Derived Algebraic Geometry
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Autumn Semester 2022-2023

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This file compiled January 12, 2023

12 Derived Schemes

12.1 Discussion

Recall that one can define[1] an n-dimensional manifold as a topological space which locally looks like $\mathbb{R}^n$ in the sense that it is equipped with an open covering $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ and homeomorphisms $\phi_\lambda : U_\lambda \xrightarrow{\sim} V_\lambda$ to some open $V_\lambda \subseteq \mathbb{R}^n$. An equivalent definition is that an n-dimensional manifold is a coequaliser in the category of topological spaces of the form

$$\operatorname{coeq}\left(\coprod_\mu V_{1,\mu} \rightrightarrows \coprod_\lambda V_{0,\lambda}\right)$$

where the $V_{1\mu}, V_{0\lambda}$ are open subsets of a copy some $\mathbb{R}^n$, and the induced $V_{1\mu} \to V_{0\lambda}$ are (isomorphic to) open immersions.

In classical algebraic geometry, instead of open subsets of $\mathbb{R}^n$, one would like to work with things that locally look like affine varieties. That is, sets of the form $V(f) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid f_j(z) = 0,\ j = 1, \dots, c\}$ for some $f_1, \dots, f_c \in \mathbb{C}[z_1, \dots, z_n]$. As we mentioned in the first lecture, in order to keep track of multiplicities, it is useful to use the quotient ring $A = \mathbb{C}[z_1, \dots, z_n]/\langle f_1, \dots, f_c\rangle$ instead of the set $V(f)$.[2]

It turns out we would also like to perform various commutative algebraic operations (e.g., completion), and modern algebraic geometry ends up allowing any commutative ring as a local neighbourhood.

So. In order to define schemes we want a category which fully faithfully contains $\mathcal{R}\mathrm{ing}^{op}$ the opposite of category of rings, and which admits colimits. Historically the two choices are: the category of locally ringed topological spaces, and the category of sheaves on $\mathcal{R}\mathrm{ing}^{op}$.

Using locally ringed topological spaces is useful for results such as Serre's GAGA since complex analytic spaces also have a natural structure of locally ringed topological space. On the other hand, for applications in the representation theory of

[1] Cf. [Lang, Fundamentals of Differential Geometry].
[2] The set $V(f)$ can be recovered as the set of ring homomorphisms $A \to \mathbb{C}$.

algebraic groups or moduli spaces, it can be more natural to consider sheaves on $\mathcal{R}\mathrm{ing}^{op}$, since moduli questions are more naturally stated in terms of functors.

Both approaches exist in derived geometry, for example Lurie mostly uses the former and Gaitsgory, Rozenblyum mostly use the latter. We will develop the latter, that is, a derived scheme will be a sheaf on $\mathcal{SCR}^{op}$ equipped with a covering by affine schemes. Doing this derived schemes, derived algebraic spaces, and derived stacks all exist in the same category. Incidentally, $\mathrm{Shv}(\mathcal{SCR}^{op})$ also contains ind-schemes and in particular, completions.

12.2 Representable morphisms

To begin with we want to define what it means for a morphism of presheaves to be an open immersion / étale / flat, etc.

Recall that a presheaf $F \in \mathrm{PSh}(C)$ is called *representable* if it is of the form $\mathrm{Map}(-, X)$ for some $X \in C$. We need a relative version of this.

Definition 1. Let C be an ∞-category and $f : F \to G \in \mathrm{PSh}(C)$ a morphism of presheaves. We say that f is *representable* if for every morphism of the form $\mathrm{Map}(-, T) \to G$, the fibre product $F \times_G \mathrm{Map}(-, T)$ is of the form $\mathrm{Map}(-, T')$ for some $T' \in C$.

Remark 2. One way of thinking about this definition is as follows. A morphism $\hom(-, T) \to G$ is a T-point of G. For example (all this will be explained below), if X is an algebraic variety defined by polynomials $f_1, \dots, f_c \in \mathbb{Z}[x_1, \dots, x_n]$ then a morphism $\mathrm{Spec}(k) \to X$ with k a field is the same as a point in the algebraic k-variety $\{(a_1, \dots, a_n) \in k^n \mid f_1(a) = 0, \dots, f_c(a) = 0\}$. So asking that all fibres $F \times_G \mathrm{Map}(-, T)$ are representable is asking that $G \to F$ is somehow "fibred in representables".

