Derived Algebraic Geometry
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4 Higher (co)limits

4.1 Recollections from last time

Last time we defined initial (resp. final) objects in ∞-categories. We defined the under (resp. over) categories $C_{p/}$ (resp. $C_{/p}$) associated to a morphism $p : I \to C$, and then defined colimits (resp. limits) as the initial (resp. finial) objects in $C_{p/}$ (resp. $C_{/p}$).

At the end, we defined the notion of adjunctions in the ∞-categorical setting, and saw that if an ∞-category C admits all limits indexed by some $I \in \mathcal{C}\mathrm{at}_\infty$, the constant diagram functor admits a right adjoint, which gives $\varprojlim(p)$ when evaluated on $p : I \to C$.

$$\mathrm{const.} : C \rightleftarrows \mathrm{Fun}(I, C) : \varprojlim$$

In this lecture we will give two ways to calculate limits when the infinity category is of the form $C = N\mathcal{M}^{\mathrm{cf}}$ for some combinatorial simplicial model category $\mathcal{M}$. In light of the identification $N\,\mathrm{Fun}(I, \mathcal{M})^{\mathrm{cf}} \cong \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}})$, it suffices to construct an adjunction

$$N\mathcal{M}^{\mathrm{cf}} \rightleftarrows N\,\mathrm{Fun}(I, \mathcal{M})^{\mathrm{cf}}.$$

4.2 Calculating limits I: Derived functors

To begin with we recall the injective and projective model structures on $\mathrm{Fun}(I, \mathcal{M})$.

Definition 1 (cf. [HTT, Def.A.2.8.1])**.** Suppose $\mathcal{M}$ is a model category. A morphism $\alpha : F \to G$ in $\mathrm{Fun}(I, \mathcal{M})$ is called:

1. a *weak equivalence* if $\alpha(i) : F(i) \to G(i)$ is a weak equivalence for each $i \in I$,
2. an *injective cofibration* if $\alpha(i) : F(i) \to G(i)$ is a cofibration for each $i \in I$,
3. an *injective fibration* if it has the right lifting property[1] with respect to all γ which are both a weak equivalence and injective cofibration,
4. a *projective fibration* if $\alpha(i) : F(i) \to G(i)$ is a fibration for each $i \in I$.
5. an *projective cofibration* if it has the left lifting property[2] with respect to all γ which are both weak equivalence and projective fibration.

[1] α has the right lifting property with respect to β if for every commutative square $\gamma \downarrow\rightrightarrows\downarrow \alpha$ there exists a diagonal morphism $\nearrow$ making two commutative triangles.

[2] α has the left lifting property with respect to β if for every commutative square $\alpha \downarrow\rightrightarrows\downarrow \beta$ there exists a diagonal morphism $\nearrow$ making two commutative triangles.

Proposition 2 ([HTT, A.3.3.2])**.** *Let $\mathcal{M}$ be a combinatorial simplicial model category and I a small simplicial category. Then:*

1. *The weak equivalences, projective cofibrations, and projective fibrations give* $\mathrm{Fun}(I, \mathcal{M})$ *a structure of model category.*
2. *The weak equivalences, injective cofibrations, and injective fibrations give* $\mathrm{Fun}(I, \mathcal{M})$ *a structure of model category.*

Clearly, in general, the cofibrant-fibrant objects[3] in $\mathrm{Fun}(I, \mathcal{M})$ will be difficult to describe, but in some nice cases we can give a complete characterisation.

Example 3. Consider the case $I{=}\Lambda^2_2$ and $\mathcal{M}{=}(\mathcal{S}\mathrm{et}_\Delta)_{\mathrm{Quillen}}$, so $N\mathcal{M}^{\mathrm{cf}}{=}N\mathcal{G}\mathrm{pd}_\infty{=}\mathcal{S}$. An object

$$\begin{array}{ccc} & & Z \\ & & \downarrow \\ X & \to & Y \end{array}$$

in $\mathrm{Fun}(\Lambda^2_2, \mathcal{S}\mathrm{et}_\Delta)$ is:

1. always *injectively cofibrant*,
2. *injectively fibrant* if $Y \in \mathcal{G}\mathrm{pd}_\infty$ and both morphisms are Kan fibrations.
3. *projectively cofibrant* if $X \sqcup Z \to Y$ is a monomorphism.
4. *projectively fibrant* if $X, Y, Z \in \mathcal{G}\mathrm{pd}_\infty$.

