Derived Algebraic Geometry
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Autumn Semester 2023-2024
Lecture 1, October 5th 2023

We use as motivation for the course the fact that algebraic K-theory has excision for general blowup squares, only if we work with derived schemes.

1 Smooth manifolds

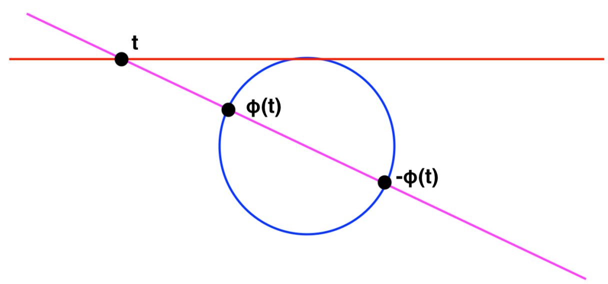
Definition 1 ([BT82, pg.20])**.** An *d-dimensional smooth manifold* is a topological space X equipped with open subsets $U_\lambda \subseteq X$, and homeomorphisms $\phi_\lambda : \mathbb{R}^d \xrightarrow{\sim} U_\lambda$ such that $\cup U_\lambda = X$ and for each λ, μ, the induced homeomorphisms

$$\underbrace{\phi_\lambda^{-1}(U_\lambda \cap U_\mu)}_{\subseteq \mathbb{R}^d} \xrightarrow{\sim} \underbrace{\phi_\mu^{-1}(U_\lambda \cap U_\mu)}_{\subseteq \mathbb{R}^d}$$

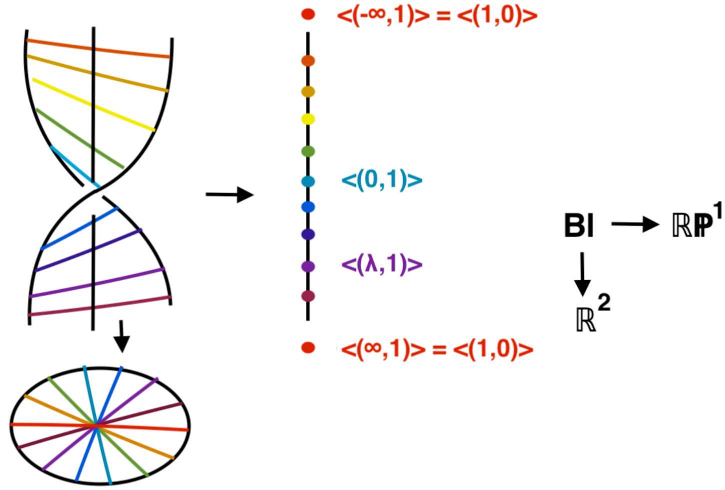
are defined by C^∞-functions.

Example 2.

1. $\mathbb{R}^d$ with the identity $\mathrm{id} : \mathbb{R}^d \xrightarrow{\sim} \mathbb{R}^d$.
2. The spheres $S^d := \{(x_0, \dots, x_d) \in \mathbb{R}^{d+1} \mid \sum x_i^2 = 1\}$ with the charts $\phi_i : \mathbb{R}^d \to S^d$; $(t_1, \dots, t_d) \mapsto \frac{1}{\sqrt{1+\sum t_i^2}}(t_1, \dots, t_i, 1, t_{i+1}, \dots, t_d)$ and $-\phi_i : \mathbb{R}^d \to S^d$ for $i = 1, \dots, d$.



3. Projective space $\mathbb{RP}^d$. As a set, this is the set of lines $L \subseteq \mathbb{R}^{d+1}$ containing the origin. The standard charts $\phi_i : \mathbb{R}^d \to \mathbb{RP}^d$ are $t \mapsto \langle (t_1, \dots, t_i, 1, t_{i+1}, \dots, t_d) \rangle$ where $\langle x \rangle = \{\lambda x \mid \lambda \in \mathbb{R}\}$ and $i = 0, \dots, n$.
4. The set $Bl_{\mathbb{R}^d}\{0\}$ of pairs $(x, L) \in \mathbb{R}^n \times \mathbb{RP}^{n-1}$ such that $x \in L$. The standard charts $\mathbb{R}^d \to Bl_{\mathbb{R}^d}\{0\}$ are $t \mapsto (t_i(t_1, \dots, t_{i-1}, 1, t_{i+1}, t_d), \langle (t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_d) \rangle)$.



