Derived Algebraic Geometry
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4 Higher algebra II: The cotangent complex

4.1 Kähler differential

References:

[Quillen, On the (co)homology of commutative rings, 1970].

[Illusie, Complexe Cotangent et Deformations I, 1971].

[Lurie, Higher Algebra].

[Lurie, Spectral Algebraic Geometry].

[Stacks Project]

We begin with a review of the classical theory. Historically, this is motivated by the following kind of questions.

Question 1.

1. Given a point $x : \mathrm{Spec}(\mathbb{C}) \to X$ in a complex variety, when can we extend it to a smooth curve $C \to X$ through x? What about the analytic germ of a smooth curve $\mathrm{Spec}(\mathbb{C}[[t]]) \to X$? Or infinitesimal curves
$$\mathrm{Spec}(\mathbb{C}[t]/(t^n)) \to X?$$
How unique are such extensions?
2. Given an affine complex variety $X = \mathrm{Spec}(A)$, a closed subvariety $Z = \mathrm{Spec}(A/I)$ and a flat quasi-coherent sheaf $\mathcal{F}$ on Z, when can we find a flat quasi-coherent sheaf $\mathcal{F}'$ on X extending $\mathcal{F}$? What about on a formal neighbourhood $X^\wedge_Z := \mathrm{Spec}(\varprojlim_n A/I^n)$ of Z? Or an an infinitesimal neighbourhood
$$i_n : Z_n = \mathrm{Spec}(A/I^n) \subseteq X; \qquad \mathcal{F}_n \text{ on } Z_n \text{ s.t. } \mathcal{F}_n|_Z \cong \mathcal{F}?$$
How unique are such extensions?

The definition of Kähler differentials appearing in most textbooks is the following.

Definition 2. Let $A \to B$ be a morphism of (classical) rings and $M \in B$-mod. An *A-derivation of B into M* is a morphism of A-modules $d : B \to M$ satisfying the Leibniz rule:

$$d(xy) = xd(y) + yd(x) \text{ for all } x, y \in B.$$

The set of all A-derivations from B into M is denoted $\mathrm{Der}_A(B, M)$.

Exercise 3. Show that $\mathrm{Der}_A(B, M)$ is equipped with a natural B-module structure.

Definition 4. The B-module of Kähler differentials is the universal A-derivation. That is, Kähler differentials are an A-derivation $d : B \to \Omega_{B/A}$ such that for any B-module composition induces an isomorphism

$$\hom_{B\text{-mod}}(\Omega_{B/A}, M) \cong \mathrm{Der}_A(B, M); \qquad \phi \mapsto \phi \circ d \tag{1}$$

Exercise 5. Using universal properties, show that

$$\Omega_{A[x_1,\ldots,x_n]/A} \cong \bigoplus_{i=1}^{n} A[x_1, \ldots, x_n]dx_i.$$

(Here the symbols dx_i represent the image of x_i under d).

Using the axioms and induction show that for any $\sum_{i=0}^{m} a_n x^n \in A[x]$ we have $d(\sum_{n=0}^{m} a_n x^n) = \sum_{n=0}^{m-1} n a_n x^{n-1} dx$.

Remark 6. The above exercise shows that Kähler differentials $\Omega_{A[x_1,\ldots,x_n]/A}$ of the ring of functions on an affine space can be identified with the cotangent bundle (relative to $\mathrm{Spec}(A)$), and $d : A \to \Omega_{A[x_1,\ldots,x_n]/A}$ identified with the derivative of a function.

Exercise 7. Textbooks usually construct $\Omega_{B/A}$ using generators and relations but here is a more categorical, but essentially equivalent way to construct it.