inj.cof.	no conditions
inj.fib.	$\substack{X\to Y \\ Z\to Y}$ Kan fibrations, Y Kan complex
proj.cof.	$X \sqcup Z{\to}Y$ monomorphism
proj.fib.	X, Y, Z Kan complexes

Consider the adjunction

$$\mathrm{const.} : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathrm{Fun}(\Lambda^2_2, \mathcal{S}\mathrm{et}_\Delta) : \varprojlim$$

Note that in the injective model structure, the left adjoint preserves cofibrant objects, and the right adjoint preserves fibrant objects. Note also that this fails for the projective model structure.

Exercise 4. Let $\mathcal{M}$ (resp $\mathcal{M}'$) be a model category with classes of cofibrations, fibrations, and weak equivalences $\mathcal{C}, \mathcal{F}, \mathcal{W}$ (resp. $\mathcal{C}'$, $\mathcal{F}'$, $\mathcal{W}'$), and suppose

$$f : \mathcal{M} \rightleftarrows \mathcal{M}' : g$$

is an adjunction. Using the characterisation of the classes $\mathcal{C}, \mathcal{C} \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F}, \mathcal{F}$ via lifting properties, show that the following are equivalent.

1. f sends morphisms in $\mathcal{C}$ (resp. $\mathcal{C} \cap \mathcal{W}$) to morphisms in $\mathcal{C}'$ (resp. $\mathcal{C}' \cap \mathcal{W}'$).
2. f sends morphisms in $\mathcal{W}' \cap \mathcal{F}'$ (resp. $\mathcal{F}'$) to morphisms in $\mathcal{W} \cap \mathcal{F}$ (resp. $\mathcal{F}$).

[3] Recall that an object X is *cofibrant* if the canonical morphism $\varnothing \to X$ from the initial object is a cofibration and *fibrant* if the canonical morphism $X \to *$ to the terminal object is a fibration.

Definition 5. An adjunction $f : \mathcal{M} \rightleftarrows \mathcal{M}' : g$ between model categories satisfying the equivalent conditions of Exer.4 is called a *Quillen adjunction.*

Proposition 6 ([HTT, Prop.5.2.4.6])**.** *Suppose*

$$f : \mathcal{M} \rightleftarrows \mathcal{M}' : g$$

is a simplicial adjunction between simplicial model categories. Then there is an induced adjunction of ∞*-categories*

$$Lf : N\mathcal{M}^{\mathrm{cf}} \rightleftarrows N(\mathcal{M}')^{\mathrm{cf}} : Rg.$$

In fact, there are commutative squares of ∞*-categories*

$$\begin{array}{ccc} N\mathcal{M}_0^{\mathrm{cof}} & \xrightarrow{f} & N(\mathcal{M}_0')^{\mathrm{cof}} \\ {\scriptstyle \mathrm{loc}}\downarrow & & \downarrow{\scriptstyle \mathrm{loc}} \\ N\mathcal{M}^{\mathrm{cf}} & \xrightarrow[Lf]{} & N(\mathcal{M}')^{\mathrm{cf}} \end{array} \qquad\qquad \begin{array}{ccc} N\mathcal{M}_0^{\mathrm{fib}} & \xleftarrow{g} & N(\mathcal{M}_0')^{\mathrm{fib}} \\ {\scriptstyle \mathrm{loc}}\downarrow & & \downarrow{\scriptstyle \mathrm{loc}} \\ N\mathcal{M}^{\mathrm{cf}} & \xleftarrow[Rg]{} & N(\mathcal{M}')^{\mathrm{cf}}. \end{array}$$

where loc *is induced by fibrant-cofibrant replacement, and* $(-)^{\mathrm{cof}}, (-)^{\mathrm{fib}}$ *means the full subcategory of cofibrant, fibrant objects, respectively.*

Now suppose that $\mathcal{M}$ is a combinatorial simplicial model category and I is a small simplicial category. Equipping $\mathrm{Fun}(I, \mathcal{M})$ with the injective model structure (so weak equivalences and cofibrations are determined objectwise, [HTT, Def.A.3.3.1]) we obtain a Quillen adjunction

$$\mathrm{const.} : \mathcal{M} \rightleftarrows \mathrm{Fun}(I, \mathcal{M})_{\mathrm{inj}} : \varprojlim .$$

