

$$\begin{array}{ccccc}
B & \xrightarrow{f} & C & \xrightarrow{s} & B \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle inc_2} & & \downarrow{\scriptstyle f} \\
C & \xrightarrow[inc_1]{} & C \otimes_B C & \longrightarrow & C
\end{array}$$

□

Exercise 20. Prove Part 1 of Theorem 19.

Example 21. Here are some more examples of representable sheaves.

1. $\mathcal{O} : \mathrm{Spec}(A) \mapsto A$.
2. $\mathcal{O}^* : \mathrm{Spec}(A) \mapsto A^*$.
3. $GL_n : \mathrm{Spec}(A) \mapsto GL_n(A)$.
4. $\mu_n : \mathrm{Spec}(A) \mapsto \{a \mid a^n = 1\}$.

4 Sheafification

Theorem 22. *For any category C, canonical inclusion* $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ *admits a left adjoint*

$$L : \mathrm{PSh}(C) \to \mathrm{Shv}(C).$$

Moreover, this left adjoint commutes with finite limits.

We sketch the proof from [Lur06, Prop.6.2.2.7] which will work in the ∞-categorical setting as well. It is not quite as tight as the 1-category case. Namely, in the 1-categorical case, one has $LF = F^{\dagger\dagger}$. For an account of the 1-category version, see [Sta18, 00ZG] for example (which unfortunately writes L for HTT's $(-)^\dagger$, and $(-)^\#$ for HTT's L).

For simplicity we assume:

(Fin) Every covering sieve $R_0 \subseteq jX$ contains a covering sieve $R_1 \subseteq R_0$ such that for any sequence $F_0 \to F_1 \to F_2 \to \dots$ of presheaves we have $\hom(R_1, \operatorname{colim} F_\lambda) = \operatorname{colim} \hom(R_1, F_\lambda)$.

This is satisfied, for example, for the Zariski topology on $\mathcal{A}\mathit{ff}$ by Exercise 16. It is made just so that we don't have to deal with transfinite compositions in the construction of L. We give a sketch of the proof. More details are given in a separate pdf on the course web page.

Sktech of proof. Given $F \in \mathrm{PSh}(C)$ and $X \in C$ define

$$F^\dagger(X) = \mathrm{colim}_{R \subseteq \hom_C(-,X)} \hom_{\mathrm{PSh}}(R, F),$$

where the colimit is over the filtered poset of covering sieves. Note that this defines a presheaf $F^\dagger$ equipped with a morphism $F \to F^\dagger$. We define[1]

$$LF = \mathrm{colim}(F \to F^\dagger \to F^{\dagger\dagger} \to \dots).$$

To show that L is the desired left adjoint, it suffices to prove:

1. If G is any sheaf then $\hom(F^\dagger, G) \to \hom(F, G)$ is an isomorphism.
2. For any presheaf F the presheaf LF is a sheaf.
3. The functor L commutes with finite limits.

Step 1. Let $Cov(C)$ be the category whose objects are covering sieves $R \subseteq jY$ and morphisms are commutative squares

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ jY & \longrightarrow & jY' \end{array}$$

There is a canonical forgetful functor $\pi : Cov(C) \to C$ sending $R \subseteq jY$ to jY (with right adjoint sending Y to $jY \subseteq jY$) and this leads to four functors

$$\mathrm{PSh}(Cov(C)) \underset{\pi_*,\ \pi^!}{\overset{\pi_!,\ \pi^*}{\rightleftarrows}} \mathrm{PSh}(C)$$

which can be described as

$$\begin{aligned} \mathrm{colim}_{R \subseteq j(Y)} F(R \subseteq jY) &= \pi_! F(Y) \\ \pi^* F(R \subseteq jY) &= F(Y) \\ F(jY \subseteq jY) &= \pi_* F(Y) \\ \pi^! F(R \subseteq jY) &= \lim_{jV \to R} F(V) \end{aligned}$$

Observe that:

(a) $F^\dagger = \pi_! \pi^! F$.

[1] If we remove the assumption (Fin), then one instead chooses a large enough regular cardinal κ and defines L as a transfinite composition $((-)^\dagger)^{\circ\kappa}$. Under the assumption (Fin), the choice $\kappa = \aleph_0$ works.

(b) Each functor is left adjoint to the one directly below it, so $\pi_! \dashv \pi^* \dashv \pi_* \dashv \pi^!$.
(c) G is a sheaf if and only if $\pi^! G = \pi^* G$.
(d) $\pi^!$ is fully faithful (because $\mathrm{id} = \pi_* \pi^!$).
Then it follows immediately that

$$\begin{aligned}
\hom_{\mathrm{PSh}(C)}(F^\dagger, G) &\overset{(a)}{=} \hom_{\mathrm{PSh}(C)}(\pi_! \pi^! F, G) \\
&\overset{(b)}{=} \hom_{\mathrm{PSh}(Cov(C))}(\pi^! F, \pi^* G) \\
&\overset{(c)}{=} \hom_{\mathrm{PSh}(Cov(C))}(\pi^! F, \pi^! G) \\
&\overset{(d)}{=} \hom_{\mathrm{PSh}(C)}(F, G)
\end{aligned}$$

