Derived Algebraic Geometry

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We do everything in the classical case in this lecture but the derived case works verbatim. Replace $\mathcal{R}\mathrm{ing}$ with $\mathcal{SCR}$ and $\mathcal{S}\mathrm{et}$ with $\mathcal{S} = N\mathcal{G}\mathrm{pd}_\infty$.

13 Quasi-coherent sheaves

Reference: [SAG, Appendix D]

13.1 Zariski sheaves

Recall that the Zariski topology on $\mathcal{R}\mathrm{ing}$ has covering families those $\{A \to B_\lambda\}_{\lambda \in \Lambda}$ such that each $A \to B_\lambda$ is an open immersion (i.e., $B_\lambda \cong A[f_\lambda^{-1}]$ for some f_λ) and for every field Ω, the morphism $\hom(B_\lambda, \Omega) \to \hom(A, \Omega)$ is surjective.

We did not do it, but we have the following nice criterion for when a presheaf is a Zariski sheaf.

Proposition 1. *A presheaf $F \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a Zariski sheaf if and only if it satisfies the following two conditions:*

1. *It sends the zero ring to the one point set.*

$$F(0) = \{*\}$$

2. *If $\{A \to B, A \to C\}$ is a two element Zariski covering, then the square*

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(B \otimes_A C) \end{array}$$

is a cartesian square.

13.2 Quasi-coherent sheaves

Definition 2. Suppose that $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a presheaf. The Zariski site X_{Zar} of X is the over category category $\mathcal{R}\mathrm{ing}_{/X}$ equipped with the induced topology from $\mathcal{R}\mathrm{ing}$.

Explicitly, objects are morphisms of presheaves $\mathrm{Spec}(A) \xrightarrow{s} X$, morphisms are commutative triangles $\mathrm{Spec}(B) \to \mathrm{Spec}(A) \to X$, and a family of morphisms

$$\left\{ \begin{array}{ccc} \mathrm{Spec}(B_\lambda) & \to & \mathrm{Spec}(A) \\ & \searrow \quad \swarrow & \\ & X & \end{array} \right\}$$

is a covering if and only if $\{A \to B_\lambda\}$ is a Zariski covering in $\mathcal{R}\mathrm{ing}$.

Example 3. If $X = \mathrm{Spec}(R)$ is representable, then X_{Zar} is equivalent to the under-category $\mathcal{R}\mathrm{ing}_{R/}$ of R-algebras. In particular, $\mathrm{Spec}(\mathbb{Z})_{\mathrm{Zar}} \cong \mathcal{R}\mathrm{ing}$.

Example 4. The structure sheaf $\mathcal{O}_X$ sends $\mathrm{Spec}(A) \xrightarrow{s} X$ to A considered as an abelian group.

Exercise 5. Show that $\mathcal{O}_X$ is a sheaf.

Definition 6. A *presheaf of $\mathcal{O}_X$-modules* is a presheaf on $\mathcal{R}\mathrm{ing}_{/X}$ such that each $F(\mathrm{Spec}(A) \xrightarrow{s} X)$ is equipped with a structure of an A-module, and given a ring homomorphism $A \to B$ the group homomorphism

$$F(\mathrm{Spec}(A) \xrightarrow{s} X) \to F(\mathrm{Spec}(B) \xrightarrow{t} X)$$

is A-linear where the target is given the A-module structure induced by $A \to B$.

A morphism of $\mathcal{O}_X$-modules is a morphism of presheaves $F \to G$ such that each

$$F(\mathrm{Spec}(A) \xrightarrow{s} X) \to G(\mathrm{Spec}(A) \xrightarrow{s} X)$$

is A-linear.

A presheaf of $\mathcal{O}_X$-modules is *quasi-coherent* if the induced morphisms

$$B \otimes_A F(\mathrm{Spec}(A) \xrightarrow{s} X) \to F(\mathrm{Spec}(B) \xrightarrow{t} X)$$

are isomorphisms. We will write

$$\mathrm{QC}_X$$

for the category of quasi-coherent sheaves on a presheaf $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$.