Exercise 3. Suppose C is an ∞-category with finite limits. Show the following.

1. Any morphism $\mathrm{Map}(-, X) \to \mathrm{Map}(-, Y)$ between representable presheaves is a representable morphism.
2. A morphism $F \to *$ towards the terminal presheaf is a representable morphism if and only if F is a representable presheaf.
3. For any presheaf F and $T \in C$ the projection $F \times \mathrm{Map}(-, T) \to F$ is representable.
4. If $F \to G$ is a representable morphism of presheaves and $H \to G$ any morphism of presheaves then $F \times_G H \to H$ is representable.
5. If $F \to G$ and $G \to H$ are representable morphisms of presheaves then the composition $F \to H$ is representable.

Definition 4. Suppose C is a category admitting fibre products. Let $f : F \to G \in \mathrm{PSh}(C)$ be a representable morphism of presheaves and $\mathcal{P}$ a property of morphisms in C. We say that f is has property $\mathcal{P}$ if for every cartesian square

$$F \times_G \mathrm{Map}(-,X) \cong \qquad \begin{array}{ccc} \mathrm{Map}(-,Y) & \xrightarrow{\hom(-,g)} & \mathrm{Map}(-,X) \\ \downarrow & & \downarrow \\ F & \longrightarrow & G \end{array}$$

the morphism $g : Y \to X$ has property $\mathcal{P}$.

Exercise 5. Let C be an ∞-category admitting fibre products, $f : Y \to X$ a morphism in C, and suppose $\mathcal{P}$ is property of morphisms in C stable under pullback. Show that the representable morphism of presheaves $\hom(-,f) : \hom(-,Y) \to \hom(-,X) \in \mathrm{PSh}(C)$ has property $\mathcal{P}$ if and only if f has property $\mathcal{P}$.

We did not prove it but it's a fact that for any $A \to B \to C \in \mathcal{R}\mathrm{ing}$, if $A \to B$ and $A \to C$ are open immersions resp. étale morphisms, then so is $B \to C$. This fact passes to presheaves.

Exercise 6. Suppose that $F \xrightarrow{f} G \xrightarrow{g} H \in \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ are representable morphisms such that g and gf are representable open immersions resp. étale morphism. Show that f is also a representable open immersion, resp. étale morphism.

12.3 Classical schemes

For reference we begin with a classical definition of schemes. Affine schemes are easy.

Definition 7. An *affine scheme* is a representable presheaf on $\mathcal{R}\mathrm{ing}$. That is, it is a presheaf of the form $\hom_{\mathcal{R}\mathrm{ing}}(A,-) : \mathcal{R}\mathrm{ing} \to \mathcal{S}\mathrm{et}$ for some ring $A \in \mathcal{R}\mathrm{ing}$. It is customary to use Spec for the coYoneda functor

$$\mathrm{Spec} : \mathcal{R}\mathrm{ing} \to \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op}),$$

that is,

$$\mathrm{Spec}(A) = \hom_{\mathcal{R}\mathrm{ing}}(A,-).$$

Exercise 8.

1. The functor which sends a ring to the underlying set of the n-fold product

$$\mathbb{A}^n : A \mapsto A^n \in \mathcal{S}\mathrm{et}.$$

   is called n-dimensional *affine space*. Show that this is an affine scheme.

2. The functor which sends a ring to its set of units is denoted

$$\mathbb{G}_m : A \mapsto A^* \in \mathcal{S}\mathrm{et}.$$

   Show that this is an affine scheme.

3. Suppose we have $f_1, \dots, f_c \in \mathbb{Z}[x_1, \dots, x_n]$. For any field k there is a unique ring homomorphism $\mathbb{Z} \to k$, and therefore a canonical way of considering the f_i as elements in $k[x_1, \dots, x_n]$. Consider the functor which sends a field k to the algebraic variety over k

$$V(f)(k) = \{(a_1, \dots, a_n) \in k^n \mid f_1(a) = 0, \dots, f_c(a) = 0\}.$$

In fact, there is no need to assume k is a field. The set $V(f)$ is well-defined for any ring A. Show that $A \mapsto V(f)(A)$ is an affine scheme.

4. If $X : \Lambda \to \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is a diagram such that each X_λ is an affine scheme, show that $\varprojlim X_\lambda$ is also an affine scheme.