1. Show that for any colimit of A-algebras

$$\mathrm{Der}_A\left(\varinjlim_{\lambda\in\Lambda} B_\lambda, M\right) = \varprojlim_{\lambda\in\Lambda} \mathrm{Der}_A(B_\lambda, M).$$

2. Show that if $\Omega_{\varinjlim B_\lambda/A}$ exists, then

$$\Omega_{\varinjlim B_\lambda/A} = \varinjlim \Omega_{B_\lambda/A}.$$

3. Show that every A-algebra can be written as a colimit of polynomial A-algebras

$$B = \varinjlim_{I_\lambda, \lambda\in\Lambda} A[x_i : i \in I_\lambda].$$

4. Using Exercise 5 deduce that $\Omega_{B/A}$ always exists.

Example 8.

1. Consider the nodal curve $A = \frac{\mathbb{C}[x,y]}{\langle y^2 - x^2(x-1)\rangle}$. The A-module Ω_A is free of rank 1 if we invert y. That is, it is a line bundle away from the singular point of $\mathrm{Spec}(A)$. If $\mathfrak{m} \subseteq A$ is the maximal ideal corresponding to the singular point, then $\Omega_A \otimes_A A/\mathfrak{m}$ is a dimension 2 vector space over $A/\mathfrak{m}$.
2. If K is a field $f(x) \in K[x]$ a monic, and $L = K[x]/\langle f\rangle$, then $\Omega_{L/K} = 0$ if and only if L is a finite product of separable field extensions of K.
3. If $A \to B$ is a ring homomorphism correspond to a morphism $f : Y \to X$ of smooth complex algebraic varieties, then $\Omega_{B/A} = 0$ if and only if $Y \to X$ is an analytic homeomorphism. That is, in the topology on X, Y induced from $\mathbb{C}^N \cong \mathbb{R}^{2N}$, for every point $y \in Y$ there is an open neighbourhod $y \in U$ such that $U \to f(U)$ is a homeomorphism.

Kähler differentials are also closely related to square zero extensions.

Definition 9. Let A be a ring and B an A-algebra. A *square zero extension* of B is a surjection $B' \to B$ of A-algebras such that $I^2 = 0$ where $I = \ker(B' \to B)$. We will write

$$\mathrm{Exal}_A(B, I)$$

for the set of isomorphism classes[1] of square zero extensions B' such that $\ker(B' \to B)$ is identified[2] with $I \in B$-mod.

Exercise 10. For any B-module I define a ring structure on the A-module $B' := B \oplus I$ by

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1).$$

Show that this defines a square zero extension of B. This is called the *trivial square zero extension.*

Exercise 11. Let $\tilde{\mathcal{O}}$ = the ring of differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \to f(x, y)$ define an equivalence relation on $\tilde{\mathcal{O}}$ by $f \sim g$ if $f(x, 0) = g(x, 0)$ and $(\partial_y f)(x, 0) = (\partial_y g)(x, 0)$ for all $x \in \mathbb{R}$. Let $\mathcal{O}$ be the ring of differentiable functions $f : \mathbb{R} \to \mathbb{R}$; $x \to f(x)$. Show that the canonical morphism $\tilde{\mathcal{O}}/\sim \to \mathcal{O}$ induced by the inclusion $\mathbb{R} = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$ is a square zero extension.

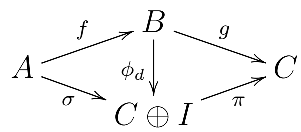
Example 12. If $P := A[x_1, \dots, x_n] \to B$ is any surjection of A-algebras with kernel J, then $P/J^2 \to B$ is a square zero extension of B by $I = J/J^2$.

Remark 13. Geometrically, square zero extensions can be thought of as a combination of the ring of functions on some variety $X = \mathrm{Spec}(B)$, together with the $\mathcal{O}_X$-module $I = \Gamma(X, \mathcal{N}_{X/Y})$ of sections of the normal bundle for some closed embedding $X \hookrightarrow Y$. Of course, if the embedding $X \hookrightarrow Y$ is not regular, then $\Gamma(X, \mathcal{N}_{X/Y})$ is not a vector bundle, just a quasi-coherent sheaf.

[1] The category (which is actually a groupoid) is written $\underline{\mathrm{Exal}}_A(B, I)$ but we won't use this.