Using the identifications $N\,\mathrm{Fun}(I, \mathcal{M})^{\mathrm{cf}} \cong \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}})$ we obtain an adjunction of ∞-categories

$$\mathrm{const.} : N\mathcal{M}^{\mathrm{cf}} \rightleftarrows \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}}) : R\varprojlim$$

where the left adjoint is given by the constant diagram functor. By uniqueness of adjoints, we deduce that the right adjoint must be the ∞-category theoretic $\varprojlim$.

Proposition 7. *Suppose that* $\mathcal{M}$ *is a combinatorial simplicial model category and* I *a small simplicial category. There is a commutative square of* ∞*-categories*

$$\begin{array}{ccc} N\mathcal{M}_0 & \xleftarrow{\varprojlim} & N(\mathrm{Fun}(I, \mathcal{M})_0)^{\mathrm{fib}}_{\mathrm{inj}} \\ {\scriptstyle \mathrm{loc}}\downarrow & & \downarrow{\scriptstyle \mathrm{loc}} \\ N\mathcal{M}^{\mathrm{cf}} & \xleftarrow[\varprojlim]{} & \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}}) \end{array}$$

Example 8. In the case $I = \Lambda^2_2$ and $\mathcal{M} = (\mathcal{S}\mathrm{et}_\Delta)_{\mathrm{Quillen}}$ our square looks like:

$$\begin{array}{ccc} N(\mathcal{S}\mathrm{et}_\Delta)_0 & \xleftarrow{(-)\times_{(-)}(-)} & N(\mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)_0)^{\mathrm{fib}}_{\mathrm{inj}} \\ \downarrow{\scriptstyle \mathrm{loc}} & & \downarrow{\scriptstyle \mathrm{loc}} \\ \mathcal{S} & \xleftarrow[(-)\times_{(-)}(-)]{} & \mathrm{Fun}(\Lambda^2_2, \mathcal{S}) \end{array}$$

Consequently, if $X \to Y \leftarrow Z$ is a pair of Kan fibrations between Kan complexes, then the pullback $X \times_Y Z$ in the 1-category $\mathcal{S}\mathrm{et}_\Delta$ is identified with the pullback in the ∞-category $\mathcal{S} = N\mathcal{G}\mathrm{pd}_\infty$ under the functor loc. Note $X \times_Y Z$ will automatically already be a Kan complex, because Kan fibrations are preserved by pullback and composition.

For more examples, see the appendix.

4.3 Calculating limits II: Weighted limits

Above we saw that if $p : I \to \mathcal{M}$ is nice enough, then $\varprojlim(p)$ is equivalent to $R\varprojlim(p)$ in the ∞-category $N\mathcal{M}^{\mathrm{cf}}$. Specifically, this works if p is injectively fibrant. However, there is no known reasonable description of injectively fibrant diagrams for a general I (even the case of finite posets starts to be complicated, cf. the examples section below). Projectively fibrant diagrams are much easier to recognise: they are those diagrams for which each $p(i)$ is fibrant. But the adjunction (const.$\dashv \varprojlim$) is not a Quillen adjunction for the projective structure. Weighted limits provide a way to modify $\varprojlim$ slightly so that it *does* become a right Quillen adjoint.

Definition 9. Suppose that $\mathcal{C}$ is a simplicial category satisfying:

(M0) The underlying classical category $\mathcal{C}_0$ admits all small limits and small colimits.

(M6) For every $X, Y \in Ob\ \mathcal{C}$ and $K \in \mathcal{S}\mathrm{et}_\Delta$ there are objects $X \otimes K$ and Y^K and isomorphisms

$$\mathrm{Map}_{\mathcal{C}}(X \otimes K, Y) \cong \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}_{\mathcal{C}}(X, Y^K)$$

which are functorial in X, Y, K.

