Derived Algebraic Geometry
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10 Higher algebra III: Smooth morphisms

We begin with something we didn't have time for at the end of the last lecture.

10.1 Square Zero Extensions

Example 1. Let A be a classical ring, and suppose that $B' \to B$ is a square zero extension of classical A-algebras with kernel

$$I = \ker(B' \to B).$$

This defines a distinguished triangle

$$I \to B' \to B \xrightarrow{\delta} I[1], \qquad \in D(A).$$

in the category of A-modules. The morphism δ can be represented concretely by the weak equivalence[1] $[I{\to}B'] \xrightarrow{\sim} B$. The the cofibre sequence is as follows.

$$\begin{bmatrix} 0 \\ \downarrow \\ I \end{bmatrix} \to \begin{bmatrix} 0 \\ \downarrow \\ B' \end{bmatrix} \to \begin{bmatrix} I \\ \downarrow \\ B' \end{bmatrix} \xrightarrow{\delta} \begin{bmatrix} I \\ \downarrow \\ 0 \end{bmatrix}, \qquad \in D(A).$$

Since $D(A)$ is stable, one sees that as an A-module, B' can be reconstructed as the fibre of δ in $D(A)$,

$$B' \cong \mathrm{fib}(B \xrightarrow{\delta} I[1])$$

and this sets up a well-known bijection

$$\hom_{D(A)}(B, I[1]) \cong \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{extensions of } B \text{ by } I \end{array} \right\}.$$

We can also recover the ring structure. The following also works in the category of simplicial rings, but it is easier to see whats going on in cdgas (=commutative differential graded algebras, [HA, Def.7.1.4.1, Def.7.1.4.8]).

Notice that since $I^2 = 0$, the chain complex $[I \xrightarrow{inc.} B']$, whose differential is inclusion, has a structure of commutative graded ring, where B' is in degree zero,

[1] Here, by $[A_1 \to A_0]$ we mean $[\cdots \to 0 \to 0 \to A_1 \to A_0]$.

and I is in degree one. Similarly, the complex of A-modules $[I \xrightarrow{0} B] = [I \to 0] \oplus [0 \to B]$ also has a commutative graded ring structure. Since the canonical morphism $[I \xrightarrow{inc.} B'] \to [I \xrightarrow{0} B]$ is surjective in each degree, it is a fibration in the category of cdgas (cf.[HA, Prop.7.1.4.10]), so the cartesian square

$$\begin{tikzcd}
B' = & \begin{bmatrix} 0 \\ \downarrow \\ B' \end{bmatrix} \arrow[rr] \arrow[dr] & & \begin{bmatrix} 0 \\ \downarrow \\ B \end{bmatrix} \arrow[dr] & = B & \\
& B \cong & \begin{bmatrix} I \\ inc. \downarrow \\ B' \end{bmatrix} \arrow[rr] & & \begin{bmatrix} I \\ \downarrow 0 \\ B \end{bmatrix} & = B \oplus I[1]
\end{tikzcd}$$

in the model category of cdgas is sent to a cartesian in the ∞-category of cdgas.[2] That is, we recover B', equipped with its ring structure, as the fibre product

$$B' = B \times_{B \oplus I[1]} B, \qquad \in \mathcal{SCR}_{A/}$$

in the ∞-category $N(CDGA^{\mathrm{cf}}_{A/})$, which by [HA, Prop.7.1.4.11] is equivalent to $\mathcal{SCR}_{A/}$ (at least when $\mathbb{Q} \subseteq A$; for more general classical rings A one can do a similar calculation using simplicial rings). As in the case of A-modules, this sets up a bijection

$$\hom_{\mathcal{SCR}_{A//B}}(B, B \oplus I[1]) \cong \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{square zero extensions} \\ \text{of } A\text{-algebras of } B \text{ by } I \end{array} \right\} \tag{1}$$
$$\left(B \xrightarrow{s} B \oplus I[1] \right) \mapsto B \times_{B \oplus I[1]} B$$

where hom means $\pi_0 \operatorname{Map}$, cf.Prop.6 below.

Definition 2 ([DAG, Def.3.3.1])**.** Let $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings, and $M \in B$-mod. A *small extension of B by M* is a pullback in $\mathcal{SCR}_{A//B}$ of the form

$$\begin{tikzcd}
\tilde{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "0"] \\
B \arrow[r, "s"'] & B \oplus M[1]
\end{tikzcd}$$

where $0 : B \to B \oplus M[1]$ is the morphism corresponding to $0 \to M[1]$ in $\mathcal{M}_B$.