[2] So, part of the structure is a choice of isomorphism $I \cong \ker(B' \to B)$ but this makes the notation cumbersome, so we implicitly assume it is given.

Exercise 14. Given morphisms $A \xrightarrow{f} B \xrightarrow{g} C \in \mathcal{R}\mathrm{ing}$, $I \in C$-mod, $d \in \mathrm{Der}_A(B, I)$, define a morphism $B \to C \oplus I$ by $\phi_d : b \mapsto (gb, db)$. Show that this fits into a commutative diagram of ring homorphisms



where $\sigma(a) = (f(a), 0)$ and $\pi(c, m) = c$. Show that any morphism ϕ fitting into such a diagram is of the form ϕ_d for some $d \in \mathrm{Der}_A(B, I)$. Writing $\mathcal{R}\mathrm{ing}_{A//C}$ for the category of rings B equipped with ring homomorphisms $A \to B \to C$, show that

$$\hom_{\mathcal{R}\mathrm{ing}_{A//C}}(B, C \oplus I) \cong \mathrm{Der}_A(B, I). \tag{2}$$

Remark 15 ([Quillen, pg.72])**.** Combining Eq.1 and Eq.2 (and the canonical isomorphism $\hom_{B\text{-mod}}(M, N|_B) \cong \hom_{C\text{-mod}}(M \otimes_B C, N)$) we obtain functorial isomorphisms:

$$\hom_{\mathcal{R}\mathrm{ing}_{A//C}}(B, C \oplus I) \cong \hom_{C\text{-mod}}(\Omega_{B/A} \otimes_B C, I).$$

That is, we have an adjunction

$$\Omega_{(-)/A} \otimes_{(-)} C : \mathcal{R}\mathrm{ing}_{A//C} \rightleftarrows C\text{-mod} : C \oplus (-). \tag{3}$$

In fact, this adjunction can be identified with the free/forgetful adjunction between $\mathcal{R}\mathrm{ing}_{A//C}$ and abelian group objects[3] in the category $\mathcal{R}\mathrm{ing}_{A//C}$.

The following proposition can be proved "by hand".

Proposition 16. *Suppose that $A \to B \to C$ is a sequence of ring homomorphisms and $J \in C$-mod. Note that we can also regard J as a B-module. Then there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Der}_B(C, J) \to \mathrm{Der}_A(C, J) \to \mathrm{Der}_A(B, J) \\ \to \mathrm{Exal}_B(C, J) \to \mathrm{Exal}_A(C, J) \to \mathrm{Exal}_A(B, J). \end{aligned} \tag{4}$$

This sequence is functorial in J.

Exercise 17. *(Long).* Prove the above proposition.

Remark 18. One (not necessarily immediate)[4] consequence of the first part of the first part of (4) is that there is an exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0 \tag{5}$$

of C-modules. In particular, $\Omega_{B/A}$ is covariantly functorial in both A and B, or said another way, it is functorial in the morphism $A \to B \in \mathrm{Fun}(\Delta^1, \mathcal{R}\mathrm{ing})$.

[3] An abelian group object in a category C admitting finite products is an object M equipped with morphisms $\eta : * \to M$ and $\mu : M \times M \to M$ satisfying the axioms of an abelian group.

[4] Going from Eq.5 to Eq.4 is straight-forward, just apply $\hom_{C\text{-mod}}(-, J)$. To go the other way, use functoriality of J to get an exact sequence of functors from C-mod to $\mathcal{A}\mathrm{b}$, and then note that since these functors are representable we get an exact sequence of the objects that represent them.

4.2 The cotangent complex

Let A be a simplicial ring, and recall the following combinatorial simplicial model categories.

