Derived Algebraic Geometry
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4 Higher algebra I: Stable ∞-categories

The goal in this lecture is define the ∞-category $D(R)$ for a simplicial ring R. We begin with a closer look at Ch_R for R a classical ring.

4.1 The derived category

Definition 1. Suppose R is a classical ring. The *derived category* of R is the ∞-category

$$D(R) := N(\mathrm{Ch}_R)^{\mathrm{cf}}$$

associated to the combinatorial simplicial model category of chain complexes Ch_R.

Example 2.

Products. Suppose that $\{X_\lambda\}_{\lambda\in\Lambda}$ is a collection of chain complexes in Ch_R. Since every chain complex is fibrant, the product $\prod_{\lambda\in\Lambda} X_\lambda$ in Ch_R is quasi-isomorphic to the product of the images in $D(R)$. That is, $Ch_R \to D(R)$ sends products to products.

Coproducts. To calculate the coproduct, we take cofibrant replacements $QX_\lambda \to X_\lambda$ of each X_λ. Then $\coprod QX_\lambda$ is quasi-isomorphic to a coproduct in $D(R)$ (in fact, since $\coprod QX_\lambda$ will be fibrant-cofibrant, it *is* a coproduct in $D(R)$).

Biproducts. In the special case that Λ is finite (e.g. empty) $\coprod QX_\lambda \to \coprod X_\lambda$ is a quasi-isomorphism since homology commutes with finite sums. That is, $\mathrm{Ch}_R \to D(R)$ sends finite products (resp. finite sums) to finite products (resp. finite sums).

Zero object. A special case of this is that 0 is both an initial and a final object of $D(R)$.

Example 3. Let us calculate the pullback of a diagram of the form $0 \to C \stackrel{g}{\leftarrow} B$. Since every complex is fibrant, every such diagram is projectively fibrant, so we can always use the weighted pullback, for example associated to the cofibrantly projective diagram $W : \Lambda^2_2 \to \mathcal{S}\mathrm{et}_\Delta$; $W = ([0] \to \Delta^1 \leftarrow [1])$. We have

$$NR\Delta^0 = (\cdots \to 0 \to R \to 0 \to \cdots)$$

concentrated in degree zero and

$$NR\Delta^1 = (\cdots \to R \stackrel{d}{\to} R \oplus R \to 0 \to \cdots)$$

concentrated in degrees zero and one where the nonzero differential d is $r \mapsto (r, -r)$. The two inclusions $NR\Delta^0 \rightrightarrows NR\Delta^1$ are clear. The weighted limit is

$$0 \overset{W}{\times}_C B = \mathrm{eq}\left(0 \times C^{\Delta^1} \times B \rightrightarrows C \times C\right)$$

and doing the calculation we find that

$$(0 \overset{W}{\times}_C B)_n = C_{n+1} \oplus B_n, \qquad d_{\left(0 \overset{W}{\times}_C B\right)} = \begin{bmatrix} d_C & g \\ 0 & d_B \end{bmatrix}$$

This $0 \overset{W}{\times}_C B$ will always be quasi-isomorphic to a pullback of the image of $0 \to C \leftarrow B$ in $N(\mathrm{Ch}_R)^{\mathrm{cf}}$ (with no conditions on the objects B, C or morphism $B \to C$).

Exercise 4.

1. Show that $(A \otimes \Delta^1)_n = A_n \oplus A_{n-1} \oplus A_n$ with differential $\begin{bmatrix} d & 0 & 0 \\ \mathrm{id} & d & -\mathrm{id} \\ 0 & 0 & d \end{bmatrix}$.
2. Show that $(C^{\Delta^1})_n = C_n \oplus C_{n+1} \oplus C_n$ with differential $\begin{bmatrix} d & \mathrm{id} & 0 \\ 0 & d & 0 \\ 0 & -\mathrm{id} & d \end{bmatrix}$.

(You may end up with different signs. This is fine).

Before the next example, we record the following consequence of the Snake Lemma and the Five Lemma:

Lemma 5. *Suppose*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is a diagram in Ch_R *whose rows are short exact sequences. Then if two of* a, b, c *are quasi-isomorphisms, so is the third.*

Exercise 6. Prove the above lemma using the Snake Lemma and the Five Lemma.