[2] Recall that in general, if $X \to Y \leftarrow Z$ are fibrations with Y fibrant, then the diagram is injectively fibrant and therefore $X \times_Y Z$ models the pullback in the associated ∞-category. Moreover, one can show that in fact, it suffices that X, Y, Z are fibrant and one of the morphisms is a fibration.

Exercise 3. Continuing with the notation from Def.2, let $\tilde{B} = B \times_{B \oplus M[1]} B$. Note that the forgetful functor $\mathcal{SCR}_{A//B} \to \mathcal{M}_A^{\mathrm{cn}}$ commutes with fibre products[3] Applying the octahedral axiom to the sequence $B \oplus B \xrightarrow{s+0} B \oplus M[1] \to B$ of A-modules, show that there is a distinguished triangle of the form $\tilde{B} \to B \to M[1] \to \tilde{B}[1]$ and therefore a long exact sequence

$$\dots \pi_{n+1}(B) \xrightarrow{\sigma} \pi_n(M) \to \pi_n(\tilde{B}) \to \pi_n(B) \xrightarrow{\sigma} \pi_{n-1}(M) \to \dots$$

where the morphisms σ are induced by $s : B \to B \oplus M[1]$.

Example 4. Lets go back to the classical case in Example 1. By adjunction and the definition of $L_{B/A}$, the equivalence of Eq.1 becomes

$$\hom_{D(B)}(L_{B/A}, I[1]) \cong \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{square zero extensions} \\ \text{of } B \text{ by } I \end{array} \right\} \tag{2}$$

In general, categories of modules are easier than categories of rings because they are linear. So one prefers a morphism $L_{B/A} \to I[1]$ of modules to a morphism $B \to B \oplus I[1]$ of rings. In this form, the $\to$ direction of Eq.(2) can be made explicit as follows.

Choose a surjection $P \twoheadrightarrow B$ such that $P \cong A[x_\lambda : \lambda \in \Lambda]$ for some Λ, and let $J = \ker(P \to B)$. It turns out that the chain complex of B-modules

$$NL_{B/A} = [J/J^2 \to \Omega_{P/A} \otimes_P B] \tag{3}$$

induced by the differential $J \subseteq P \xrightarrow{d} \Omega_{P/A}$ is a representative of the truncation $\tau_{\leq 1} L_{B/A}$ (the complex Eq.(3) is called the *naïve cotangent complex* on the Stacks Project). Since $I[1]$ is concentrated in degree ≤ 1 we have $\mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(L_{B/A}, I[1]) \cong \mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(NL_{B/A}, I[1])$. Possibly adjoining more variables to P, we can assume that our map $s \in \mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(NL_{B/A}, I[1])$ is represented by a morphism of chain complexes

$$\begin{bmatrix} J/J^2 \\ \downarrow \\ \Omega_{P/A} \otimes_P B \end{bmatrix} \xrightarrow{s} \begin{bmatrix} I \\ \downarrow \\ 0 \end{bmatrix}, \qquad \in D(B)$$

such that $J/J^2 \to I$ is surjective. We claim that the corresponding morphism of cdgas is

$$\begin{bmatrix} & J/J^2 \\ inc. & \downarrow \\ & P/J^2 \end{bmatrix} \to \begin{bmatrix} I & \\ \downarrow & 0 \\ B & \end{bmatrix}, \qquad \in \mathcal{SCR}_{A//B}$$

where $P/J^2 \to B$ is induced by the surjection $P \twoheadrightarrow B$ and $J/J^2 \to I$ is the surjection from s. This morphism is term-wise surjective, hence a fibration, so one easily calculates that

$$B \times_{B \oplus I[1]} B \cong P/K,$$

concentrated in degree zero, where $K = \ker(J \to J/J^2 \to I)$. Note $J^2 \subseteq K \subseteq J$, so this is indeed a square zero extension of $B = P/J$.

[3] This can be checked on the underlying model categories for example.

Definition 5. A simplicial ring resp. connective module is called *discrete* if $\pi_n = 0$ for $n \neq 0$. If B is a simplicial ring, the full subcategory of discrete B-modules is written $\mathcal{M}_B^\heartsuit \subseteq \mathcal{M}_B^{\mathrm{cn}}$. Similarly, the full subcategory of B-modules such that $\pi_n = 0$ for $n \neq k$ is written $\mathcal{M}_B^\heartsuit[k] \subseteq \mathcal{M}_B^{\mathrm{cn}}$.