Exercise 7. Suppose $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a presheaf and $\mathcal{F} \in \mathrm{QC}_X$ a quasi-coherent presheaf of $\mathcal{O}_X$-modules. Show that $\mathcal{F}$ is a sheaf.

Exercise 8. Suppose $X = \mathrm{Spec}(R)$ is a representable presheaf. Show that the assignment $\Omega : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ to $\Omega_{A/R}$ has a canonical structure of quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Exercise 9. Suppose $X = \mathrm{Spec}(R)$ is a representable presheaf. Show that there is a canonical equivalence of categories

$$\mathrm{QC}_{\mathrm{Spec}(R)} \cong R\text{-mod}.$$

Exercise 10. Let $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ be a presheaf and suppose that $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is a family of sheaves of $\mathcal{O}_X$-modules. Show that the assignment sending $\mathrm{Spec}(A)\overset{s}{\to}X$ to $\bigoplus_{\lambda\in\Lambda}\mathcal{F}_\lambda(s)$ has a structure of quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Exercise 11. Let $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ be a presheaf and suppose that $\mathcal{F} \to \mathcal{G}$ is a morphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that the assignment sending $\mathrm{Spec}(A)\overset{s}{\to}X$ to $\mathrm{coker}(\mathcal{F}(s) \to \mathcal{G}(s))$ has a structure of quasi-coherent sheaf of $\mathcal{O}_X$-modules.

Remark 12. In general, a category of sheaves $\mathrm{Shv}(C)$ considered as a subcategory of $\mathrm{PSh}(C)$ is closed under limits but no colimits. The above exercises show that QC_X is special.

13.3 Glueing modules

Definition 13. Let $\{A \to B, A \to C\}$ be a two element Zariski covering in $\mathcal{R}\mathrm{ing}$. Define DD to be the category whose objects are triples (M_B, M_C, ϕ) where

1. M_B is a B-module,
2. M_C is a C-module, and
3. ϕ is an isomorphism $M_B \otimes_B (B \otimes_A C) \overset{\sim}{\to} M_C \otimes_C (C \otimes_A B)$.

A morphism $(M_B, M_C, \phi) \to (N_B, N_C, \psi)$ of DD is a pair (f_B, f_C) where

1. f_B is a morphism of B-modules,
2. f_C is a morphism of C-modules, and
3. the induced square

$$\begin{array}{ccc} M_B \otimes_A C & \longrightarrow & M_C \otimes_A B \\ \downarrow & & \downarrow \\ N_B \otimes_A C & \longrightarrow & N_C \otimes_A B \end{array}$$

is commutative.

Note that there are canonical functors

$$A\text{-mod} \to DD; \qquad M \mapsto (M \otimes_A B, M \otimes_A C, \phi)$$

where ϕ is the canonical isomorphism

$$M \otimes_A B \otimes_B (B \otimes_A C) \overset{\sim}{\to} M \otimes_A C \otimes_C (C \otimes_A B)$$

and

$$DD \to A\text{-mod}; \qquad (M_B, M_C, \phi) \mapsto M_B \times_{M_C \otimes_C (C \otimes_A B)} M_C$$

where the morphism $M_B \to M_C \otimes_C (C \otimes_A B)$ is induced by ϕ.

Exercise 14. Show that the two functors defined in Def.**??** are inverse equivalences of categories. Hint.[1] Hint.[2] Hint.[3]

[1] Note that $0 \to M \to M \otimes_A B \times M \otimes_A C \to M \otimes_A B \otimes_A C$ is an exact sequence for any A-module M.

[2] Note that since $A \to B$ is an open immersion $- \otimes_A B$ preserves finite limits and finite colimits. The same is true for $- \otimes_A C$.

[3] Note that since $A \to B$, $A \to C$ are open immersions, we have $B \otimes_A B \cong B$ and $C \otimes_A C \cong C$.