Example 7. Now consider a diagram of the form $0 \leftarrow A \xrightarrow{f} B$. If A and B are cofibrant then the weighted colimit will be quasi-isomorphic to a pushout of the image in the ∞-category $D(R)$. Using the same W as above, we have

$$0 \overset{W}{\sqcup}_A B = \mathrm{coeq}\left(A \oplus A \rightrightarrows 0 \oplus (A \otimes \Delta^1) \oplus B\right)$$

and doing the calculation we find that

$$(0 \overset{W}{\sqcup}_A B)_n = A_{n-1} \oplus B_n, \qquad d_{\left(0 \overset{W}{\sqcup}_A B\right)} = \begin{bmatrix} d_A & 0 \\ f & d_B \end{bmatrix}$$

Note that there is a canonical short exact sequence

$$0 \to B \to 0 \overset{W}{\sqcup}_A B \to A[1] \to 0.$$

where $A[1]_n = A_{n-1}$. Using Lemma 5 applied to this short exact sequence, one sees that if $Q0 \overset{Q0}{\leftarrow} QA \overset{Qf}{\to} QB$ is an injectively cofibrant replacement of the original $0 \leftarrow A \to B$, then the induced morphism $Q0 \overset{W}{\sqcup}_{QA} QB \to 0 \overset{W}{\sqcup}_A B$ is always a quasi-isomorphsim (with no conditions on A or B).

Hopefully you noticed that $0 \overset{W}{\sqcup}_A B$ looks very similar to $0 \overset{W}{\times}_C B$. Certainly, we always have a commutative diagram of the form:

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \longrightarrow & 0 \overset{W}{\sqcup}_A B & \longrightarrow & A[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
(0 \overset{W}{\times}_C B) & \longrightarrow & B & \xrightarrow[g]{} & C & \longrightarrow & (0 \overset{W}{\times}_C B)[1]
\end{array}$$

and by the short exact sequences

$$0 \to B \to 0 \overset{W}{\sqcup}_A B \to A[1] \to 0$$

$$0 \to C[-1] \to 0 \overset{W}{\times}_C B \to B \to 0$$

our commutative diagram gives rise to a morphism of long exact sequences

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(0 \overset{W}{\sqcup}_A B) & \longrightarrow & H_{n-1}(A) & \longrightarrow & \dots \\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_{n+1}(0 \overset{W}{\times}_C B) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \longrightarrow & H_n(0 \overset{W}{\times}_C B) & \longrightarrow & \dots
\end{array}$$

Exercise 8. Show that the boundary morphism $H_n(A) \to H_n(B)$ associated to the short exact sequence $0 \to B \to 0 \overset{W}{\sqcup}_A B \to A[1] \to 0$ is actually the morphism induced by $f : A \to B$.

Consequently, by the Five Lemma, we obtain:

Proposition 9. *Suppose that $A \overset{f}{\to} B \overset{g}{\to} C$ are two morphisms in Ch_R with $gf = 0$ so that we get induced morphisms*

1. *$0 \overset{W}{\sqcup}_A B \to C$.*
2. *$A \to 0 \overset{W}{\times}_C B$.*

If one of these is a quasi-isomorphism, then so is the other. Consequently, the image of the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & C \end{array}$$

in the ∞-category $D(R)$ is a pullback square if and only if it is a pushout square.

By the equivalence $\mathrm{Map}(\Delta^1 \times \Delta^1, N(\mathrm{Ch}_R)^{\mathrm{cf}})) \cong N(\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathrm{Ch}_R))^{\mathrm{cf}}$, every commutative square in $D(R)$ is equivalent to one in the image of Ch_R.

4.2 Stable ∞-categories

Definition 10 ([HA, Def.1.1.1.1, Def.1.1.1.9])**.** An ∞-category C is said to be *pointed* if it satisfies:

1. C has an object 0 which is both an initial object and a final object.

$$0 := \varnothing \cong *.$$

Such an object is called a *zero* object.

An ∞-category C is said to be *stable* if it is pointed and satisifes:

2. A commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

is cartesian if and only if it is cocartesian.

3. Every morphism f (resp. g) fits into a cocartesian (resp. cartesian) square of the above form. The corner in such a square is called the *cofibre* (resp. *fibre*).