1. $(\mathcal{R}\mathrm{ing}_\Delta)_{A/}$ is the category of morphisms $A \to B$ in $\mathcal{R}\mathrm{ing}_\Delta$. A morphism $B \to B'$ in $(\mathcal{R}\mathrm{ing}_\Delta)_{A/}$ is a weak equivalence (resp. fibration), if the underlying morphism of simplicial abelian groups $UB \to UB'$ is a weak equivalence (resp. fibration, i.e., term-wise surjection). Cofibrations are determined by the left lifting property.
   An A-algebra B is cofibrant-fibrant if it is a retract of a "free" A-algebra, where *free* means there is sequence of sets $\{K_n\}_{n\in\mathbb{N}}$ such that B_n is the polynomial ring $B_n = A_n[x_k : k \in \sqcup_{[n]\twoheadrightarrow[i]} K_i]$ with one variable for every pair of a surjection $[n] \twoheadrightarrow [i]$ and a $k \in K_i$, and the degeneracy morphisms are induced by the canonical inclusions.
   The structure of simplicial category is determined by $(B \otimes K)_n = \otimes_{k\in K_n} B_n$ for $B \in (\mathcal{R}\mathrm{ing}_\Delta)_{A/}$ and $K \in \mathcal{S}\mathrm{et}_\Delta$.
2. A-Mod_Δ is the category of simplicial A-modules. That is, simplicial abelian groups $M \in \mathcal{A}\mathrm{b}_\Delta$ such that each M_n is an A_n-module, and each $M_n \to M_m$ is a morphism of A_n-modules via the ring homomorphism $A_n \to A_m$. Weak equivalences and fibrations are determined by the forgetful functor to A-mod $\to \mathcal{A}\mathrm{b}_\Delta$.
   An A-module M is cofibrant-fibrant if each M_n is a projective A_n-module.
   The structure of simplicial category is determined by $(M \otimes K)_n = \oplus_{k\in K_n} M_n$ for $M \in (\mathcal{R}\mathrm{ing}_\Delta)_{A/}$ and $K \in \mathcal{S}\mathrm{et}_\Delta$.

Definition 19 ([DAG, §3.1])**.** Let $A \in \mathcal{R}\mathrm{ing}_\Delta$: The ∞-category associated to the combinatorial simplicial model category $(\mathcal{R}\mathrm{ing}_\Delta)_{A/}$ is denoted

$$\mathcal{SCR}_{A/} := N(\mathcal{R}\mathrm{ing}_\Delta)^{\mathrm{cf}}_{A/}.$$

The ∞-category of (connective) modules over A is

$$\mathcal{M}^{\mathrm{cn}}_A := N(A\text{-}\mathrm{Mod}_\Delta)^{\mathrm{cf}}.$$

Its stabilisation is

$$\mathcal{M}_A := St(\mathcal{M}^{\mathrm{cn}}_A).$$

Definition 20 ([Quillen, Eq.4.4], [Ill, 1.2.5.1], [Lurie, DAG, pg.35])**.** Let $A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. The adjunction of (3) is a Quillen adjunction, and so induces an adjunction of ∞-categories.

$$L\Big(B \otimes_{(\cdot)} \Omega_{(\cdot)/A}\Big) : \mathcal{SCR}_{A//B} \rightleftarrows \mathcal{M}^{\mathrm{cn}}_B : R\Big(B \oplus (\cdot)\Big)$$

The cotangent complex is left derived functor of this functor evaluated at the terminal object $A \to B \stackrel{\mathrm{id}}{\to} B$.

$$L_{B/A} := L\Big(B \otimes_{(\cdot)} \Omega_{(\cdot)/A}\Big)(B) \in \mathcal{M}^{\mathrm{cn}}_B.$$

Remark 21. Quillen used the notation $\mathcal{A}\mathrm{b}$ is because via $B \oplus -$ we think of B-modules as abelian group objects in $(\mathcal{R}\mathrm{ing}_\Delta)_{A//B}$, cf.Remark 15.

Remark 22. There is a purely ∞-categorical way of constructing the above adjunction used in [HA] which proceeds by identifying $N(B\text{-mod})^{\mathrm{cf}}$ with the category of connective loop space objects in $N(\mathcal{R}\mathrm{ing}_\Delta)^{\mathrm{cf}}_{A//B}$. Then Lurie uses the adjunction $\Sigma^\infty : \mathcal{SCR}_{A//B} \rightleftarrows Sp(\mathcal{SCR}_{A//B}) : \Omega^\infty$. In fact he does more. Since he wants to take care of functoriality questions in a unified way, he starts with the categorical fibration $ev_1 : \mathrm{Fun}(\Delta^1, \mathcal{SCR}_{A/}) \to \mathcal{SCR}_{A/}$, and stabilises all fibres at once, and uses the notion of relative adjunction.

