Derived Algebraic Geometry
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11 Higher topoi

11.1 The classical version

Suppose that X is a topological space, and $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is an open covering. Then giving a continuous morphism

$$f : X \to \mathbb{R}$$

is the same thing as giving a collection of continuous morphisms $f_i : U_\lambda \to \mathbb{R}$ that agree on the intersections. That is, such that for every μ, λ we have

$$f_\lambda|_{U_\lambda\cap U_\mu} = f_\mu|_{U_\lambda\cap U_\mu}.$$

Said another way the collection of f_λ is in the equaliser of the two canonical restriction maps

$$\prod_{\lambda\in\Lambda} \hom_{cont.}(U_\lambda, \mathbb{R}) \rightrightarrows \prod_{\lambda,\mu\in\Lambda} \hom_{cont.}(U_\lambda \cap U_\mu, \mathbb{R}).$$

As we mentioned last week, we can also do this with local homeomorphisms. If $p : Y \to X$ is a continuous morphism of topological spaces such that for every $y \in Y$ there is an open neighbourhood $y \in V \subseteq Y$ such that $p(V) \subseteq X$ is open and $V \to p(V)$ is a homeomorphism, then to give a continuous morphism $f : X \to \mathbb{R}$ is the same thing as giving a continuous morphism $g : Y \to \mathbb{R}$ that is constant on fibres. That is, such that

$$\pi(y_1) = \pi(y_2) \Rightarrow g(y_1) = g(y_2).$$

Said another way, g is in the equaliser of the two maps

$$\hom_{cont.}(Y, \mathbb{R}) \rightrightarrows \hom_{cont.}(Y \times_X Y, \mathbb{R})$$

induced by the two projections $pr_i : Y \times_X Y \to Y$; $(y_1, y_2) \mapsto y_i$ where $i = 1$ or 2.

We could also have done this discussion in other settings. Instead of $\mathbb{R}$, we could have used any topological space. We could also have assumed Y, X were differential manifolds, or complex analytic varieties with the appropriate notion of local homeomorphism, and used some other $F(-)$ instead of $\hom_{cont.}(-, \mathbb{R})$.

Grothendieck topologies are an abstraction and generalisation of these.

Definition 1 ([]). Suppose that C is a classical category. A *topology*[1] T on C is a collection of families $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ of morphisms, called *coverings* satisfying the following conditions.

1. Every singleton
$$\{Y \xrightarrow{\sim} X\}$$
containing an isomorphism is a covering.[2]
2. If $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering and $Y \to X$ is a morphism, then the pullbacks $Y \times_X U_\lambda$ exist[3] in C and
$$\{Y \times_X U_\lambda \to U_\lambda\}_{\lambda\in\Lambda}$$
is a covering.
3. If $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering and for each λ we have a covering $\{V_{\lambda\mu} \to U_\lambda\}_{\mu\in M_\lambda}$, the the family of compositions
$$\{V_{\lambda\mu} \to U_\lambda \to X\}_{\lambda\in\Lambda,\mu\in M_\lambda}$$
is a covering.

Exercise 2. Show that the following are topologies.

1. C is the category of topological spaces and T is the collection of families $\{p_\lambda : U_\lambda \to X\}_{\lambda\in\Lambda}$ such that each $U_\lambda \to X$ is an open immersion and $\sqcup_{\lambda\in\Lambda} p_\lambda(U_\lambda) \to X$ is surjective.
2. C is the category of topological spaces and T is the collection of families $\{p_\lambda : U_\lambda \to X\}_{\lambda\in\Lambda}$ such that each $U_\lambda \to X$ is a local homeomorphism and $\sqcup_{\lambda\in\Lambda} p_\lambda(U_\lambda) \to X$ is surjective.
3. C is the opposite of the category of rings, and T is the collection of families $\{p_\lambda : B_\lambda \leftarrow A\}_{\lambda\in\Lambda}$ such that each $B_\lambda \leftarrow A$ is a étale and $\prod_{\lambda\in\Lambda} \hom(B_\lambda, \Omega) \to \hom(A, \Omega)$ is surjective for every separably closed field Ω.