We restrict our attention to separated schemes so that we don't have to talk about open subfunctors.

Definition 9 ([Jantzen, §1.9])**.** A *separated scheme* is a Zariski sheaf $X : \mathcal{R}\mathrm{ing} \to \mathcal{S}\mathrm{et}$ for which there exists a collection of affine schemes U_λ and representable open immersions $U_\lambda \to X$ such that for every field Ω the map

$$\coprod_{\lambda \in \Lambda} U_\lambda(\Omega) \to X(\Omega)$$

is surjective. Such a family $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ is called an *open affine covering*.

Example 10. Last week we claimed without proof that every representable functor on $\mathcal{R}\mathrm{ing}^{op}$ is an fppf-sheaf, and therefore a Zariski sheaf. Consequently, every affine scheme is a separated scheme. So we have the following fully faithful inclusions.

$$\left\{ \begin{array}{c} \text{affine} \\ \text{schemes} \end{array} \right\} \subseteq \left\{ \begin{array}{c} \text{separated} \\ \text{schemes} \end{array} \right\} \subseteq \mathrm{Shv}(\mathcal{R}\mathrm{ing}^{op}) \subseteq \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$$

Example 11 (cf.[Jantzen, pg.15])**.** The easiest example of a non-affine separated scheme is $\mathbb{P}^n$ for $n > 0$. One way of defining $\mathbb{P}^n$ is as the functor sending a ring A to the set of equivalence classes

$$\mathbb{P}^n(A) = \left\{ \begin{array}{c} \text{surjections of } A\text{-modules} \\ A^{\oplus n+1} \twoheadrightarrow L \\ \text{such that } L \text{ is locally free of rank one} \end{array} \right\} / \sim$$

Here, *locally free of rank one* means there exists an open Zariski covering $\{A \to B_\lambda\}_{\lambda \in \Lambda}$ and isomorphisms of B_λ-modules $L \otimes_A B_\lambda \cong B_\lambda$. Two surjections $A^{\oplus n+1} \twoheadrightarrow L$ and $A^{\oplus n+1} \twoheadrightarrow L'$ are equivalent if there exists an isomorphism $L \cong L'$ making a commutative triangle. Given a ring homomorphism $A \to B$ the morphism $\mathbb{P}^n(A) \to \mathbb{P}^n(B)$ sends $A^{\oplus n+1} \twoheadrightarrow L$ to $B^{\oplus n+1} \cong B \otimes_A A^{\oplus n+1} \twoheadrightarrow B \otimes_A L$.

Exercise 12. For each $i = 0, \dots, n$ define $U_i \subseteq \mathbb{P}^n$ to be the subfunctor

$$U_i(A) = \{A^{\oplus n+1} \twoheadrightarrow L \mid A \xrightarrow{\iota_i} A^{\oplus n+1} \twoheadrightarrow L \text{ is an isomorphism } \}$$

where ι_i is the inclusion of the ith component. Define an isomorphism of functors

$$U_i \cong \mathbb{A}^n$$

and show that $\{U_i \to \mathbb{P}^n\}_{i=0}^n$ is an open affine covering.

Exercise 13. (Harder). Suppose $f \in \mathbb{Z}[x_0, \dots, x_n]$ is a homogeneous polynomial of degree d. That is, an element of the free $\mathbb{Z}$-module $\mathrm{Sym}^d(\mathbb{Z}x_0 \oplus \cdots \oplus \mathbb{Z}x_n)$ of rank $\binom{n+d}{d}$. Given any ring A the canonical morphism $\mathbb{Z} \to A$ induces an element $f|_A$ of $\mathrm{Sym}^d(A^{\oplus n+1})$, and if we also have a morphism $A^{\oplus n+1} \to L$ to some other A-module L, we get an element $f|_{\mathrm{Sym}^d(L)}$ of $\mathrm{Sym}^d(L)$. Now consider the subfunctor $Z \subseteq \mathbb{P}^n$ which sends a ring A to the set

$$Z(A) = \{A^{\oplus n+1} \twoheadrightarrow L \mid f|_{\mathrm{Sym}^d(L)} = 0\} \subseteq \mathbb{P}^n(A).$$