Remark 23. A canonical model for $L_{B/A}$ is constructed in [Ill, (1.2.5.1)] as follows. Cf.also[Stacks Project, 08PM]. For each classical ring A, the free/forgetful adjunction

$$F : \mathcal{S}\mathrm{et} \rightleftarrows A\text{-alg} : U$$

induces an endomorphism $P = FU : A\text{-alg} \to A\text{-alg}$ sending an A-algebra C to the polynomial ring $P(C)_0 := A[C] := A[x_c : c \in C]$ with one variable for each element of C. Notice that we have a counit ring homomorphism $A[C] \to C$; $x_c \mapsto c$ and a unit morphism of sets $C \to A[C]$; $c \mapsto x_c$. This unit induces a ring homomorphism $P(C)_0 = A[C] \to A[A[C]] =: P(C)_1$. Iterating these, we obtain a simplicial ring with $P(C)_n = A[A[\dots A[C]\dots]]$ and face and degeneracy morphisms defined using the unit and counit of the adjunction (F, U). Now given a morphism of simplicial rings $A \to B$, the simplicial ring with nth term $P(B_n)_n$ gives a factorisation $A \to P \to B$, which in fact is a cofibrant replacement in $(\mathcal{R}\mathrm{ing}_\Delta)_{A//B}$. Then $L_{B/A}$ can be defined as the B-module with nth term $(\Omega_{P(B_n)_n/A_n}) \otimes_{P(B_n)_n} B_n$.

Remark 24 (Functoriality. [Qui, pg.73], [Ill, 1.2.3])**.** Given a (not necessarily co-cartesian) commutative square of simplicial rings

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

We have a commutative square of Quillen adjunctions (it is easiest to show that the right adjoints commute and are right Quillen functors).

$$\begin{array}{ccccccccc} P & & B' \times_B P & & (\mathcal{R}\mathrm{ing}_\Delta)_{A//B} & \rightleftarrows & B\text{-mod} & & M & & M'|_B \\ \downarrow & & \uparrow & & \downarrow\uparrow & & \downarrow\uparrow & & \downarrow & & \uparrow \\ A' \otimes_A P & & P' & & (\mathcal{R}\mathrm{ing}_\Delta)_{A'//B'} & \rightleftarrows & B'\text{-mod} & & B' \otimes_B M & & M' \end{array}$$

It follows that we have canonical morphism[5] in $\mathcal{M}^{\mathrm{cn}}_{B'}$.

$$B' \otimes^L_B L_{B/A} \to L_{B'/A'} \tag{6}$$

[5]This morphism can be constructed explicitly the canonical resolutions described in Rem.23.

and by adjunction, a canonical morphism in $\mathcal{M}_B^{\mathrm{cn}}$.

$$L_{B/A} \to L_{B'/A'}|_B. \tag{7}$$

Moreover, since the above square is a commutative square of Quillen adjunctions one sees that if the square is cartesian in the sense that $B' = A' \otimes_A^L B$ then the morphism (7) is an equivalence.

Exercise 25. Show that the functor $P' \mapsto P' \times_{B'} B$ sends morphisms in $\mathcal{F}$ (resp. $\mathcal{W} \cap \mathcal{F}$ to morphisms in $\mathcal{F}$ (resp. $\mathcal{W} \cap \mathcal{F}$). Hint.[6]

Remark 26. It follows directly from the definition that

$$\mathrm{Map}_{\mathcal{SCR}_{A//B}}(B, B \oplus M) \cong \mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(L_{B/A}, M)$$