Proposition 6 ([DAG, Prop.3.3.5])**.** *Let $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings such that B is discrete, and consider the full subcategory $(\mathcal{M}_B^\heartsuit[1])_{L_{B/A}/} \subseteq (\mathcal{M}_B^{\mathrm{cn}})_{L_{B/A}/}$ of the undercategory whose target lies in $\mathcal{M}_B^\heartsuit[1]$.*

The functor

$$\begin{aligned}(\mathcal{M}_B^\heartsuit[1])_{L_{B/A}/} &\to \mathcal{SCR}_{A//B} \\ (L_{B/A} \xrightarrow{s} M) &\mapsto B \times_{B \oplus M} B\end{aligned}$$

is fully faithful, and its essential image consists of those discrete $\tilde{B} \to B$ in $\mathcal{SCR}_{A//B}$ such that $\pi_0 \tilde{B}$ is a square zero extension of $\pi_0 B$ by I.

Proposition 7 ([DAG, Prop.3.3.5])**.** *Choose $k > 0$, let $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings such that $\pi_n B = 0$ for $n > k$ and consider the full subcategory $(\mathcal{M}_B^\heartsuit[k{+}1])_{L_{B/A}/} \subseteq (\mathcal{M}_B^{\mathrm{cn}})_{L_{B/A}/}$ of the undercategory whose target lies in $\mathcal{M}_B^\heartsuit[k{+}1]$. The functor*

$$\begin{aligned}(\mathcal{M}_B^\heartsuit[k{+}1])_{L_{B/A}/} &\to \mathcal{SCR}_{A//B} \\ (L_{B/A} \xrightarrow{s} M) &\mapsto B \times_{B \oplus M} B\end{aligned}$$

is fully faithful, and its essential image consists of those $\tilde{B} \to B$ in $\mathcal{SCR}_{A//B}$ such that

1. *$\pi_i \tilde{B} \xrightarrow{\sim} \pi_i B$ for $i \neq k$, and,*
2. *there is a short exact sequence of $\pi_0 B$-modules*

$$0 \to \pi_k M \to \pi_k \tilde{B} \to \pi_k B \to 0$$

Remark 8. Suppose $A \to B$ is a morphism of simplicial rings. We can apply the above proposition to $\tau_{\leq k-1} B$ and $M = (\pi_k B)[k]$. In this case, the proposition tells us that we can build $\tau_{\leq k} B$ out of the triple

$$(\tau_{\leq k-1} B, \quad \pi_k B, \quad L_{B/A} \to (\pi_k B)[k]).$$

In $\mathcal{SCR}_{A/}$ we have $B = \varprojlim_{n \in \mathbb{N}} \tau_{\leq n} B$ so in fact, B is equivalent to the data of the discrete A-algebra $\pi_0 B$, together with the sequence

$$\underset{\in \tau_{\leq 0} B\text{-mod}}{L_{\tau_{\leq 0} B/A} \to (\pi_1 B)[1]}, \quad \underset{\in \tau_{\leq 1} B\text{-mod}}{L_{\tau_{\leq 1} B/A} \to (\pi_2 B)[2]}, \quad \underset{\in \tau_{\leq 2} B\text{-mod}}{L_{\tau_{\leq 2} B/A} \to (\pi_3 B)[3]}, \quad \ldots$$

That is, the A-algebra B is determined by the discrete A-algebra $\pi_0 B$ and purely "linear" data, where "linear" means contained in some category of modules.

10.2 Flat morphisms and open immersions

Here is the classical definition of flatness.

Definition 9. Let A be a classical ring. A classical A-module M is *flat* if the functor $M \otimes_A - : A\text{-mod} \to A\text{-mod}$ sends monomorphisms to monomorphisms. A classical A-algebra B is flat if it is flat as an A-module.

Remark 10. Flatness is a slightly mysterious property. It does not have a very clear geometric interpretation, although there are some results in this direction. For example, if M is a finitely generated flat A-module, then the function $\mathfrak{p} \mapsto \dim_{k(\mathfrak{p})} M \otimes k(\mathfrak{p})$ is locally constant on $\mathrm{Spec}(A)$. That is, it is constant on each connected component of $\mathrm{Spec}(A)$. See also [Stacks Project, 0D4H, 0D4J].