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \mathrm{cof}(f) \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{fib}(g) & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

Example 11. Examples of pointed categories are easy to find. For any ∞-category C and any object $c \in C$, the over-under category $C_{c//c} \subseteq \mathrm{Fun}(\Delta^2_1, C)$ has zero object c. In particular, if C has a terminal object $* \in C$, then the under category $C_{*/}$ is pointed. Specifically, the category of pointed ∞-groupoids $\mathcal{S}_{*/} = (N\mathcal{G}\mathrm{pd}_\infty)_{*/}$ is a pointed ∞-category.

Categories satisfying the second condition are rarer, but not uncommon. Later we will see a procedure for turning any ∞-category into a stable ∞-category.

Example 12.

1. The ∞-category $D(R)$ is stable, as we saw above.
2. The subcategory $D(R)_{\geq 0}$ of complexes K with $H_n K = 0$ for all $n < 0$ is also not stable. In $D(R)_{\geq 0}$ we have $0 \sqcup_X 0 \cong X[1]$, but $0 \times_X 0 \cong \tau_{\geq 0} X[-1]$ where

$$\tau_{\geq 0} X[-1] = (\cdots \to X_3 \to X_2 \to \ker(d) \to 0 \to \dots).$$

   In particular if X is concentrated in degree zero, then $0 \times_X 0 \cong 0$ in $D(R)_{\geq 0}$.
3. The ∞-category $\mathcal{S}_* = (N\mathcal{G}\mathrm{pd}_\infty)_{*/}$ of pointed ∞-groupoids has a zero object, but is not stable. In the category of pointed spaces $\mathcal{S}_* = (N\mathcal{G}\mathrm{pd}_\infty)_{*/}$ we have $* \sqcup_{\partial\Delta^1} * \cong S^1$ but $* \times_{S^1} * \cong \mathbb{Z}$ where $\mathbb{Z}$ is considered as a discrete space pointed at zero.

Definition 13 ([HA, Pg.23])**.** Let C be a stable ∞-category. For $X \in C$ define

$$\Sigma X := 0 \sqcup_X 0, \qquad \Omega X := 0 \times_X 0.$$

These are called respectively the *suspension* and *loop* functors.

One rigorous way of making this functorial is as follows. Consider the full sub-∞-category $E \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, C)$ consisting of those bicartesian squares

$$\begin{array}{ccc} X & \longrightarrow & 0' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y \end{array}$$

such that 0 and $0'$ are both zero objects.[1] By [HTT.4.3.2.15], the evaluation at $(0,0) \in \Delta^1 \times \Delta^1$ functor e_{00} is a trivial fibration, and therefore there exists a section $s_{00} : C \to E$. Composition with the evaluation at $(1,1) \in \Delta^1 \times \Delta^1$ functor e_{11} determines an endomorphism

$$\Sigma : C \stackrel{s_{00}}{\to} E \stackrel{e_{11}}{\to} C.$$

Dually, if we choose a section s_{11} to e_{11} we obtain an endomorphism

$$\Omega : C \stackrel{s_{11}}{\to} E \stackrel{e_{00}}{\to} C.$$

Remark 14. By functoriality of pushouts / pullbacks, there are natural transformations $\mathrm{id}_E \to s_{11} e_{11}$ and $s_{00} e_{00} \to \mathrm{id}_E$ inducing natural transformations

$$\Sigma\Omega \to \mathrm{id}_C \ \text{ and } \ \mathrm{id}_C \to \Omega\Sigma.$$

Since any two choices of pullback / pushout are equivalent via any of the canonical comparison morphisms, these two natural transformations are equivalences.

Remark 15. If C is a pointed ∞-category admitting pushouts and pullbacks, then the above natural transformations make (Σ, Ω) a pair of adjoints, even if they are not equivalences [HA, Rem.1.1.2.8].

[1] For example, in $D(R)$ this means they are chain complexes K, not necessarily zero, but such that $H_n K = 0$ for all n.

Definition 16. Let C be a stable ∞-category. By abuse of notation we write

$$-[n] : C \to C$$

for any composition $\Sigma^{i_1}\Omega^{j_1}\Sigma^{i_2}\Omega^{j_2}\ldots\Sigma^{i_n}\Omega^{j_n}$ such that $\sum i_k - \sum j_k = n$. Such functors are called *shift* functors.