Definition 3. Let C be a category equipped with a topology T. A *presheaf* is a functor $F : C^{op} \to \mathcal{S}\mathrm{et}$. A presheaf is a *sheaf* if for every covering $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ we have
$$F(X) = \mathrm{eq}\left(\prod_{\lambda\in\Lambda} F(U_\lambda) \rightrightarrows \prod_{\lambda,\mu} F(U_\lambda \times_X U_\mu)\right).$$

Example 4.

1. For any topological space E, the presheaf $\hom_{cont.}(-, E)$ on the category of topological spaces with the canonical topology is a sheaf.

[1] We actually give the definition of a pretopology. But since pretopologies have a canonically associated topology which gives rise to the same category of sheaves, people often call pretopologies topologies.

[2] By the next axiom, only assuming that identities are coverings gives the same notion, since pullbacks are only defined up to isomorphism.

[3] One can easily avoid assuming that these pullbacks exists, but it is standard to assume their existence, and all our examples will satisfy this, so we do.

2. For any ring A, the presheaf $\hom(A, -)$ on the opposite of the category of rings is a sheaf for the étale topology.

Definition 5. A *topos* is a category of the form $\mathrm{Shv}(C)$ for some category C equipped with some topology T.

Remark 6. For any category equipped with a topology, the canonical inclusion $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ admits a left adjoint, called *sheafification*. There are a number of explicit descriptions of this adjoint. Here is one. Given a presheaf F, define $F^+(X) = \varinjlim \mathrm{eq}(\prod_{\lambda\in\Lambda} F(U_\lambda) \to \prod_{\lambda,\mu} F(U_\lambda \times_X U_\mu))$ where the colimit is over coverings. This is functorial in X, as well as F, so defines a functor $\mathrm{PSh}(C) \to \mathrm{PSh}(C)$. Then it turns out that applying this twice gives the left adjoint to inclusion. That is, for any presheaf F and sheaf G, the presheaf F^{++} is a sheaf, and we have $\hom(F, G) = \hom(F^{++}, G)$.

11.2 Higher topoi

The notion of topology on an ∞-category is the same as that on a classical category.

Definition 7 ([]). Suppose that C is an ∞-category. A *topology*[4] T on C is a collection of families $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ of morphisms, called *coverings* satisfying the following conditions.

1. Every singleton
$$\{Y \xrightarrow{\sim} X\}$$
containing an equivalence is a covering.[5]
2. If $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering and $Y \to X$ is a morphism, then the pullbacks $Y \times_X U_\lambda$ exist[6] in C and
$$\{Y \times_X U_\lambda \to U_\lambda\}_{\lambda\in\Lambda}$$
is a covering.
3. If $\{U_\lambda \to X\}_{\lambda\in\Lambda}$ is a covering and for each λ we have a covering $\{V_{\lambda\mu} \to U_\lambda\}_{\mu\in M_\lambda}$, the the family of compositions
$$\{V_{\lambda\mu} \to U_\lambda \to X\}_{\lambda\in\Lambda,\mu\in M_\lambda}$$
is a covering.

Example 8.

[4]We actually give the definition of a pretopology. But since pretopologies have a canonically associated topology which gives rise to the same category of sheaves, people often call pretopologies topologies.

[5]By the next axiom, only assuming that identities are coverings gives the same notion, since pullbacks are only defined up to isomorphism.

[6]One can easily avoid assuming that these pullbacks exists, but it is standard to assume their existence, and all our examples will satisfy this, so we do.

1. The coverings of the *étale topology* on the opposite category $\mathcal{SCR}^{op}$ are families of morphisms $\{A \to B_\lambda\}_{\lambda\in\Lambda}$ such that each $A \to B_\lambda$ is étale and for every separably closed field Ω the morphism $\prod_{\lambda\in\Lambda} \hom(B_\lambda, \Omega) \to \hom(A, \Omega)$ is surjective.
2. The coverings for the *fppf topology* on the opposite category $\mathcal{SCR}^{op}$ are families of morphisms $\{A \to B_\lambda\}_{\lambda\in\Lambda}$ such that each $A \to B_\lambda$ is flat, of finite presentation, and for every algebraically closed field Ω the morphism $\prod_{\lambda\in\Lambda} \hom(B_\lambda, \Omega) \to \hom(A, \Omega)$ is surjective.