Proposition 27 (cf.[Quillen, pg.73])**.** *Suppose that $A \in \mathcal{R}\mathrm{ing}_\Delta$ is a simplicial ring and $B : I \to (\mathcal{R}\mathrm{ing}_\Delta)_{/A}$ is a diagram of A-algebras. Then there is a canonical equivalence*

$$\varinjlim_{i \in I}(\Omega_{B_i/A} \otimes_{B_i} B) \cong \Omega_B$$

in $\mathcal{M}_B^{\mathrm{cn}}$ where $B = \varinjlim B_i$. Relatedly, there is a canonical equivalence

$$\varinjlim_{i \in I}\left(\Omega_{B_i/A}|_A\right) \cong \Omega_B|_A$$

in $\mathcal{M}_A^{\mathrm{cn}}$.

Proof. They corepresent the same functors. For example:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(\varinjlim_{i \in I}(L_{B_i/A} \otimes_{B_i} B), M) &\cong \varprojlim \mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(L_{B_i/A} \otimes_{B_i} B, M) \\
&\cong \varprojlim \mathrm{Map}_{\mathcal{M}_{B_i}^{\mathrm{cn}}}(L_{B_i/A}, M|_{B_i}) \\
&\cong \varprojlim \mathrm{Map}_{\mathcal{SCR}_{A//B_i}}(B_i, B_i \oplus M|_{B_i}) \\
&\cong \varprojlim \mathrm{Map}_{\mathcal{SCR}_{A//B}}(B_i, B \oplus M) \\
&\cong \mathrm{Map}_{\mathcal{SCR}_{A//B}}(B, B \oplus M) \\
&\cong \mathrm{Map}_{\mathcal{M}_B^{\mathrm{cn}}}(L_{B/A}, M)
\end{aligned}$$

□

Theorem 28 (Transitivity, [Quillen, Thm.5.1], [Ill, Prop.II.2.1.2])**.** *Suppose that $A \to B \to C \in \mathcal{R}\mathrm{ing}_\Delta$ are two composable morphisms. Then there is a canonical pushout square in C-mod of the form*

$$\begin{array}{ccc}
C \otimes_B L_{B/A} & \longrightarrow & L_{C/A} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & L_{C/B}
\end{array}$$

[6] For $\mathcal{W} \cap \mathcal{F}$ use the characterisation in terms of the right lifting property and the fact that limits commute with the forgetful functor $\mathcal{R}\mathrm{ing} \to \mathcal{S}\mathrm{et}$.

Finally, we note that the long exact sequence from the first section is obtained naturally from the cotangent complex.

Proposition 29 ([Quillen, Eq.4.3, Eq.4.5])**.** *Suppose $A \to B$ is a morphism of classical rings and M is a B-module. Then there are canonical identifications*

$$\hom_{h\mathcal{M}_B^{\mathrm{cn}}}(L_{B/A}, M) \cong \mathrm{Der}_A(B, M)$$
$$\hom_{h\mathcal{M}_B^{\mathrm{cn}}}(L_{B/A}, M[1]) \cong \mathrm{Exal}_A(B, M)$$

Combined with the long exact sequence associated to Thm.28 we recover the long exact sequence of Prop.16.

4.3 Finiteness conditions

Consider the following classes of objects in $D(R)_{\geq 0}$ for R a classical ring.

1. Chain complexes of the form

$$(\cdots \to 0 \to 0 \to R^{\oplus n_i} \to \cdots \to R^{\oplus n_1} \to R^{\oplus n_0}) \tag{8}$$

   where $R^{\oplus n_0}$ is in degree zero. Starting with $R = (\ldots \to 0 \to R)$ concentrated in degree zero, and repeatedly applying finite sum and cofibre (i.e., repeatedly taking finite colimits in the derived category), we can construct any chain complex quasi-isomorphic to one of the form (8).
2. Chain complex of the form