Exercise 17. Using the pushout and pullback calculations above show for $X_\bullet \in \mathrm{Ch}_R$ and $n \in \mathbb{Z}$, an object $X_\bullet[n]$ in $D(R)$ is quasi-isomorphic to the complex $X_\bullet$ with the indices shifted by n.

Lemma 18 ([HA, pg.24])**.** *Suppose that C is a stable ∞-category. Then for any two objects X, Y the hom set $\hom_{hC}(X, Y)$ in the homotopy category hC is naturally equipped with an abelian group structure, compatible with composition.*

Proof. Since C is stable, there exists Z such that $Y \cong \Omega\Omega Z$. Then we have $\mathrm{Map}_C(X, Y) \cong \mathrm{Map}_C(X, \Omega\Omega Z) \cong \Omega\Omega\,\mathrm{Map}_C(X, Z)$ where $\mathrm{Map}_C(X, Z) \in (N\mathcal{G}\mathrm{pd}_\infty)_{*/}$ is pointed by a morphism $X \to 0 \to Z$ factoring through a zero object. Hence, $\pi_0 \mathrm{Map}_C(X, Y) \cong \pi_2 \mathrm{Map}_C(X, Z)$. □

Definition 19 ([HA, Def.1.1.2.11])**.** Suppose that C is a stable ∞-category. A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in the homotopy category is called a *distinguished triangle* if there exists a diagram $\Delta^1 \times \Delta^2 \to C$

$$\begin{array}{ccccc} X & \xrightarrow{\tilde f} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \tilde g} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\tilde h} & W \end{array}$$

such that $0, 0'$ are zero objects, both squares are cocartesian, the morphisms $\tilde f$, $\tilde g$ represent f, g respectively, and $\tilde h$ represents h up to an equivalence $W \xrightarrow{\cong} X[1]$ determined by the outer cocartesian square.

Example 20.

1. In the derived category $D(\mathbb{Q})$ of $\mathbb{Q}$-vector spaces, a diagram

   $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

   is a distinguished triangle if and only if

   $$H_n(Y) \cong \mathrm{coker}\Big(H_{n+1}Z \to H_n X\Big) \oplus \ker\Big(H_n Z \to H_{n-1} X\Big)$$

   for all n, and similar for X and Z.

2. In the derived category of abelian groups $D(\mathbb{Z})$, for any short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ there is an induced distinguished triangle
$$A \to B \to C \xrightarrow{h} A[1].$$
Moreover, there is a canonical identification[2] $\hom_{hD(\mathcal{A}\mathrm{b}))}(C, A[1]) \cong \mathrm{Ext}^1(C, A)$ under which the morphism h corresponds to the extension B.

Lemma 21 ([HA, pg.28])**.** *Suppose that C is a stable ∞-category and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle. Then*
$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$
and
$$Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$
are distinguished triangles.

Proof.
$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0' & \longrightarrow & Z & \longrightarrow & W \\ & & \downarrow & & \downarrow \\ & & 0'' & \longrightarrow & V \end{array}$$
□

Lemma 22. *Suppose that C is a stable ∞-category and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle. Then for any other object A there are long exact sequences of abelian groups*
$$\cdots \to \hom_{hC}(A, X) \to \hom_{hC}(A, Y) \to \hom_{hC}(A, Z) \to \hom_{hC}(A, X[1]) \to \dots$$
and
$$\cdots \to \hom_{hC}(X[1], A) \to \hom_{hC}(Z, A) \to \hom_{hC}(Y, A) \to \hom_{hC}(X, A) \to \dots$$

Proof. By Lemma 21 and duality it suffices to show exactness at $\hom_{hC}(A, Y)$. But this follows from the fact that
$$\begin{array}{ccc} \mathrm{Map}(A, X) & \longrightarrow & \mathrm{Map}(A, Y) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Map}(A, Z) \end{array}$$
is a cartesian in the category $(\mathcal{G}\mathrm{pd}_\infty)_{*/}$ pointed ∞-groupoids. □

[2]Ever abelian group C admits a presentation $0 \to F_1 \xrightarrow{d} F_0 \to C \to 0$ where F_0, F_1 are free abelian groups (possibly of infinite rank). Then $\hom_{hD(\mathcal{A}\mathrm{b}))}(C, A[1])$ can be calculated as $\mathrm{coker}\left(\hom(F_0, A) \overset{d}{\to} \hom(F_1, A)\right)$. The extension corresponding to a morphism $h \in \hom_{hD(\mathcal{A}\mathrm{b}))}(C, A[1])$ is then $B := F_0 \sqcup_{F_1} A = \mathrm{coker}(F_1 \xrightarrow{(d,h)} F_0 \oplus A)$.