Remark 11. As a left adjoint, the functor $M \otimes_A -$ automatically preserves colimits, so asking for it to preserve monomorphisms is the same as asking for it to preserve short exact sequences. In this case, $M \otimes_A - : D(A) \to D(A)$ preserves $\mathcal{M}_A^{\heartsuit}$. In fact, one can show fairly easily that $M \otimes_A - : D(A) \to D(A)$ preserves $\mathcal{M}_A^{\heartsuit}$ if and only if M is flat in the above sense, cf.Thm.17(2) below.

Indeed, if it doesn't then there is some module N with projective resolution $\dots \to P_1 \to P_0 \to N$ such that $H_n(M \otimes_A P_\bullet) \neq 0$ for some $n \neq 0$. That is, $\frac{ker(M\otimes_A P_n \to M\otimes_A P_{n-1})}{im(M\otimes_A P_{n+1} \to M \otimes_A P_n)}$ is non-zero. But if $M \otimes_A -$ preserves short exact sequences, then this is $M \otimes_A \frac{ker(P_n \to P_{n-1})}{im(P_{n+1} \to P_n)}$ which is zero by definition of projective resolution.

Exercise 12. Suppose that A is a classical ring and $A \to B$ is surjective, flat, and the kernel is finitely generated. Show that $A \cong B \times C$ for some A-algebra C. Hint.[4] Hint.[5] Hint.[6]

Examples of flat modules are plentiful.

Exercise 13. Show the following.

1. The ring A considered as an A-module over itself is flat.
2. Any finite direct sum of flat modules is flat.
3. Any filtered colimit of flat modules is flat.
4. In particular, for any $f \in A$, the localisation $M[f^{-1}] = \varinjlim(M \xrightarrow{f} M \xrightarrow{f} \dots)$ is flat, and any polynomial ring $A[x_\lambda : \lambda \in \Lambda]$ is flat.

In fact, the above list of examples is exhaustive.

Theorem 14 (Lazard's theorem, cf.[Stacks Project, 058G])**.** *An A-module M is flat if and only if it is a filtered colimit of finitely generated free A-modules.*

[4] Show that $I = I^2$.
[5] Use Nakayama's Lemma to show that $I = (f)$ for some f such that $f^2 = f$.
[6] Notice that $e = 1 - f$ is also an idempotent, and $1 = e + f$ and $ef = 0$ in A.

Sketch of proof. The "if" direction is straight-forward since sums and filtered colimits commute with tensor products and preserve monomorphisms. For the "only if" direction it suffices to show that if M is flat, then the category $A\text{-mod}^{f.g.free}/M$ of finitely generated free modules over M is filtered, and has colimit M. The latter follows from the former since every element $m \in M$ is in the image of some $A^{\oplus n} \to M$. The former is the hardest part of the proof, but is in some sense just the equational criterion for flatness, [Stacks Project, 00HK]. $\square$

In these notes we take Lazard's Theorem as the definition in the higher setting, although there is also a higher version of Def.9, cf.Theorem 17.

Definition 15 ([DAG, pg.21])**.** Let $A \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring. A module M is *flat* if it is in the smallest subcategory of $\mathcal{M}_A^{\mathrm{cn}}$ which:

1. contains A,
2. is closed under finite sums,
3. is closed under filtered colimits.

An A-algebra $A \to B$ is *flat* if its image under the forgetful functor $\mathcal{SCR}_{A/} \to \mathcal{M}_A^{\mathrm{cn}}$ is flat.

Exercise 16. Let A be a simplicial ring.

1. Show that the $\mathbb{Z}[x]$-algebra $\mathbb{Z}[x, x^{-1}]$ is flat in the sense of Def.15.
2. Show that the $\mathbb{Z}$-algebra $\mathbb{Z}[x_1, \ldots, x_n]$ is flat in the sense of Def.15.
3. Let $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings, and consider the adjunction $- \otimes_A B : \mathcal{M}_A^{\mathrm{cn}} \rightleftarrows \mathcal{M}_B^{\mathrm{cn}} : (-)|_A$ Show that if M is a flat A-module in the sense of Def.15, then $M \otimes_A B$ is a flat B-module in the sense of Def.15.