1. Show that Z is a functor, that is, the morphism $\mathbb{P}^n(A) \to \mathbb{P}^n(B)$ associated to a ring homomorphism $A \to B$ sends $Z(A)$ into $Z(B)$.
2. Let $\mathbb{A}^n \cong U_i \subseteq \mathbb{P}^{n+1}$ be a standard chart as in the Exercise 12. Describe $\mathbb{A}^n \times_{\mathbb{P}^n} Z \subseteq \mathbb{A}^n$

Exercise 14.
1. Show that if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of separated schemes, then the product $\prod_{\lambda \in \Lambda} X_\lambda$ (taken in $\mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$) is a separated scheme.
2. Show that if $Y \to X$ is a morphism of separated schemes and $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ is an open affine covering, then there exists an open affine covering $\{V_{\lambda\mu} \to Y\}_{\lambda \in \Lambda, \mu \in M}$ and commutative squares

$$\begin{array}{ccc} V_{\lambda\mu} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U_\lambda & \longrightarrow & X \end{array}$$

3. (Harder). Show that if $Y \to X$, $Z \to X$ are two morphisms of separated schemes then the fibre product $Y \times_X Z$ (taken in $\mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$) is a separated scheme. Hint.[3] Hint.[4] Hint.[5] Hint.[6]
4. Using the above parts, show that the subcategory $\{\text{separated schemes}\} \subseteq \mathrm{PSh}(\mathcal{R}\mathrm{ing}^{op})$ is closed under all small limits.

[3] Using the previous part choose compatible open affine coverings $\{U_\lambda \to X\}, \{V_{\lambda\mu} \to Y\}, \{W_{\lambda\nu} \to Z\}$.

[4] Show that the morphisms $V_{\lambda\mu} \to Y$ (resp. $W_{\lambda\nu} \to Z$) factor via representable open immersions $V_{\lambda\mu} \to Y \times_X U_\lambda$ (resp. $W_{\lambda\nu} \to U_\lambda \times_X Z$).

[5] Show that the morphisms $V_{\lambda\mu} \times_{U_\lambda} W_{\lambda\nu} \to Y \times_X Z$ are representable open immersions by factoring them as $V \times_U W \to (Y \times_X U) \times_U W = Y \times_X W \to Y \times_X (U \times_X Z) \to Y \times_X X \times_X Z = Y \times_X Z$

[6] Show that $\{V_{\lambda\mu} \times_{U_\lambda} W_{\lambda\nu} \to Y \times_X Z\}$ is an open affine covering.

12.4 Derived schemes

The definition of derived schemes is a straight-forward generalisation.

Definition 15 (cf.[GR, I.2.3.1.1, I.2.3.1.2, I.2.3.1.3])**.** An *affine derived scheme* is a representable functor $\mathrm{Map}(A, -) : \mathcal{SCR} \to \mathcal{S} = N(\mathcal{G}\mathrm{pd}_\infty)$. As in the classical case, we write

$$\mathrm{Spec} : \mathcal{SCR} \to \mathrm{PSh}(\mathcal{SCR}^{op})$$

for the coYoneda functor.

A *separated derived scheme* is a Zariski sheaf $X : \mathcal{SCR} \to \mathcal{S}$ for which there exists a collection of affine schemes U_λ and representable open immersions $U_\lambda \to X$ such that

$$\coprod_{\lambda \in \Lambda} U_\lambda(\Omega) \to X(\Omega)$$

is surjective for every field Ω.

Example 16. As in the classical case, every affine derived scheme is a separated derived scheme. In the derived setting the affine space $\mathbb{A}^n = \mathrm{Spec}(\mathbb{Z}[t_1, \dots, t_n])$ sends a simplicial ring R to the n-fold product of its underlying homotopy type

$$\mathbb{A}^n : A \mapsto A^n \in \mathcal{S}.$$

Exercise 17 (cf.[DAG, Prop.4.6.3])**.** Do Exercise 14 for separated derived schemes.

Every derived scheme has a corresponding classical scheme and just as $\mathcal{SCR}$ contains $\mathcal{R}$ing fully faithfully as the subcategory of 0-truncated rings, the category of derived schemes contains the category of classical schemes fully faithfully. More generally, we can promote the notion of n-truncated derived ring to n-truncated derived scheme.