$$P = (\cdots \to 0 \to 0 \to P_i \to \cdots \to P_1 \to P_0) \tag{9}$$

   where P_0 is in degree zero, and there is a finite set of elements $f_1, \ldots, f_n \in R$ such that $\{\mathrm{Spec}(R[f_i^{-1}]) \to \mathrm{Spec}(R)\}_{i \in I}$ is a Zariski covering and each localisation $P \otimes_R R[f_i^{-1}] \in D(R[f_i^{-1}])$ is of the form (8). In other words, each P_i is a projective R-module of finite rank [Stacks Project, 00NX]. Note, for any complex of the form (9) there exists a complex C of the form (8) such that $C = P \oplus Q$ for some Q. That is, we can build any P using finite colimits, and direct summands starting from $R = (\ldots \to 0 \to R)$.
3. Chain complexes C that we can approximate by ones of the form (9). Concretely, C such that for all n there is P of the form (9) and an equivalence $\tau_{\leq n} C \cong \tau_{\leq n} P$. That is, chain complexes quasi-isomorphic to one of the form

$$P = (\cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0) \tag{10}$$

   where each P_i is a projective R-module of finite rank.

Exercise 30. Prove the above claims that $M \in D(R)_{\geq 0}$ is in the smallest subcategory containing R and closed under finite colimits, if and only if M is quasi-isomorphic to a complex of the form (8). (Note a subcategory of $D(R)_{\geq 0}$ is closed under finite colimits if and only if it is closed under cofibres and finite sums).

These three classes all have their uses. The second two are related to compact objects in the following way.

Recall that an ∞-category K is *filtered* if every finite diagram $D \to K$ admits a (not necessarily colimit) cone $D^{\triangleright} \to K$. An object c in an ∞-category C is *compact* if $\mathrm{Map}(c, -)$ commutes with filtered colimits. That is, for every filtered category K and diagram $X : K \to C$ admitting a colimit, we have

$$\mathrm{Map}(c, \varinjlim X_k) \cong \varinjlim \mathrm{Map}(c, X_k).$$

Exercise 31.

1. (Easy) Show that a $K \in \mathcal{S}\mathrm{et}$ is compact if and only if it is finite. Show that every set is a filtered colimit of finite sets.
2. (Harder) Show that an R-module $M \in R$-mod (for a classical ring R) is compact if and only if it is isomorphic to a finite rank projective R-module. Show that every R-module is a filtered colimit of finite rank projective R-modules.
3. (Harder) Let I be a small classical category and consider the classical category $\mathrm{Fun}(I^{op}, \mathcal{S}\mathrm{et})$ of presheaves of sets. Show that a presheaf is compact if and only if it is a retract of a finite colimit of representable presheaves. Show that every presheaf is a filtered colimit of retracts of finite colimits of representable presheaves

Proposition 32 ([DAG, pg.19], [HA, ***])**.** *Let $A \in \mathcal{R}\mathrm{ing}_{\Delta}$ be a simplicial ring and $M \in D(A)$ an object in the derived category. The following are equivalent.*

1. *$M \in D(A)$ is compact.*
2. *M is in the smallest subcategory of $D(A)$ which:*
   - (a) *contains A,*
   - (b) *is closed under finite colimits,*
   - (c) *is closed under direct summand, and*
   - (d) *is stable.*

For the third class above, we need to discuss truncation. Recall that there is a canonical forgetful functor $U : D(A) \to D(\mathbb{Z}) = N\,\mathrm{Ch}_{\mathbb{Z}}^{\mathrm{cf}}$. The *homology groups* $H_n M$ of $M \in D(A)$ are the homology groups of $UM \in D(\mathbb{Z})$. For every n we can consider the ∞-category $D(A)_{\leq n} \subseteq D(R)$ of objects M such that $H_i M = 0$ for $i > n$. The inclusion has a right adjoint $\tau_{\leq n} : D(A) \to D(A)_{\leq n}$. In the case of a classical ring A this functor sends $(\cdots \to M_i \to \dots)$ to

$$(\cdots \to 0 \to 0 \to \mathrm{coker}(d) \to M_{n-1} \to M_{n-2} \to \dots).$$

The sub-∞-category $D(A)_{\leq n}$ continues to have all colimits and limits. As a right adjunction, the inclusion commutes with limits. That is, the image of the limit of a diagram $K \to D(A)_{\leq n}$ in $D(A)$ is the limit of the diagram $K \to D(A)_{\leq n} \to D(A)$. The inclusion does not commute with colimits, but truncation does. That is, the colimit of a diagram $K \to D(A)_{\leq n}$ is the image under $\tau_{\leq n}$ of the colimit of the diagram $K \to D(A)_{\leq n} \to D(A)$.