Theorem 17 (Derived Lazard Theorem [DAG, Thm.2.5.2])**.** *Let $A \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring and $M \in \mathcal{M}_A^{\mathrm{cn}}$ a connective A-module. The following are equivalent.*

1. *M is flat in the sense of Def.15.*
2. *$N \in \mathcal{M}_A^{\heartsuit} \Rightarrow M \otimes_A N \in \mathcal{M}_A^{\heartsuit}$.*
3. *The $\pi_0 A$-module $\pi_0 M$ is flat in the classical sense, $\pi_i M \cong \pi_i A \overset{cl}{\otimes}_{\pi_0 A} \pi_0 M$ for all i, where $\overset{cl}{\otimes}$ means the tensor product of classical modules.*
4. *The $\pi_0 A$-module $\pi_0 A \otimes_A M$ is discrete, and flat in the classical sense.*

Exercise 18. Let A be a classical ring, and suppose that M is a flat classical $A[t]$-module. Show that

$$t^n M / t^{n+1} M \cong ((t^n)/(t^{n+1})) \otimes_A (M/tM),$$

where we consider A as the $A[t]$-algebra $A[t]/(t)$. Hint.[7] Hint.[8]

Exercise 19. Let $A \to B \to C$ and $A \to D \in \mathcal{R}\mathrm{ing}_\Delta$ be morphisms of simplicial rings.

[7] First show that $(t^n) \otimes_{A[t]} M \cong t^n M$.

[8] Notice that in general, if M is an R-module and N is an R/I-module then $N \otimes_{R/I} (M/IM) \cong N \otimes_{R/I} R/I \otimes_R M \cong N \otimes_R M$.

1. Show that if D is a flat A-algebra, then $D \otimes_A B$ is a flat B-algebra.
2. Show that if $A \to B$ is a flat A-algebra and $B \to C$ is a flat B-algebra then $A \to C$ is a flat A-algebra.
3. Suppose that $f : \mathbb{Z}[x] \to A \in \mathcal{R}\mathrm{ing}_\Delta$ is a morphism of simplicial rings where $\mathbb{Z}[x]$ means the "constant" simplicial ring. Form the pushout in $\mathcal{SCR}$

$$\mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}[x]} A =: A[f^{-1}].$$

   Show that this is a flat A-algebra.

10.3 Smooth morphisms and étale morphisms

Smooth morphisms have a more geometric interpretation than flat morphisms. The differential geometric analogue is *submersion* and the idea is that locally, a submersion $Y \to X$ looks like a projection $U \times \mathbb{R}^m \to U$. That is, for every point $y \in Y$ there exists an open neighbourhood $y \in V \subseteq Y$ and a commutative diagram

$$\begin{tikzcd} Y \arrow[d] & V \arrow[l] & \cong & f(V) \times \mathbb{R}^m \arrow[d, "\pi"] \\ X & & & f(V) \arrow[lll, "\iota"'] \end{tikzcd}$$

where π is the canonical projection and ι is an open immersion. This is also true in algebraic geometry but "locally" needs some adjustment since polynomials do not provide enough charts.

The key observation is that (in complex geometry, topology, differential geometry, ...) local homeomorphisms (i.e., submersions as above such that $m = 0$) work just as well as open immersions for many purposes.

A robust algebro-geometric notion of local homeomorphism is the following.

Definition 20. A morphism of rings $A \to B$ is *étale* if it is of finite presentation,[9] flat, and the diagonal $B \otimes_A B \to B$ is flat.

Remark 21. The term étale is a French word which means spread out, in the sense that you would spread out dough with a rolling pin, or nutella on bread with a knife. More poetically, it can be used to describe the sea. *** PICTURES ***

Remark 22. This definition looks very algebraic, but it is in fact very geometric. First note that the diagonal $B \otimes_A B \to B$ is always surjective (it has two canonical sections). So $\Delta : \mathrm{Spec}(B) \to \mathrm{Spec}(B \otimes_A B) = \mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$ is always a closed immersion. Since the image of any flat morphism is open (cf.Ex.12 above) asking that the diagonal is flat is the same as asking that $Y \times_X Y \cong \text{something} \sqcup Y$ (here $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$), at least when $A \to B$ is of finite presentation. Since $\mathrm{Spec}(B \otimes_A B)$ parameterises pairs of points $\mathrm{Spec}(K) \rightrightarrows \mathrm{Spec}(B)$ in the same

[9] Some authors only require locally of finite presentation.

fibre, asking that Δ is a connected component says geometrically that we cannot find a "path" from a pair of distinct points to a pair of identical points. That is, there are no *branch* points. Since $A \to B$ is also flat, there are also no sudden "jumps" in the number of points in the fibres.