Exercise 3. Describe the subsets $\phi_i^{-1}(U_i \cap U_j) \subseteq \mathbb{R}^d$ in the case of $\mathbb{RP}^d$ and $Bl_{\mathbb{R}^d}\{0\}$. Describe the transition functions $\phi_i^{-1}(U_i \cap U_j) \to \phi_j^{-1}(U_i \cap U_j)$ in these two cases.

Exercise 4 (Harder)**.** Equip the set $Gr(n,k)$ of k-dimensional subspaces of $\mathbb{R}^n$ with the structure of a manifold. Hint. Consider determinants of submatricies.

Example 5. Blowups are often used to desingularise things which are not smooth manifolds. For example, consider the image of the map $\mathbb{R} \to \mathbb{R}^2$; $t \mapsto (t^2-1, t^3-t)$ this cannot be a smooth manifold because close to the origin it looks like the axes. However, it has a canonical lift through the blowup $\mathbb{R} \to Bl_{\mathbb{R}^2}\{0\} \to \mathbb{R}^2$ and the image in $Bl_{\mathbb{R}^2}\{0\}$ *is* a smooth manifold.

$$- - - - \textit{picture} - - - -$$

2 Vector bundles

The map

$$Bl_{\mathbb{R}^d}\{0\} \xrightarrow{\pi} \mathbb{RP}^d$$
$$(x, L) \mapsto L$$

has the special property that each fibre $\pi^{-1}L$ has a structure of vector space, and these vary smoothly as L varies.

Definition 6. A *vector bundle* of rank r over a manifold X is map $\pi : E \to X$ together with a structure vector space on each fibre $\pi^{-1}\{x\}$, such that there exists an open covering $\{U_\lambda \subseteq X\}_{\lambda \in \Lambda}$ and commutative triangles

$$\begin{array}{ccc} \pi^{-1}U_\lambda & \longrightarrow & U_\lambda \times \mathbb{R}^r \\ & \searrow \quad \swarrow & \\ & U_\lambda & \end{array}$$

which are vector space isomorphisms on each fibre.

Classifying vector bundles is a major question in various areas of mathematics.

Question 7. Given a smooth manifold X, classify the set of isomorphism classes of vector bundles.

Example 8.

1. For any r we always have the trivial bundle $X \times \mathbb{R}^r$ which we sometimes write just as $\mathbb{R}^r$.
2. The tangent space $TX \to X$ to any smooth manifold X is a vector bundle.
3. For any two vector bundles $\pi : M \to X$, $\pi' : M' \to X$ of ranks r, r', the space
$$M \times_X M' = \{(m, m') \in M \times M' \mid \pi(m) = \pi'(m')\}$$
has a canonical structure of vector bundle of rank $r + r'$ it is written
$$M \oplus M'.$$
4. All vector bundles over $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ are of the form $\mathbb{R} \oplus \cdots \oplus \mathbb{R} \oplus M$ where M is the Möbius bundle $Bl_{\{0\}}\mathbb{R}^2 \to \mathbb{RP}^1 \cong S^1$. Notice that $M \oplus M \cong \mathbb{R}^2$.
5. For a rather complete answer to the classification problem, see [BT82, Prop.23.14].

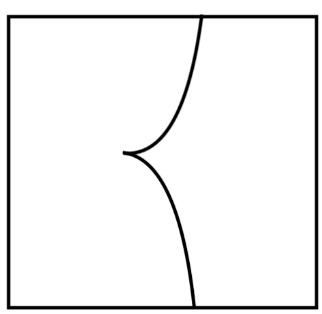
3 Affine algebraic geometry

In complex geometry and algebraic geometry one allows more general charts.

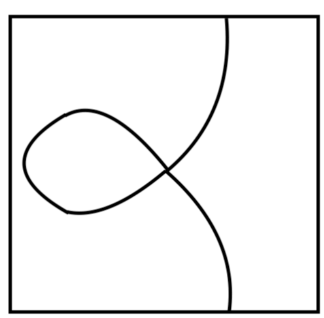
Definition 9. An *affine* $\mathbb{C}$*-variety* is a subset $V \subseteq \mathbb{C}^n$ of the form
$$V = \{z \in \mathbb{C}^n \mid f_1(z) = 0, \dots, f_c(z) = 0\}$$
for some polynomials $f_1, \dots, f_c \in \mathbb{C}[x_1, \dots, x_n]$. A *morphism* of varieties $V_1 \to V_2$ is a map $z \mapsto (g_1(z), \dots, g_{n_2}(z))$ defined by polynomials $g_1, \dots, g_{n_2} \in \mathbb{C}[x_1, \dots, x_{n_1}]$.