For the Zariski topology we need a notion of open immersion.

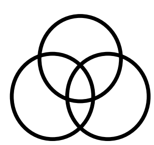
Definition 9. A morphism $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ of simplicial rings is an *open immersion* if it is flat, of finite presentation, and the diagonal $B \otimes_A B \to B$ is an equivalence.

Exercise 10.
1. For any simplicial ring A and any morphism $\mathbb{Z}[x] \to A$ the morphsim $A \to A \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, x^{-1}]$ is an open immersion.
2. Show that if $A \to B$ is an open immersion and $B \to C$ is an open immersion then $A \to C$ is an open immersion.
3. Show that if $A \to B$ is an open immersion and $A \to D$ any morphism then $D \to D \otimes_A B$ is an open immersion.

Example 11. The coverings for the *Zariski topology* on the opposite category $\mathcal{SCR}^{op}$ are families of morphisms $\{A \to B_\lambda\}_{\lambda\in\Lambda}$ such that each $A \to B_\lambda$ is an open immersion and for every field Ω the morphism $\prod_{\lambda\in\Lambda} \hom(B_\lambda, \Omega) \to \hom(A, \Omega)$ is surjective.

The notion of sheaf is more subtle. To see why, let's go back to a classical site.

Example 12. Consider a topological space X equipped with an open covering U_0, U_1, U_2 such that all of U_λ, $U_\lambda \cap U_\mu$ and $U_\lambda \cap U_\mu \cap U_\nu$ are contractible for distinct λ, μ, ν.



Consider the sheaf of complexes of abelian groups concentrated in degree zero $F : U \mapsto \hom(U, \mathbb{R})$ where $\mathbb{R}$ is given the discrete topology. Let's try and imitate Def.3 for the covering U_0, U_1, U_2. In the ∞-category $Cplx_{\mathbb{R}}$, the equaliser

$$\mathrm{eq}\left(\prod_{i=0,1,2} F(U_i) \rightrightarrows \prod_{i,j=0,1,2} F(U_i \cap U_j)\right)$$

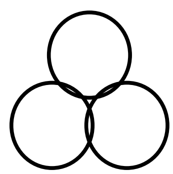
is the complex concentrated in homological degrees 0 and -1

$$\left[\prod_{i=0,1,2} \mathbb{R} \to \prod_{i,j=0,1,2} \mathbb{R}\right]$$

with morphism $(a_0, a_1, a_2) \mapsto \begin{pmatrix} 0 & a_1 - a_0 & a_2 - a_0 \\ a_0 - a_1 & 0 & a_2 - a_1 \\ a_0 - a_2 & a_1 - a_2 & 0 \end{pmatrix}$. The H_0 of this complex is $\{(a, a, a)\} \cong \mathbb{R}$, agreeing wth the H_0 of $F(X)$, but the H_{-1} is $\mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R} \neq 0$. The first factor $\mathbb{R}^3$ is the missing diagonal, the second factor $\mathbb{R}$ is the symmetry between the upper left and lower right triangles. But there is a remaining factor of $\mathbb{R}$ corresponding to the fact that the diagram

$$\begin{array}{ccccc} & & U_{01} & & \\ & \swarrow & & \searrow & \\ & U_0 & & U_1 & \\ \nearrow & & & & \nwarrow \\ U_{02} & \longrightarrow & U_2 & \longleftarrow & U_{12} \end{array}$$

is essentially an unfilled circle. To remove the unwanted factor we need to also take into account the triple intersection $U_0 \cap U_1 \cap U_2$. Note that if this triple intersection was empty, we would want this extra factor in H_{-1}, since our X would be homotopic to a circle.