Let

$$\iota_{\leq n} : \mathcal{SCR}_{\leq n} \subseteq \mathcal{SCR}$$

denote the full subcategory of those simplicial commutative rings A such that $\pi_i A = 0$ for $i > n$. The inclusion admits $\iota_{\leq n}$ admits a left adjoint $\pi_{\leq n}$. In fact, we want to consider the opposite functor

$${}^{\leq n}\gamma = \iota_{\leq n}^{op} : \mathcal{SCR}_{\leq n}^{op} \subseteq \mathcal{SCR}^{op}.$$

As discussed in the lecture on topoi, ${}^{\leq n}\gamma$ induces an adjunction of presheaf categories

$${}^{\leq n}\gamma^* : \mathrm{PSh}(\mathcal{SCR}_{\leq n}^{op}) \rightleftarrows \mathrm{PSh}(\mathcal{SCR}^{op}) : {}^{\leq n}\gamma_*$$

where ${}^{\leq n}\gamma_*$ is composition with ${}^{\leq n}\gamma$ and the left adjoint ${}^{\leq n}\gamma^*$ is compatible with Yoneda in the sense that

$${}^{\leq n}\gamma^* \,\mathrm{Map}(A, -) \cong \mathrm{Map}({}^{\leq n}\gamma A, -).$$

Remark 18. Apart from knowing that ${}^{\leq n}\gamma^*$ preserves representables and colimits, it's not easy to say much more about it.

Definition 19. The *truncation functor* is the composition

$$\tau^{\leq n} = {}^{\leq n}\gamma^* {}^{\leq n}\gamma_* : \mathrm{PSh}(\mathcal{SCR}^{op}) \to \mathrm{PSh}(\mathcal{SCR}^{op}).$$

Note that it comes equipped with a natural transformation

$$\tau^{\leq n} \to \mathrm{id}_{\mathrm{PSh}(\mathcal{SCR}^{op})}.$$

and since ${}^{\leq n}\gamma$ is fully faithful, the unit $\mathrm{id} \to {}^{\leq n}\gamma_* {}^{\leq n}\gamma^*$ of adjunction is an equivalence, and therefore $\tau^{\leq n}\tau^{\leq n} \cong \tau^{\leq n}$. In fact, since for any $m \leq n$ the inclusions factor $\mathcal{SCR}_{\leq m} \subseteq \mathcal{SCR}_{\leq n} \subseteq \mathcal{SCR}$ one can check that we have $\tau^{\leq m}\tau^{\leq n} \cong \tau^{\leq m}$ and from this we obtain a tower of natural transformations

$$\tau^{\leq 0} \to \tau^{\leq 1} \to \tau^{\leq 2} \to \cdots \to \mathrm{id}$$

Exercise 20. Using the adjunction $(\pi_{\leq n}, \iota_{\leq n})$ and the fact that left Kan extensions (e.g., γ^*) send representables to representables, show that if $\mathrm{Spec}(A)$ is an affine derived scheme. Then

$$\tau^{\leq n}\,\mathrm{Spec}(A) = \mathrm{Spec}(\tau_{\leq n}A)$$

where $\tau_{\leq n} = \iota_{\leq n}\pi_{\leq n}$. In particular, $\tau^{\leq n}\,\mathrm{Spec}(A)$ is again affine.

12.5 Convergence

Recall from the lecture on the cotangent complex that for any morphism of simplicial rings $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$, the A-algebra B can be built from its truncation $\pi_0 B$ and various linear information encoded in cotangent complexes. In particular, we claimed that

$$B = \varprojlim \tau_{\leq n} B.$$

We can upgrade this to schemes.

Definition 21 (cf.[GR, I.2.1.4.1], cf.[DAG, Def.3.4.1])**.**] A presheaf $F \in \mathrm{PSh}(\mathcal{SCR}^{op})$ is called *convergent* [DG] or *nilcomplete* [DAG] if for every $A \in \mathcal{SCR}$ the canonical map

$$F(A) \to \varprojlim_n F(\tau_{\leq n}A)$$

is an equivalence.

Exercise 22. Using the fact that for any simplicial ring A we have $A \cong \varprojlim_n \tau_{\leq n}A$ in $\mathcal{SCR}$, show that every affine derived scheme is convergent.

The above is also true of schemes.

Proposition 23 ([GR, I.2.3.4.2])**.** *Every separated derived scheme is convergent (as an object of* $\mathrm{PSh}(\mathcal{SCR}^{op})$*).*