Example 10.

1. Cusp $= \{(x, y) \in \mathbb{C}^2 \mid y^2 - x^3 = 0\}$
2. Node $= \{(x, y) \in \mathbb{C}^2 \mid y^2 - x^2(x + 1) = 0\}$



Cusp



Node

Associated to any affine $\mathbb{C}$-variety V as above we have the ring of polynomial functions

$$\mathcal{O}(V) = \frac{\mathbb{C}[x_1, \dots, x_n]}{\langle f_1, \dots, f_c \rangle}.$$

Moreover, the points of V are in canonical bijection withhomomorphisms of $\mathbb{C}$-algebras $\mathcal{O}(V) \to \mathbb{C}$, and more generally, there is a bijection

$$\hom_{\mathcal{V}\mathrm{ar}_{/\mathbb{C}}}(V_1, V_2) \cong \hom_{\mathcal{R}\mathrm{ing}_{/\mathbb{C}}}(\mathcal{O}(V_2), \mathcal{O}(V_1))$$

So the modern point of view is to just treat every ring as an affine scheme.

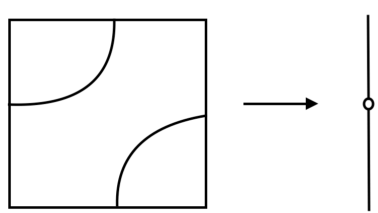
Definition 11. The category of *affine schemes* is the opposite of the category of rings.

$$\mathcal{A}\mathrm{ff} := \mathcal{R}\mathrm{ing}^{\mathrm{op}}.$$

The affine scheme associated to a ring $R \in \mathcal{R}\mathrm{ing}^{\mathrm{op}}$ is denoted $\mathrm{Spec}(R) \in \mathcal{A}\mathrm{ff}$.

One can show that a morphism $V_1 \to V_2$ of $\mathbb{C}$-varieties is an open immersion (for the topology induced from the usual topology on $\mathbb{C}^n \cong \mathbb{R}^{2n}$) if and only if there exists $f \in \mathbb{C}[x_1, \dots, x_n]$ such that $V_1 \cong \{x \in V_2 \mid f(x) \neq 0\}$. Equivalently, if $\mathcal{O}(V_1) \cong \mathcal{O}(V_2)[f^{-1}]$ for some $f \in \mathcal{O}(V_1)$. Similarly, a family $\{U_\lambda \to V\}_{\lambda \in \Lambda}$ of such open immersions satisfies $V = \cup U_\lambda$ if and only if $\langle f_\lambda \rangle = \mathcal{O}(V)$ where $\mathcal{O}(U_\lambda) \cong \mathcal{O}(V)[f_\lambda^{-1}]$. That is, if there exist $g_1, \dots, g_m \in \mathcal{O}(V)$ and λ_i such that $1 = f_{\lambda_1} g_1 + \dots + f_{\lambda_m} g_m$, cf. the partitions of unity from [BT82, pg.21].

Example 12. The subset $U = \{x \in \mathbb{C} \mid x \neq 0\} \subseteq \mathbb{C}$ is the image of the canonical projection $\{(x, y) \in \mathbb{C}^2 \mid xy - 1 = 0\} \to \mathbb{C}; (x, y) \mapsto x$.



Note that $\mathcal{O}(U) = \mathbb{C}[x, y]/\langle xy - 1 \rangle \cong \mathbb{C}[x, x^{-1}]$.

Definition 13. A morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ of affine schemes is an *open immersion* if $A \cong B[f^{-1}]$ for some $f \in B$. A family of open immersions $\{\mathrm{Spec}(B[f_\lambda^{-1}]) \to \mathrm{Spec}(B)\}_{\lambda \in \Lambda}$ is a *covering family* if $B = \langle f_\lambda \rangle$. That is, if the f_λ generate the unit ideal.

It is equivalent, and often easier to work with projective modules.

Definition 14. A *projective module* of rank r over a ring R is an R-module M such that there exists a covering $\{\operatorname{Spec}(R[f_\lambda^{-1}]) \to \operatorname{Spec}(R)\}_{\lambda\in\Lambda}$ and isomorphisms $M[f_\lambda^{-1}] \cong R[f_\lambda^{-1}]^{\oplus r}$ of $R[f_\lambda^{-1}]$-modules for each λ.