A related example is the de Rham complex on $\mathbb{R}^2$ or on $\mathbb{R}^2 \setminus \{0\}$. By Poincaré's Lemma on contractible opens $U \subseteq \mathbb{R}^2$ the de Rham complex is quasi-isomorphic to $\mathbb{R}$ concentrated in degree zero $\mathbb{R} \cong \Omega^\bullet(U)$. On the other hand, $\Omega^\bullet(\mathbb{R}^2 \setminus \{0\})$ is quasi-isomorphic to $[\mathbb{R} \xrightarrow{0} \mathbb{R}]$. That is, this is essentially the same as the example above.

Definition 13. Let C be an ∞-category equipped with a topology T and E an ∞-category admitting limits (the canonical choice is $E = \mathcal{S}$ the category of spaces). A *presheaf* with values in E is a functor $F : C^{op} \to E$. A presheaf is a *sheaf* if for every covering $\{U_\lambda \to X\}_{\lambda \in \Lambda}$ we have

$$F(X) = \varprojlim_{n \in \Delta} \prod_{(\lambda_0, \ldots, \lambda_n) \in \Lambda^{n+1}} F(U_{\lambda_0} \times_X \cdots \times_X U_{\lambda_n}).$$

The category of sheaves is the full subcategory

$$\mathrm{Shv}(C, E) \subseteq \mathrm{PSh}(C, E)$$

consisting of those presheaves which are sheaves.

Remark 14. It follows directly from the definition that if we have two topologies T_1, T_2 on an ∞-category, and all T_1 coverings are T_2 coverings, then any T_2-sheaf is a T_1-sheaf. In particular, any fppf sheaf is an étale sheaf, and any étale sheaf is a Zariski sheaf.

Example 15.

1. For any simplicial ring A, the presheaf $\mathrm{Map}(A, -) : \mathcal{SCR} \to \mathcal{S}$ is an fppf sheaf (and therefore also an étale sheaf and a Zariski sheaf).
2. The presheaf which sends a simplicial ring A to $\mathcal{M}_A$ is an fppf sheaf, [DAG, Exam.4.2.5].
3. Fixing a simplicial ring $A \in \mathcal{R}\mathrm{ing}_\Delta$, the presheaf $\mathcal{SCR}_{A/} \to \mathcal{M}_A^{\mathrm{cn}}$; $B \mapsto L_{B/A}|_A$ is an étale sheaf.
4. For a simplicial ring R let $\mathrm{Pic}(R) \subseteq \mathcal{M}_R^{\mathrm{cn}}$ denote the non-full subcategory whose objects are locally free of rank 1, and morphisms are weak equivalences (and containing all higher morphisms). Given $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$, let $\widetilde{Pic}_{B/A}$ denote the presheaf $\mathcal{SCR}_{A/} \to \mathcal{S}$; $A' \mapsto \mathrm{Pic}(A' \otimes_A B)$. Then $\widetilde{Pic}_{B/A}$ is an étale sheaf [DAG, Thm.8.2.2].
5. Let $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. Given an A-algebra $A \to A'$, let $\mathrm{Hilb}_{B/A}(A')$ denote the ∞-groupoid of $B' = A' \otimes_A B$-algebras C almost of finite presentation such that $\pi_0 B' \to \pi_0 C$ is surjective, $A' \to C$ is flat, and $\pi_0 A' \to \pi_0 C$ is finite in the classical sense. Then $\mathrm{Hilb}_{B/A} : \mathcal{SCR}_{A/} \to \mathcal{S}$ is an étale sheaf [DAG, Thm.8.3.3].

As in the classical case, the inclusion of sheaves into presheaves admits a left adjoint.

Proposition 16 ([HTT, Prop.6.2.2.7])**.** *Suppose C is an ∞-category equipped with a topology. Then the canonical inclusion* $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ *admits a left adjoint.*

Remark 17. The sheafification functor exists for abstract reasons but the adjoint functor theorem, but one can also give a more concrete description of it similar to Rem.6 above. One added complication is that instead of applying $(-)^+$ twice, one must apply it κ-many times for some ordinal κ which depends on the site, cf. the proof of [HTT, Prop.6.2.2.7].