Before there was a widely accepted notion of stable ∞-category, triangulated categories were a popular choice. The following theorem says that the homotopy category of any stable ∞-category is a triangulated category. Note that by (TR3), the cofibre of $X \xrightarrow{f} Y$ in (TR1)(a) is unique up to isomorphism, but a priori, this isomorphism is highly non-unique. This is one of the major advantages of ∞-categories: being able to choose $\mathrm{cof}(X \xrightarrow{f} Y)$ in an appropriately functorial way.

Theorem 23 ([HA, Thm.1.1.2.14])**.** *Suppose that C is a stable ∞-category.*

(TR0)

1. *The homotopy category hC is an additive category:*
   (a) *For every pair of objects X, Y the set $\hom_{hC}(X, Y)$ has a canonical structure of abelian group, compatible with composition.*
   (b) *For any pair of objects X, Y the canonical morphism $X \sqcup Y \to X \times Y$ is an equivalence. That is, finite products and finite coproducts agree. [HA, Rmk.1.1.3.5]*
2. *The canonical morphisms $X[-1][1] \to X \to X[1][-1]$ are equivalences. That is, the functor $X \mapsto X[1]$ is an equivalence of categories.*

(TR1) (a) *Every morphism $X \xrightarrow{f} Y$ of hC can be completed to a distinguished triangle.*

(b) *The collection of distinguished triangles is stable under isomorphsim.*

(c) *For every object X the diagram*

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$$

is a distinguished triangle.

(TR2) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a distinguished triangle if and only if the rotated diagrm

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.

(TR3) Given a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow & & \vdots & & \downarrow{\scriptstyle f[1]} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1]
\end{array}$$

in which both horizontal rows are distinguished triangles, there exists a dotted arrow rendering the entire diagram commutative.

(TR4) Suppose given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

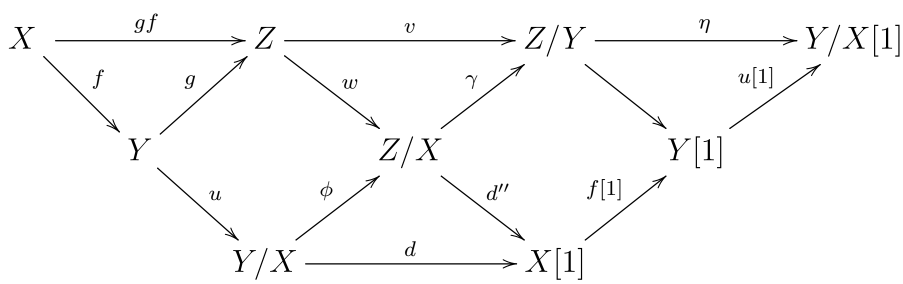
$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

in hC. Then there exists a fourth distinguished triangle

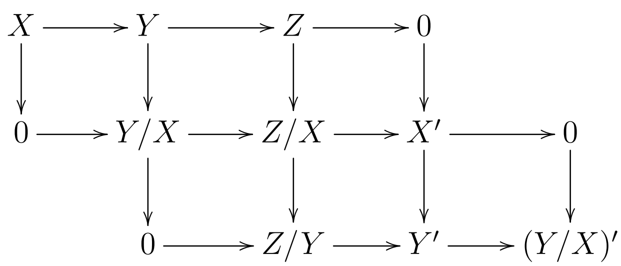
$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\gamma} Z/Y \xrightarrow{\eta} Y/X[1]$$

such that the diagram



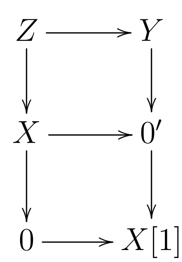
commutes

Remark 24. We have already proven some of this theorem above. Much of it is more or less straight forward. The trickiest part is (TR4). This is proven using a diagram of the form



Proposition 25 ([HA, Prop.1.1.3.4])**.** *Let C be a stable ∞-category. Then C admits all finite limits and finite colimits.*

Proof. By duality it suffices to show that C admits all finite limits. For this it suffices to show that it admits pairwise products and equalisers. For products, use the diagram



The lower and outside cartesian squares exists by the axioms of a stable ∞-category. It follows that the upper square is cartesian, i.e., that $Z \cong X \times Y$ since $0'$ is a final object.