*** PICTURE ***

Exercise 23.

1. Show that $A \to A[x_1, \ldots, x_n]$ is étale if and only if $n = 0$.
2. Suppose K is a field, and let $f(x) \in K[x]$ be a monic. Show that $K \to K[x]/\langle f \rangle =: A$ is étale if and only if $f(x)$ has no repeated roots in an algebraic closure $\overline{K}/K$. That is, $K \to A$ is étale if and only if A is a product of finite separable extensions of K. Hint.[10]
3. Show that $\mathbb{C}[x] \to \mathbb{C}[x]$; $x \mapsto x^2$ is not étale. Show that $\mathbb{C}[x, x^{-1}] \to \mathbb{C}[x, x^{-1}]$; $x \mapsto x^2$ is étale. Hint.[11]

Now that we have a good notion of "local" we can give a good definition of smooth.

Definition 24. A morphism $A \to B$ of classical rings is *smooth* if étale locally it is of the form $A \to A[x_1, \ldots, x_n]$ for some n in the following sense: for every prime $\mathfrak{q} \in \mathrm{Spec}(B)$ there exists $a \in A$, $b \in B$ and a factorisation

$$\begin{tikzcd} B \arrow[r] & B[b^{-1}] & A[a^{-1}][x_1, \ldots, x_n] \arrow[l, "\phi"'] \\ A \arrow[u] \arrow[rr] & & A[a^{-1}] \arrow[u] \end{tikzcd}$$

such that ϕ is étale.

Just as in the case of differentiable manifolds, we can recognise étale morphisms (analogue of local diffeomorphisms) and smooth morphisms (analogue of submersions) by looking at the tangent space.

Proposition 25. *Let $f : A \to B$ be a morphism of classical rings. Then f is étale (resp. smooth) if and only if f is flat, of finite presentation, and $\Omega_{B/A} = 0$ (resp. $\Omega_{B/A}$ is a locally free B-module).*

Proposition 25 is often taken as the definition of étale (resp. smooth) in the classical case, and [DAG] uses it as the definition in the higher setting.

Definition 26 ([DAG, Def.3.4.1, Def.3.4.3, Def.3.4.7, Prop.3.4.9])**.** Let $f : A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. We say that f is:

1. *étale* if f is almost of finite presentation and $L_{B/A} = 0$.

[10] Note that a K-algebra B is étale if and only if $\overline{K} \to \overline{K} \otimes_K B$ is étale.

[11] Consider the connected components of the tensor square.

2. *smooth* if f is almost of finite presentation and $L_{B/A}$ is finite, locally free. That is, there exist $b_1, \ldots, b_n \in \pi_0 B$ generating the unit ideal such that $L_{B[b_i^{-1}]/A} \cong B[b_i^{-1}]^{\oplus n}$ for each i.

Exercise 27. For any simplicial ring A, show that the simplicial ring

$$A[x_1, \ldots, x_n] := A \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \ldots, x_n]$$

is smooth for any n.

Proposition 28 ([DAG, Prop.3.4.9])**.** *Let $f : A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. The following conditions are equivalent.*

1. *$L_{B/A}$ is perfect and discrete, and B is locally of finite presentation over A.*
2. *$L_{B/A}$ is perfect and discrete, and B is almost of finite presentation over A.*
3. *$L_{B/A}$ is perfect and discrete, and $\pi_0 B$ is a finitely presented $\pi_0 A$-algebra (in the classical sense).*
4. *f is flat, and $\pi_0 A \to \pi_0 B$ is smooth (in the classical sense).*

Corollary 29 ([DAG, Cor.3.4.10])**.** *Let $f : A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. The following conditions are equivalent.*

1. *$L_{B/A} = 0$ and B is of finite presentation over A.*
2. *$L_{B/A} = 0$ and B is locally of finite presentation over A.*
3. *$L_{B/A} = 0$ and B is almost of finite presentation over A.*
4. *$L_{B/A} = 0$ and $\pi_0 B$ is a finitely presented $\pi_0 A$-algebra (in the classical sense).*
5. *f is flat, and $\pi_0 A \to \pi_0 B$ is étale (in the classical sense).*