Suppose $p : I \to \mathcal{C}_0$ is a functor from a small classical category and $W : I \to \mathcal{S}\mathrm{et}_\Delta$ is any functor. The *weighted limit* (with respect to W) is defined as

$$\varprojlim{}^W(p) = \mathrm{eq}\left(\prod_{i \in Ob\ I} p(i)^{W(i)} \rightrightarrows \prod_{\substack{i \xrightarrow{u} j \\ \in Arr\ I}} p(j)^{W(i)} \right)$$

where the two morphisms are induced by $p_u^{W(i)} : p_i^{W(i)} \to p_i^{W(i)}$ and $p_i^{W(u)} : p_i^{W(j)} \to p_i^{W(i)}$.

Exercise 10. Show that if W is the constant functor with value $* \in \mathcal{S}\mathrm{et}_\Delta$ then $\varprojlim^W = \varprojlim$. That is, in this case the weighted limit is the same as the usual classical limit in the 1-category $\mathcal{C}_0$.

Exercise 11. Show that there is an adjunction

$$- \otimes W(-) : \mathcal{C}_0 \rightleftarrows \mathrm{Fun}(I, \mathcal{C}_0) : \varprojlim^W$$

where the left adjoint sends an object X to the functor $i \mapsto X \otimes W(i)$.

Exercise 12 ([Hirschorn, Model categories and their localisations, Prop.9.3.7])**.** Suppose that $\mathcal{M}$ is a simplicial model category. Show that the corner axiom:

(M7) If $i : A \to B$ is in $\mathcal{C}$ and $p : X \to Y$ is in $\mathcal{F}$, then

$$\mathrm{Map}_{\mathcal{M}}(B, X) \to \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$$

is in $\mathcal{F}$. If either i or p are in $\mathcal{W}$ then so is the above map.

is equivalent to:

(M7') If $i : A \to B$ is in $\mathcal{C}$ and $j : L \to K$ is a monomorphism of simplicial sets, then then

$$A \otimes K \coprod_{A \otimes L} B \otimes L \to B \otimes K$$

is in $\mathcal{C}$. If either i or j are in $\mathcal{W}$ then so is the above map.

Suppose we have a cofibrantly projective diagram $W : I \to \mathcal{S}\mathrm{et}_\Delta$. Using Exercise 12 one can show that $- \otimes W(-) : \mathcal{C}_0 \to \mathrm{Fun}(I, \mathcal{C}_0)$ is a left Quillen functor. Consequently, by Exercise 11 and Exercise 4 the functor $\varprojlim^W$ is a right Quillen functor. If moreover, W is weakly equivalent to the constant diagram $* : I \to \mathcal{S}\mathrm{et}_\Delta$, then the induced derived functor $N\mathcal{M}^{\mathrm{cf}} \to \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}})$ will be equivalent to the constant diagram functor. Now by uniqueness of adjoints, it follows that the right adjoint $R\varprojlim^W : \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}}) \to N\mathcal{M}^{\mathrm{cf}}$ is equivalent to the ∞-category theoretic limit.

Proposition 13. *Suppose that $\mathcal{M}$ is a combinatorial simplicial model category, I a small simplicial category, and $W : I \to \mathcal{S}\mathrm{et}_\Delta$ a projectively cofibrant diagram which is weakly equivalent to the constant diagram $W \sim *$. Then there is a commutative square of ∞-categories*

$$\begin{array}{ccc} N\mathcal{M}_0 & \xleftarrow{\varprojlim^W} & N(\mathrm{Fun}(I, \mathcal{M})_0)^{\mathrm{fib}}_{\mathrm{proj}} \\ {\scriptstyle \mathrm{loc}}\downarrow & & \downarrow{\scriptstyle \mathrm{loc}} \\ N\mathcal{M}^{\mathrm{cf}} & \xleftarrow[\varprojlim]{} & \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}}) \end{array}$$

Example 14. Suppose that $I = \Lambda^2_2$ and $\mathcal{M} = (\mathcal{S}\mathrm{et}_\Delta)_{\mathrm{Quillen}}$. Consider the diagram $I \to \mathcal{S}\mathrm{et}_\Delta$ given by $(\{0\}{\to}\Delta^1{\leftarrow}\{1\})$. We claim this is projectively cofibrant (certainly it is weakly equivalent to the constant diagram $*$). Indeed, consider the evaluation functors $ev_i : \mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)_{\mathrm{proj}}{\to}\mathcal{S}\mathrm{et}_\Delta$; $X \mapsto X_i$. By definition of the projective model structure, these are right Quillen functors. Hence, their left adjoints