Corollary 18. *Let C be an ∞-category equipped with a topology. The category* $\mathrm{Shv}(C)$ *admits all small colimits and small limits. The inclusion* $\mathrm{Shv}(C) \subseteq \mathrm{PSh}(C)$ *preserves limits. That is, if $F_- : K \to \mathrm{Shv}(C)$; is a diagram of sheaves, then for any $X \in C$ the canonical morphism $(\varprojlim F_\lambda)(X) \to \varprojlim(F_\lambda(X))$ is an equivalence.*

11.3 Morphisms of topoi

The following is essentially identical in the classical case and higher setting so we go directly to the higher setting.

We begin with a proposition which should be in the lecture on limits.

Proposition 19 ([HTT, Def.4.3.2.2, Prop.4.3.2.17, Rem.5.5.3.10])**.** *Let $\phi : C \to C'$ be a functor between small ∞-categories. Then the functor $\phi_* : \mathrm{PSh}(C') \to \mathrm{PSh}(C)$ induced by composition admits both a left ϕ^* and right $\phi^!$ adjoint. These can be concretely "calculated" by the formulas*

$$\phi^* F(X) = \varinjlim_{Y \in C_{X/}} F(Y)$$

$$\phi^! F(X) = \varprojlim_{Y \in C_{/X}} F(Y)$$

Remark 20. Sometimes the three functors $\phi^*, \phi_*, \phi^!$ are denoted $\phi_!, \phi^*, \phi_*$. This basically depends on whether one is thinking in terms of categories or topoi, since lower indicies indicate covariance and upper indicies indicate contravariance.

Remark 21. Since the inclusion $\iota : \mathrm{Shv}(C) \to \mathrm{PSh}(C)$ is fully faithful, the counit $a \circ \iota \to \mathrm{id}$ of the adjunction (a, ι) is an equivalence. It follows that for any functor $\phi : C \to C'$, the functor $\mathrm{Shv}(C) \to \mathrm{PSh}(C) \to \mathrm{PSh}(C') \to \mathrm{Shv}(C')$ fits into a commutative square

$$\begin{array}{ccc} \mathrm{PSh}(C) & \longrightarrow & \mathrm{PSh}(C') \\ \downarrow & & \downarrow \\ \mathrm{Shv}(C) & \longrightarrow & \mathrm{Shv}(C'). \end{array}$$

Proposition 22. *Suppose C, C' are ∞-categories equipped with topologies T, T', and $\phi : C \to C'$ is a functor. If ϕ sends T-coverings to T'-coverings, then restriction* $\mathrm{PSh}(C') \to PSh(C)$ *sends sheaves to sheaves, and the induced functor is right adjoint to the canonical functor* $\mathrm{Shv}(C) \to \mathrm{Shv}(C')$. *So we obtain an adjunction*

$$\mathrm{Shv}(C) \rightleftarrows \mathrm{Shv}(C').$$

We will call such a functor $\phi : C \to C'$ a morphism of sites*.*

Example 23.

1. Suppose C is an ∞-category equipped with a topology T and $X \in C$ is an object. Say a family $\{U_\lambda \to Y\}$ in $C_{/X}$ is a covering family if its image under the forgetful functor $C_{/X} \to C$ is a covering. Then $C_{/X} \to C$ is a morphism of sites.
2. If moreover $Y \to X$ is an morphism in C, then composition induces a functor $C_{/Y} \to C_{/X}$ which is also a morphism of sites.
3. Suppose moreover that C admits pullbacks. Then pullback induces a morphism of sites $C_{/X} \to C_{/Y}$.
4. Suppose that $C_0 \subseteq C$ is a full subcategory such that every covering family of an object in C_0 is contained in C_0. Equip C_0 with the topology whose coverings are those families which are coverings in C. Then $C_0 \to C$ is a morphism of sites.