For equalisers, the idea is to show that there is a distinguished triangle of the form

$$\begin{array}{ccc} \mathrm{eq}(X \overset{f}{\underset{g}{\rightrightarrows}} Y) & \longrightarrow & X \\ \downarrow & & \downarrow {\scriptstyle f-g} \\ 0 & \longrightarrow & Y \end{array}$$

□

4.3 Stabilisation

The approach to stabilisation we will take is the following. We want to turn the endomorphism $\Omega : C \to C$ into an equivalence of categories. In the land of ∞-categories there is a simple way to do this: just take the inverse limit[3] of the iterated loop functor.

Definition 26 ([Def.1.4.2.8, Prop.1.4.2.21])**.** Suppose C is an ∞-category with finite limits. The category of spectra is the inverse limit

$$\varprojlim(\dots \overset{\Omega}{\to} C \overset{\Omega}{\to} C \overset{\Omega}{\to} C). \tag{1}$$

Here are the main properties of the stabilisation.

Proposition 27 ([HA, Cor.1.4.2.17])**.** *Let C be an ∞-category which admits finite limits. Then the ∞-category $Sp(C)$ of spectrum objects is stable.*

Definition 28. The functor $Sp(C) = \varprojlim(\dots \overset{\Omega}{\to} C \overset{\Omega}{\to} C \overset{\Omega}{\to} C) \to C$ projecting to the last copy of C is denoted

$$\Omega^\infty : Sp(C) \to C.$$

Proposition 29 ([HA, Prop.1.4.2.21])**.** *Let C be an ∞-category which admits finite limits. Then C is stable if and only if $\Omega^\infty : Sp(C) \to C$ is an equivalence of ∞-categories.*

Corollary 30 ([Cor.1.4.2.23])**.** *Let C be an ∞-category which admits finite limits, and T a stable ∞-category. Then composition with the functor $\Omega^\infty : Sp(C) \to C$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}^{Lex}(T, Sp(C)) \to \mathrm{Fun}^{Lex}(T, C)$$

where Fun^{Lex} *means the full subcategory of functors sending finite limits to finite limits.*[4]

Example 31. Suppose R is a classical ring. Then $\mathrm{Ch}_R \cong Sp((\mathrm{Ch}_R)_{\geq 0})$.

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[3] One could also consider the colimit $\varinjlim(C \overset{\Sigma}{\to} C \overset{\Sigma}{\to} C \overset{\Sigma}{\to} \dots)$ but this does not behave well with respect to "large" objects such as infinite sums.

[4] Such functors are called *left exact*.

4.4 Dold-Kan

Definition 32 ([HA, Def.1.2.3.9.])**.** Let R be a classical ring. Given a simplicial R-module $M \in R\text{-Mod}_\Delta$, one defines the normalised chain complex as

$$NM_n := \bigcap_{i=1}^{n} \left(M_n \xrightarrow{d_i} M_{n-1} \right)$$

(and $NM_0 = M_0$). The face maps $d_0 : M_n \to M_{n-1}$ define a structure of chain complex on these modules.

Example 33. We have

$$(NR\Delta^n)_m = \bigwedge^{m+1} (R^{\oplus n+1})) \cong R^{\oplus \binom{n+1}{m+1}}$$

the $(m+1)$th exterior power with differential acting on the canonical basis by $e_{j_0} \wedge \cdots \wedge e_{j_m} \mapsto \sum_{k=0}^{m} (-1) e_{j_0} \wedge \ldots \widehat{e_{j_k}} \cdots \wedge e_{j_m}$ where $\widehat{e_{j_k}}$ means omit the kth element.[5]

Exercise 34.