Remark 15. One can show that the category DD is actually a model for the fibre product

$$B\text{-mod} \times^{L}_{B \otimes_A C\text{-mod}} C\text{-mod}$$

in the ∞-category of ∞-categories. So the Exericse 14 essentially says that the "presheaf" $A \mapsto A$-mod of categories is a Zariski sheaf. We put "presheaf" in quotes because this assignment is only functorial up to the canonical isomorphisms $C \otimes_B B \otimes_A - \cong C \otimes_A -$. We will come back to this problem later.

13.4 Glueing presheaves

Suppose that $f : Y \to X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a representable morphism of presheaves. By definition, for every $\mathrm{Spec}(A) \to X$ the pullback $Y \times_X \mathrm{Spec}(A)$ is representable, so we obtain a functor $X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$. As such we obtain an adjunction

$$f^* : \mathrm{PSh}(X_{\mathrm{Zar}}) \rightleftarrows \mathrm{PSh}(Y_{\mathrm{Zar}}) : f_*.$$

Since $X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$ has the left adjoint sending $\mathrm{Spec}(B) \to Y$ to the composition $\mathrm{Spec}(B) \to Y \to X$, this adjunction is particularly easy to describe.

1. f_* is composition with pullback $X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$ (this is always true).
2. f^* is composition with "composition" $Y_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$.

It is not too hard to show that both pullback $X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$ and composition $Y_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$ send Zariski coverings to Zariski coverings, and so both f^* and f_* preserve Zariski sheaves. That is, the induced adjunction

$$f^* : \mathrm{Shv}(X_{\mathrm{Zar}}) \rightleftarrows \mathrm{Shv}(Y_{\mathrm{Zar}}) : f_*$$

has the same description as above.

Exercise 16. Show that the functor $X_{\mathrm{Zar}} \to Y_{\mathrm{Zar}}$ sends covering families to covering families.

Exercise 17. Show that f^* and f_* send quasi-coherent sheaves to quasi-coherent sheaves.

Now we upgrade the above descent theorem to the presheaf setting.

Definition 18. Let $\{U \to X, V \to X\}$ is a pair of representable open immersions of presheaves such that for every field Ω the map $U(\Omega) \sqcup V(\Omega) \to X(\Omega)$ is surjective. We will use the Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{\iota} & V \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle k} \\ U & \xrightarrow[i]{} & X \end{array}$$

Define DD to be the category whose objects are triples $(\mathcal{F}_U, \mathcal{F}_V, \phi)$ where

1. $\mathcal{F}_U$ is a sheaf on U_{Zar},
2. $\mathcal{F}_V$ is a sheaf on V_{Zar}, and
3. ϕ is an isomorphism $\kappa^*\mathcal{F}_U \xrightarrow{\sim} \iota^*\mathcal{F}_V$.

A morphism $(\mathcal{F}_U, \mathcal{F}_V, \phi) \to (\mathcal{G}_U, \mathcal{G}_V, \psi)$ of DD is a pair (f_U, f_V) where

1. f_U is a morphism of sheaves on U_{Zar},
2. f_V is a morphism of sheaves on V_{Zar}, and
3. the induced square

$$\begin{array}{ccc} \kappa^*\mathcal{F}_U & \longrightarrow & \iota^*\mathcal{F}_V \\ \downarrow & & \downarrow \\ \kappa^*\mathcal{G}_U & \longrightarrow & \iota^*\mathcal{G}_V \end{array}$$

is commutative.

Note that there are canonical functors

$$\text{Shv}(X_{\text{Zar}}) \to DD; \qquad \mathcal{F} \mapsto (i^*\mathcal{F}, k^*\mathcal{G}, \phi)$$

where ϕ is the canonical isomorphism

$$\kappa^*\iota^* \xrightarrow{\sim} \iota^*k^*$$

and

$$DD \to \text{Shv}(X_{\text{Zar}}); \qquad (\mathcal{F}_U, \mathcal{F}_V, \phi) \mapsto i_*\mathcal{F}_U \times_{k_*\iota_*\iota^*\mathcal{F}_V} k_*\mathcal{F}_V$$

where the morphism $i_*\mathcal{F}_U \to k_*\iota_*\iota^*\mathcal{F}_V$ is the morphism

$$i_*\mathcal{F}_U \to i_*\kappa_*\kappa^*\mathcal{F}_U = k_*\iota_*\kappa^*\mathcal{F}_U \xrightarrow{\phi} k_*\iota_*\iota^*\mathcal{F}_V.$$