$$\begin{aligned}\gamma_0(X) &= (X{=}X{\leftarrow}\varnothing)\\ \gamma_1(X) &= (\varnothing{\to}X{=}X)\\ \gamma_2(X) &= (\varnothing{\to}X{\leftarrow}\varnothing)\end{aligned}$$

are left Quillen functors. Hence, the two diagrams $(\{0\}{=}\{0\}{\leftarrow}\varnothing)$ and $(\varnothing{\to}\{1\}{=}\{1\})$ and their disjoint union $K_0 := (\{0\}{\to}\partial\Delta^1{\leftarrow}\{1\})$ are projectively cofibrant. Similarly, $\gamma_2(\partial\Delta^1 \to \Delta^1)$ is a projective cofibration. Consequently, the pushout

$$\begin{array}{cccccc} \gamma_2(\partial\Delta^1) = & (\varnothing{\to}\partial\Delta^1{\leftarrow}\varnothing) & \longrightarrow & (\{0\}{\to}\partial\Delta^1{\leftarrow}\{1\}) & = K_0 \\ & \downarrow & & \downarrow & \\ \gamma_2(\Delta^1) = & (\varnothing{\to}\Delta^1{\leftarrow}\varnothing) & \longrightarrow & (\{0\}{\to}\Delta^1{\leftarrow}\{1\}) & \end{array}$$

is projectively cofibrant.

It follows that for *any* diagram $X : \Lambda^2_2 \to \mathcal{G}\mathrm{pd}_\infty$ of Kan complexes, the simplicial set

$$X_0 \times_{X_2} \mathrm{Fun}(\Delta^1, X_2) \times_{X_2} X_1$$

is a fibre product for $(X_0{\to}X_2{\leftarrow}X_1)$ in the ∞-category $\mathcal{S}$ of spaces.

See below for more examples.

4.4 Main properties of (co)limits in ∞-categories

We now summarise the main properties of (co)limits. All proofs are omitted but we give references to [HTT] for the interested reader.

Proposition 15 ([HTT, Lem.4.4.2.1] 2-out-of-3 for Cartesian squares)**.** *Let $C \in \mathcal{C}\mathrm{at}_\infty$ and $X : \Delta^2 \times \Delta^1 \to C$ a diagram:*

$$\begin{array}{ccccc} X_{00} & \longrightarrow & X_{10} & \longrightarrow & X_{20} \\ \downarrow & & \downarrow & & \downarrow \\ X_{01} & \longrightarrow & X_{11} & \longrightarrow & X_{21} \end{array}$$

Suppose that the right square is a pullback in C. Then the left square is a pullback if and only if the outer square is a pullback.

Definition 16. We say a diagram $p : K \to C$ is *finite* or *$\aleph_0$-small* if the simplicial set K has finitely many non-degenerate[4] simplicies. More generally, if κ is an uncountable regular cardinal[5] a diagram is called *κ-small* if each K_n is in $\mathcal{S}\mathrm{et}_{<\kappa}$.

[4] Recall a simplex $\sigma \in K_n$ is non-degenerate if it is not in the image of any $K_{n-1} \to K_n$.

[5] A cardinal κ is *regular* if $I \in \mathcal{S}\mathrm{et}_{<\kappa}$ and $\{K_i\}_{i\in I} \subseteq \mathcal{S}\mathrm{et}_{<\kappa}$ implies $\varinjlim_{i\in I} K_i \in \mathcal{S}\mathrm{et}_{<\kappa}$ where $\mathcal{S}\mathrm{et}_{<\kappa}$ is the category of sets of size $< \kappa$.

Proposition 17 ([HTT, Prop.4.4.3.2] Limits = products + equalisers)**.** *A ∞-category $C \in \mathcal{C}\mathrm{at}_\infty$ has all finite limits if and only if it has equalisers and all finite products. More generally, C has all κ-small limits if and only if it has equalisers and all κ-small products.*

Exercise 18. Prove the classical version of the above proposition in the category of sets. That is, show that every limit can be written as an equaliser of products.

Proposition 19 ([HTT, Prop.4.4.2.6] Limits = products + pullbacks)**.** *The ∞-category C has all finite limits if and only if it has pullbacks and all finite products. More generally, C has all κ-small limits if and only if it has pullbacks and all κ-small products.*

Exercise 20. Show that in the category of sets every equaliser can be written as a fibre product, and conversely, every fibre product can be written as an equaliser.