Exercise 15. Let P be a projective module of rank r over a ring R. Let $A = \operatorname{Sym}(P) := \oplus_{n=0}^{\infty} \operatorname{Sym}_R^n P$ be the free R-algebra generated by P. So $\operatorname{Sym}_R^n P$ is the quotient of $P \otimes_R \cdots \otimes_R P$ by the action of the symmetric group $p_1 \otimes \cdots \otimes p_n \sim p_{\sigma 1} \otimes \cdots \otimes p_{\sigma n}$. Show that:

($*$) There exists an open covering $\{\operatorname{Spec}(R[f_\lambda^{-1}])\}$ and isomorphisms $\phi_\lambda : A[f_\lambda^{-1}] \xrightarrow{\sim} R[f_\lambda^{-1}][x_1, \ldots, x_r]$, such that for each λ, μ, the induced isomorphisms

$$\phi_{\lambda\mu} = \phi_\mu \circ \phi_\lambda^{-1} : R[f_\lambda^{-1}][f_\mu^{-1}][x_1, \ldots, x_r] \xrightarrow{\sim} R[f_\mu^{-1}][f_\lambda^{-1}][x_1, \ldots, x_r]$$

are linear, in the sense that for each λ, μ we have $\phi_{\lambda\mu}(x_i) = a_1 x_1 + \cdots + a_r x_r$ for some $a_i \in R[f_\mu^{-1}][f_\lambda^{-1}]$.

Exercise 16 (Harder)**.** Given an R-algebra A satisfying ($*$) from Exercise 15, show that there exists a projective module P and an isomorphism $A \cong \oplus_{n=0}^{\infty} \operatorname{Sym}_R^n P$. Hint. Find a sub-$R$-module $P \subseteq A$ of "homogeneous elements of degree one". Use the facts that:

1. Any morphism of R-modules $M \to A$ induces a unique R-algebra homomorphism $\operatorname{Sym}(M) \to A$.
2. A morphism of R-algebras $A_1 \to A_2$ is an isomorphism if and only if it induces isomorphisms $A_1[f_\lambda^{-1}] \cong A_2[f_\lambda^{-1}]$ over all $R[f_\lambda^{-1}]$ in some open covering.

Example 17.

1. For any local ring (e.g., a field) every projective module is free, i.e., of the form $R^{\oplus r}$.
2. If R is a Noetherian ring of Krull dimension one, then every projective module is of the form $R^{\oplus(r-1)} \oplus L$ for some projective module L of rank one.
3. If $R = \mathbb{C}[x, y]/\langle y^2 - x^2(x + 1)\rangle$ is the ring associated to the node, then isomorphism classes of rank one projective modules are in bijection with units $\mathbb{C}^*$.

$$\operatorname{Pic}(R) \cong \mathbb{C}^*$$

   The module associated to $\lambda \in \mathbb{C}^*$ can be described as $\{f(t) \in \mathbb{C}[t] \mid f(1) = \lambda f(-1)\} \subseteq \mathbb{C}[t]$ where the R-module structure is via the ring homomorphism map $R \to \mathbb{C}[t]; x \mapsto t^2 - 1, y \mapsto t(t^2 - 1)$. Geometrically, we are taking the trivial bundle on the affine line, and glueing the fibre at -1 to the fibre at 1 using λ.
4. If $R = \mathbb{C}[x, y]/\langle y^2 - x^3\rangle$ is the ring associated to the cusp, then isomorphism classes of rank one projective modules are in bijection with elements of $\mathbb{C}$.

$$\mathbb{C} \cong \operatorname{Pic}(R).$$

   The module associated to $\lambda \in \mathbb{C}$ can be described as $\{\sum a_i t^i \in \mathbb{C}[t] \mid a_1 = \lambda a_0\} \subseteq \mathbb{C}[t]$ where the R-module structure is via the ring homomorphism map $R \to \mathbb{C}[t]; x \mapsto t^2, y \mapsto t^3$.

Question 18. Given a ring R classify the projective modules over R.

As in the case of smooth manifolds, the direct sum of two projective modules is projective. This makes the set of isomorphism classes of projective modules into an abelian monoid. It is often nicer to work with a group so one takes the group completion.

Definition 19. The group $K_0(R)$ is the quotient

$$K_0(R) := \frac{\mathbb{Z}\{\text{ projective modules of finite rank }\}}{\langle [P \oplus P'] = [P] + [P'] \rangle}$$

of the free abelian group generated by isomorphism classes of vector bundles, modulo the relation $[P \oplus P'] = [P] + [P']$.