13.5 Cotangent sheaf

Definition 19 ([DAG, pg.35])**.** Let $X \in \text{PSh}(\mathcal{R}\text{ing}^{op})$ be a functor, and $\text{Spec}(A) \xrightarrow{s} X$ a morphism of presheaves. Consider the functor that sends an A-module M to the set of factorisations

$$\Omega_X(s, -) : A\text{-mod} \to \mathcal{S}\text{et}; \qquad M \mapsto \left\{ \begin{array}{ccc} \text{Spec}(A) & \xrightarrow{s} & X \\ {\scriptstyle\pi}\downarrow & \nearrow & \\ \text{Spec}(A \oplus M) & & \end{array} \right\}$$

where $\pi : A \oplus M \to A$ is the canonical projection. If this functor is corepresentable by an A-module, we will write $\Omega_X(s)$ for this A-module. That is,

$$\hom_{A\text{-mod}}(\Omega_X(s), M) = \{\text{Spec}(A \oplus M) \xrightarrow{t} X \mid t \circ \pi = s\}$$

functorially in M.

Exercise 20. Suppose that $X = \text{Spec}(R)$ is an affine scheme. Show that all $\Omega_X(s, -)$ are representable.

Exercise 21. Suppose that $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a functor for which all $\Omega_X(s,-)$ are representable. Show that the assignment

$$\Omega_X : \mathrm{Spec}(A) \xrightarrow{s} X \mapsto \Omega_X(s)$$

has a canonical structure of presheaf of $\mathcal{O}_X$-modules. That is, show that every commutative triangle $t : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \xrightarrow{s} X$ of presheaves induces a morphism $B \otimes_A \Omega_X(s) \to \Omega_X(t)$ of B-modules, and this assignment is compatible with composition and identities. Hint.[4] Hint.[5]

Definition 22. Suppose $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a functor. We say X *admits a cotangent sheaf* if all $\Omega_X(s,-)$ are representable, and Ω_X is a quasi-coherent sheaf. That is, if all comparison morphisms

$$B \otimes_A \Omega_X(s) \to \Omega_X(t)$$

are isomorphisms.

Exercise 23. Suppose that $X = \mathrm{Spec}(R)$ is an affine scheme. Show that $\mathrm{Spec}(A)$ admits a cotangent sheaf.

Definition 24. Recall that a separated scheme is a presheaf $X \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ that admits an open affine covering $\{U_\lambda \to X\}_{\lambda\in\Lambda}$. If X admits an open affine covering where Λ is finite, then we say that X is *quasi-compact*.

Proposition 25. *Suppose that X is a quasi-compact separated scheme. Then X admits a cotangent sheaf.*

Proof. We work by induction on the size of open affine covering that X admits. If X admits an open affine covering of size one, then it is representable, and the result is Exercise 23. Suppose all schemes admitting open affine coverings of size n admit cotangent sheaves, and let X be a scheme admitting an open affine covering of size $n+1$, say $\{U_i \to X\}_{i=0}^n$. Define V to be the Zariski sheafification of the presheaf union $V = \cup_{i=1}^n U_i$. Note that $U \times_X V$ admits the open affine covering $\{U \times_X U_i \to U \times_X V\}_{i=1}^n$. Since U, V and $U \cap V$ admit cotangent sheaves, by the glueing we explained above, it follows that X admits a cotangent sheaf. □

[4] Consider the functors B-mod $\to \mathcal{S}$et that $B \otimes_A \Omega_X(s)$ and $\Omega_X(t)$ corepresent.

[5] Note there is a commutative square of rings

$$\begin{array}{ccc} B & \longleftarrow & A \\ \uparrow & & \uparrow \\ B \oplus M & \longleftarrow & A \oplus M|_A \end{array}$$