1. Show that the face maps d_0 send NM_n into NM_{n-1}.
2. Show that $d_0 \circ d_0 = 0$.
3. Deduce that N defines a functor

$$N : R\text{-Mod}_\Delta \to \text{Ch}_R .$$

Definition 35. Given a chain complex $M \in \text{Ch}_R$, one defines a simplicial object by setting

$$KM_n = \hom_{\text{Ch}_R}(NR\Delta^n, M).$$

This defines a functor

$$R\text{-Mod}_\Delta \leftarrow \text{Ch}_R : K$$

Theorem 36 ([HA, Thm.1.2.3.7 (Dold-Kan Correspondence)])**.** *Let R be a classical ring. Then the functors*

$$N : R\text{-Mod}_\Delta \rightleftarrows (\text{Ch}_R)_{\geq 0} : K$$

are inverse equivalences of (classical) categories. Moreover, homotopy groups (at 0) on the left correspond to homology groups on the right, and hence, the model structures are the same on both sides. Consequently, we obtain inverse equivalences of ∞-categories

$$N(R\text{-Mod}_\Delta)^{\text{cf}} \cong N((\text{Ch}_R)_{\geq 0})^{\text{cf}}.$$

[5] This can be proven using [HA, 1.2.3.17]. Let Kos denote the complex made from the exterior algebra. Sending $e_{j_0} \wedge \cdots \wedge e_{j_m}$ (where $j_0 < \cdots < j_m$) to the element of $(R\Delta^n)_m$ corresponding to $[m] \hookrightarrow [n]; j \mapsto j_m$. Defines a map of chain complexes $Kos \to CR\Delta^n$ which one composes with the projection $CR\Delta^n \to NR\Delta^n$.

Warning 37. The above equivalence is not monoidal! The category $R\text{-Mod}_\Delta$ has a canonical monoidal structure given by term-wise tensor product $(M \otimes N)_n = M_n \otimes N_n$. On the other hand, the category $(\mathrm{Ch}_R)_{\geq 0}$ also has a canonical monoidal structure given by $(M \otimes N)_n = \oplus_{i+j=n} M_i \otimes M_j$. There are canonical natural transformations

$$\nabla : N(A) \otimes N(B) \to N(A \otimes B)$$

and

$$\Delta : N(A \otimes B) \to N(A) \otimes N(B)$$

called respectively, the Eilenberg-Zilber map and the Alexander-Whitney map. The composition $\Delta \circ \nabla$ is the identity, but $\nabla \circ \Delta$ is just a chain homotopy equivalence in general.

A consequence of this is the the categories of commutative monoid objects in $R\text{-Mod}_\Delta$ and $(\mathrm{Ch}_R)_{\geq 0}$, i.e., the categories of simplicial R-algebras, and commutative differential graded R-algebras, are not equivalent (unless $\mathbb{Q} \subseteq R$).

Let $R \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring considered as a ring object in the category of presheaves $\mathrm{Fun}(\Delta^{op}, \mathcal{S}\mathrm{et})$. An R-module is an R-module in the category of presheaves $\mathrm{Fun}(\Delta^{op}, \mathcal{S}\mathrm{et})$. In other words, an R-module is a simplicial abelian group M such that for each n, the group M_n has a structure of R_n-module, and for every $[n] \to [m]$ the maps $M_m \to M_n$ are morphisms of R_m-modules, where the R_m-module structure on M_n is via the corresponding map $R_m \to R_n$.

Example 38. Every simplicial set $X \in \mathcal{S}\mathrm{et}_\Delta$ determines a simplicial R-module RX which is the free R_n-module $R_n^{\oplus X_n}$ in degree n.

As in the classical case, the category of R-modules is equipped with a simplicial model structure where fibrations and weak equivalences are detected by the forgetful functor $U : R\text{-mod} \to \mathcal{A}\mathrm{b}_\Delta$, and the simplicial structure is defined using the $R\Delta^n$ from the above example. As in the classical case, the cofibrant-fibrant objects are those simplicial R-modules M such that each M_n is a projective R_n-module.

Definition 39. Let $R \in \mathcal{R}\mathrm{ing}_\Delta$ be a simplicial ring. The derived category $D(R)$ is the stabilisation of the ∞-category associated to simplicial R-modules

$$D(R) := Sp(N(R\text{-mod})^{\mathrm{cf}}).$$