Proposition 33 ([DAG, Prop.2.5.7]). *Let $A \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring, $M \in D(A)$ an object in the derived category, and $n \in \mathbb{Z}$. The following are equivalent.*

1. *There truncation $\tau_{\leq n}M$ is a compact object of $D(A)_{\leq n}$.*
2. *There exists a compact object $N \in D(A)_{\leq n}$ and an equivalence $\tau_{\leq n}N \cong \tau_{\leq n}M$.*

Definition 34 ([DAG, pg.19, pg.23]). Let $R \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring and $M \in D(A)$. We say that M is:

1. *finitely presented* if it is in the smallest subcategory of $D(A)$ which:
   (a) contains R,
   (b) is closed under finite colimits,
   (c) is stable.
2. *perfect* if it is a compact object of $D(A)$.
3. *almost perfect* if every truncation $\tau_{\leq n}M$ is a compact object of $D(A)_{\leq n}$.

Definition 35 ([DAG, pg.32]). Let $f : A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings.

1. We say that B is a *finitely presented A-algebra* if it lies in the smallest subcategory of $\mathcal{SCR}_{A/}$ which contains $A[x]$ and is stable under the formation of finite colimits.
2. We say that B is *locally finitely presented* if it is a compact object of $\mathcal{SCR}_{A/}$. That is, $\mathrm{Map}_{\mathcal{SCR}_{A/}}(B, -)$ commutes with filtered colimits.
3. We say that B is *almost finitely presented* if it is a compact object of $\mathcal{SCR}_{A/}$. That is, $\mathrm{Map}_{\mathcal{SCR}_{A/}}(B, -)$ commutes with filtered colimits.

We also want to consider the analogous categories in $\mathcal{SCR}$.

Definition 36 (Cf.[DAG, Prop.3.1.5]). Let $f : A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. One says B is:

1. a *finitely presented A-algebra* if it lies in the smallest subcategory of $\mathcal{SCR}_{A/}$ which contains $A[x]$ and is closed under finite colimits.
2. a *locally finitely presented A-algebra* if it is a compact object of $\mathcal{SCR}_{A/}$; in other words, if $\mathrm{Map}_A(B, -)$ commutes with filtered colimits.
3. *almost of finite presentation* if for every n there exists a finitely presented A-algebra B' and a morphism $B' \to B$ of A-algebras inducing isomorphisms $\pi_m(B') \cong \pi_m(B)$ for $m \leq n$.

Proposition 37 ([DAG, Prop.3.2.14, Prop.3.2.18]). *Let $f : A \to B \in \mathcal{R}\mathrm{ing}_\Delta$ be a morphism of simplicial rings. Then $L_{B/A}$ exists and is connective. Moreover, we have the following equivalences.*

$$\begin{aligned} f \text{ is of finite presentation} &\iff (FP_0) \text{ holds and } L_{B/A} \text{ is of finite presentation.} \\ f \text{ is locally of finite presentation} &\iff (FP_0) \text{ holds and } L_{B/A} \text{ is perfect.} \\ f \text{ is almost of finite presentation} &\iff (FP_0) \text{ holds and } L_{B/A} \text{ is almost perfect.} \end{aligned}$$

Where (FP_0) means $\pi_0 A \to \pi_0 B$ is of finite presentation in the classical sense. I.e., $\pi_0 B = (\pi_0 A)[x_1, \ldots, x_n]/\langle f_1, \ldots, f_c\rangle$ for some $f_i \in \pi_0 A[x_1, \ldots, x_n]$.