Proposition 21 ([HTT, Cor.5.1.2.3] Limits of presheaves are calculated object wise)**.** *Let $K, S \in \mathcal{S}\mathrm{et}_\Delta$ and suppose $C \in \mathcal{C}\mathrm{at}_\infty$ admits K-indexed limits. Then*

1. *The ∞-category $\mathrm{Fun}(S, C)$ admits K-indexed limits.*
2. *A map $K^\triangleleft \to \mathrm{Fun}(S, C)$ is a limit diagram if and only if for each vertex $s \in S$, the induced map $K^\triangleleft \to C$ is a limit diagram.*

That is, for $F : K \to \mathrm{Fun}(S, C)$ and $s \in S_0$ we have

$$(\varprojlim_K F_k)(s) = \varprojlim_K (F_k(s)).$$

Proposition 22 ([HTT, 5.1.3.2], Yoneda preserves limits)**.** *Let $C \in \mathcal{C}\mathrm{at}_\infty$ be a small ∞-category and $j : C \to \mathrm{Fun}(C^{op}, \mathcal{S})$ the Yoneda embedding. Then j preserves all small limits which exists in C.*

Proposition 23 ([HTT, Lem.5.1.5.3], Every presheaf is the colimit of its sections)**.** *Suppose $C \in \mathcal{S}\mathrm{et}_\Delta$, let $j : C \to \mathrm{PSh}(C) = \mathrm{Fun}(C^{op}, \mathcal{S})$ denote the Yoneda embedding, and take $F \in \mathrm{PSh}(C)$. Consider the slice category $C_{/F} = C \times_{\mathrm{PSh}(C)} (\mathrm{PSh}(C)_{/F})$ whose objects are the morphisms $j(c) \to F$ for $c \in C$. The canonical cocone $C_{/F}^\triangleright \to \mathrm{PSh}(C)$ exhibits F as a colimit over $C_{/F}$:*

$$F = \varinjlim_{c \in C_{/F}} j(c).$$

Proposition 24 ([HTT, Prop.5.2.3.5])**.** *Let $f : C \to D \in \mathcal{C}\mathrm{at}_\infty$ be a functor which admits a right adjoint $g : D \to C$. Then f preserves all colimits which exist in C and g preserves all limits which exists in D.*

Proposition 25 ([HTT, Prop.5.3.3.3] Filtered colimits commute with finite limits)**.** *Suppose that I is an ∞-category. Then the following are equivalent.*

1. *K is* cofiltered. *That is, every finite diagram $D \to K$ admits a (not necessarily limit) cone $D^\triangleleft \to K$.*

2. *The limit functor* $\varprojlim : \mathrm{Fun}(K, \mathcal{S}) \to \mathcal{S}$ *preserves finite colimits.*

$$\varinjlim_D \varprojlim_K p = \varprojlim_K \varinjlim_D p.$$

Proposition 26 ([HTT, Lem.5.5.2.3] Limits commute with limits)**.** *Let K, L be simplicial sets, let $p : (K^{\triangleleft}) \times (L^{\triangleleft}) \to C$ be a diagram. Suppose that:*

1. *For every vertex $k \in K^{\triangleleft}$, the associated map $p_k : L^{\triangleleft} \to C$ is a limit diagram.*
2. *For every vertex $l \in L$, the associated map $p_l : K^{\triangleleft} \to C$ is a limit diagram.*

Then the restriction $p_0 : K^{\triangleleft} \to C$ is a limit diagram, where $0 \in K^{\triangleleft}$ is the cone point. That is,

$$\varprojlim_{k \in K} \varprojlim_{l \in L} p(k, l) = \varprojlim_{l \in L} \varprojlim_{k \in K} p(k, l).$$

Proposition 27 ([HTT, Def.6.1.1.2, Lem.6.1.3.14], Colimits are universal in $\mathcal{S}$)**.** *For any morphism $X \to Y$ in the ∞-category of spaces $\mathcal{S}$ the associated pullback functor $\mathcal{S}^{/Y} \to \mathcal{S}^{/X}$ preserves (small) colimits. That is, for any diagram $p : K \to \mathcal{S}^{/Y}$, we have*

$$X \times_Y \left(\varinjlim_{k \in K} p(k) \right) = \varinjlim_{k \in K} (X \times_Y p(k))$$

where the colimits are taken in $\mathcal{S}$.