Step 2. We want to show that for any covering sieve $R \subseteq jX$, we have $LF(X) = \lim_{jY \to R} LF(Y)$. Note that we can replace C with $C_{/X}$, and doing so, assume that $X = *$ is a terminal object of C. Now consider a modified version of the functors from Step 1. Namely, given a fixed sieve $R_0 \subseteq *$ of the terminal object, let $Cov(C)_0 \subseteq Cov(C)$ be the full subcategory whose objects are those covering sieves $jY \times R_0 \subseteq R \subseteq jY$ containing the pullback $jY \times R_0$ of our fixed $R_0 \subseteq *$. Via the composition $\rho : Cov(C)_0 \subseteq Cov(C) \overset{\pi}{\to} C$ we get another four functors

$$\mathrm{PSh}(Cov(C)_0) \; \overset{\rho_!}{\underset{\rho_*}{\rightrightarrows}} \; \mathrm{PSh}(C), \qquad \rho^*, \rho^! : \mathrm{PSh}(C) \to \mathrm{PSh}(Cov(C)_0)$$

which can be explicitly computed as

$$\begin{aligned}
F(jY \times R_0 \subseteq jY) &= \rho_! F(Y) \\
\rho^* F(R \subseteq jY) &= F(Y) \\
F(jY \subseteq jY) &= \rho_* F(Y) \\
\rho^! F(R \subseteq jY) &= \lim_{jV \to R} F(V)
\end{aligned}$$

In particular we have

$$\rho_! \rho^! F(Y) = \hom(jY \times R_0, F).$$

The inclusion $Cov(C)_0 \subseteq Cov(C)$ induces a canonical factorisation

$$F \to \rho_! \rho^! F \to \pi_! \pi^! F.$$

Now the crucial observation is that for any $jY \to R_0$ we have $jY \times R_0 = jY$, so $\rho_! \rho^! F(Y) = F(Y)$ in this case. Consequently,

$$\begin{aligned}
\hom(R_0, \rho_! \rho^! F) &= \lim_{jV \to R_0} \rho_! \rho^! F(V) \\
&= \lim_{jV \to R_0} F(V) \\
&= \hom(R_0, F) \\
&= \lim_{jV \to R_0} F(V) \\
&= \rho_! \rho^! F(*)
\end{aligned}$$

So the morphism Φ in the diagram

$$\begin{array}{ccccc} F(*) & \longrightarrow & \rho_!\rho^!F(*) & \longrightarrow & \pi_!\pi^!F(*) \\ \big\downarrow & & \big\downarrow{\scriptstyle\Phi} & & \big\downarrow \\ \hom(R_0, F) & \longrightarrow & \hom(R_0, \rho_!\rho^!F) & \longrightarrow & \hom(R_0, \pi_!\pi^!F) \end{array}$$

is an isomorphism. As this is true for any presheaf F, we find that the morphism

$$\begin{array}{c} \operatorname{colim}\Big(F(*) \to F^{\dagger}(*) \to F^{\dagger\dagger}(*) \to \dots\Big) \\ \big\downarrow \\ \operatorname{colim}\Big(\hom(R_\lambda, F) \to \hom(R_\lambda, F^{\dagger}) \to \hom(R_\lambda, F^{\dagger\dagger}) \to \dots\Big) \end{array}$$

is a colimit of isomorphisms, and therefore itself is an isomorphism. Then since

$$\operatorname{colim}\Big(\hom(R_\lambda, F) \to \hom(R_\lambda, F^{\dagger}) \to \hom(R_\lambda, F^{\dagger\dagger}) \to \dots\Big)$$

$$= \hom\Big(R_\lambda, \operatorname{colim}(F \to F^{\dagger} \to F^{\dagger\dagger} \to \dots)\Big)$$

by the assumption (Fin), we have

$$LF(*) = \hom(R_0, LF).$$

Step 3. One sees directly from the definition that $(-)^{\dagger}$, and therefore $L(-)$ commutes with finite limits. $\square$

Corollary 23. *The category* $\operatorname{Shv}(C)$ *has all limits and colimits. Furthermore, for any small category* I *the following squares commute up to natural isomorphism*

$$\begin{array}{ccc} \operatorname{Fun}(I, \operatorname{PSh}(C)) & \xrightarrow{\operatorname{colim}} & \operatorname{PSh}(C) \\ \big\uparrow & & \big\downarrow \\ \operatorname{Fun}(I, \operatorname{Shv}(C)) & \xrightarrow[\operatorname{colim}]{} & \operatorname{Shv}(C) \end{array} \qquad \begin{array}{ccc} \operatorname{Fun}(I, \operatorname{PSh}(C)) & \xrightarrow{\lim} & \operatorname{PSh}(C) \\ \big\uparrow & & \big\uparrow \\ \operatorname{Fun}(I, \operatorname{Shv}(C)) & \xrightarrow[\lim]{} & \operatorname{Shv}(C) \end{array}$$

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