Theorem 20 (Quillen ($i > 0$), Bass ($i < 0$))**.** *There exist functors* $K_i : \mathcal{R}\mathrm{ing} \to \mathcal{A}\mathrm{b}$ *for* $i \in \mathbb{Z}$ *such that for every ring* R *and open covering of the form* $\{\mathrm{Spec}(R[f^{-1}]), \mathrm{Spec}(R[g^{-1}])\}$ *there is a long exact sequence*

$$\cdots \to K_i(R) \to K_i(R[f^{-1}]) \oplus K_i(R[g^{-1}]) \to K_i(R[(fg)^{-1}]) \to K_{i-1}(R) \to \dots$$

4 Schemes

Just as smooth manifolds a built by gluing together copies of $\mathbb{R}^n$ along open immersions, general schemes are built by gluing together affine schemes along open immersions. We will explain a precise way of doing this in a later lecture, but for now, we just note that there exist algebraic versions $\mathbb{R}^n$, $\mathbb{RP}^n$, and $Bl_{\mathbb{R}^n}\{0\}$ for a general ring R, written $\mathbb{A}^n$, $\mathbb{P}^n$, and $Bl_{\mathbb{A}^n}\{0\}$. Later we will also see a definition of vector bundle for schemes. One can also upgrade K-theory to schemes. Using the property in Theorem 20 one can show that there is a unique[1] functor on (qcqs) schemes such that for any open covering of the form $\{U \subseteq X, V \subseteq X\}$ there is a long exact sequence

$$\cdots \to K_i(X) \to K_i(U) \oplus K_i(V) \to K_i(U \cap V) \to K_{i-1}(X) \to \dots$$

In addition to the Mayer-Vietoris long exact sequence, K-theory has a number of other long exact sequences. For example, blowups give rise to one.

Theorem 21 (Thomason)**.** *For any ring* R *there exists a long exact sequence.*

$$\cdots \to K_i(\mathbb{A}^n) \to K_i(Bl_{\mathbb{A}^n}\{0\}) \oplus K_i(\{0\}) \to K_i(\mathbb{P}^n) \to K_{i-1}(\mathbb{A}^n) \to \dots$$

[1] Actually, for uniqueness, one probably needs to work with the spectrum K rather than just the homotopy groups $K_i = \pi_i K$.

More generally, suppose that $f_1, \dots, f_c \in R$ *is a regular sequence*[2] *in a Noetherian ring and* $f : \mathrm{Spec}(R) = X \to \mathbb{A}^c$ *the corresponding morphism. Let*

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

be the pullback of the blowup square along $X \to \mathbb{A}^c$. *Then there is a long exact sequence*

$$\cdots \to K_i(X) \to K_i(Y) \oplus K_i(Z) \to K_i(E) \to K_{i-1}(X) \to \dots$$

The above sequence fails in general if the sequence $f_1, \dots, f_c$ is not a regular sequence. However, we can recover it if we use derived schemes.

Theorem 22 (Kerz, Strunk, Tamme)**.** *If one takes the pullback in the sense of derived schemes, then there is a long exact sequence*

$$\cdots \to K_i(X) \to K_i(Y) \oplus K_i(Z) \to K_i(E) \to K_{i-1}(X) \to \dots$$

for any *morphism* $X \to \mathbb{A}^c$.

5 Outline

1. Classical algebraic geometry ($\sim$ 4 lectures)
   (a) affine schemes, Kähler differentials, open and closed immersions, schemes, fibre products of schemes, quasi-coherent sheaves, blow-ups, formal completions
2. ∞-category foundations ($\sim$ 4 lectures)
   (a) homotopy types (simplicial sets, topological spaces)
   (b) ∞-categories (quasi-categories, simplicial categories, model categories)
   (c) (co)limits in ∞-categories
3. Derived algebraic geometry ($\sim$ 4 lectures)
   (a) affine derived schemes
   (b) cotangent complex
   (c) derived schemes, quasi-coherent sheaves
   (d) derived blowups
   (e) derived formal completion

References

[BT82] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, vol. 82, Springer, 1982.

[2] A sequence is *regular* if f_i is a non-zero divisor in $R/\langle f_1, \dots, f_{i-1}\rangle$ for each i. The point is rather that the R/I-module I/I^2 is projective, where $I = \langle f_1, \dots, f_c\rangle \subseteq R$. This module is the algebraic version of the normal bundle of a closed immersion of manifolds.