Remark 28. Combining Prop.27 with Prop.21 we see that for any $\mathcal{C} \in \mathcal{C}\mathrm{at}_\infty$ Prop.27 also holds in $\mathrm{PSh}(\mathcal{C}, \mathcal{S})$. Moreover, if $\mathrm{PSh}(\mathcal{C}, \mathcal{S}) \to \mathcal{T}$ is any finite limit preserving functor admitting a fully faithful right adjoint, then Prop.27 also holds in $\mathcal{T}$. In fact, Prop.27 is one of the fundamental characterising properties of higher topoi.

We have not used it but for interest, we record that the following is main tools used to manipulate limits in the ∞-categorical context.

Proposition 29 ([HTT, Prop.4.4.1.1, Prop.4.4.2.2] Colimits over a colimit is a colimit)**.** *Let $C \in \mathcal{C}\mathrm{at}_\infty$.*

1. *Suppose $p : K \to C$ is a diagram such that $K = \coprod K_\alpha$, and suppose each restricted diagram $p|_{K_\alpha}$ has a colimit X_α. Then one may identify colimits of p with coproducts $\coprod X_\alpha$. That is,*

$$\coprod \varinjlim_{K_\alpha} p_\alpha = \varinjlim_{\coprod K_\alpha} p.$$

2. *Suppose $p : K \to C$ is a diagram such that $K = K' \sqcup_{L'} L$ where $L' \to L$ is a monomorphism in $\mathcal{S}\mathrm{et}_\Delta$, and suppose $p|_{K'}$ (resp. $p|_{L'}$, $p|_L$) has a colimit X (resp. Y, Z). Then one may identify colimits for p with pushouts $X \sqcup_Y Z$. That is,*

$$\left(\varinjlim_{K'} (p|_{K'}) \coprod_{\varinjlim_{L'} (p|_{L'})} \varinjlim_{L} (p|_L) \right) = \varinjlim_{K' \sqcup_{L'} L} p.$$

4.5 Appendix I: Examples of derived limits

Example 30 (Products)**.** Suppose I is discrete (that is, all morphisms are identity morphisms) and $\mathcal{M}$ is a combinatorial simplicial model category. The projective and injective model structures on $\mathrm{Fun}(I,\mathcal{M})$ are the same. Consequently, for any collection $\{X_i\}_{i\in I}$ of fibrant objects in $\mathcal{M}$, the product

$$\prod_{i\in I} X_i$$

in $\mathcal{M}_0$ is a product in $N\mathcal{M}^{\mathrm{cf}}$.

Example 31 (Pullbacks)**.** Suppose $I{=}\Lambda^2_2$ and $\mathcal{M}$ is a combinatorial simplicial model category. An object

$$\begin{array}{ccc} & & Z \\ & & \downarrow \\ X & \rightarrow & Y \end{array}$$

in $\mathrm{Fun}(\Lambda^2_2,\mathcal{M})$ is injectively fibrant if and only if Y is fibrant and both morphisms are fibrations. Consequently, for such objects, the fibre product $X\times_Y Z$ in $\mathcal{M}_0$ is sent to a fibre product in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc}: N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Remark 32. More generally, a morphism $X\to Y$ is injectively fibrant if and only if both $X_\varepsilon \to Y_\varepsilon \times_{Y_2} X_2$ are fibrations.

Example 33 (Equalisers)**.** Suppose $I{=}(0\rightrightarrows 1)$ and $\mathcal{M}$ is a combinatorial simplicial model category. An object

$$X \rightrightarrows Y$$

in $\mathrm{Fun}(I,\mathcal{M})$ is injectively fibrant if and only if Y is fibrant and $X\to Y\times Y$ is fibration. Consequently, for such objects, the equaliser $\mathrm{eq}(X\rightrightarrows Y)$ in $\mathcal{M}_0$ is a sent to an equaliser in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc}: N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Example 34 (Towers)**.** Suppose $I{=}N\mathbb{N}^{op}$ and $\mathcal{M}$ is a combinatorial simplicial model category. An object

$$(\cdots \to X_2 \to X_1 \to X_0)$$

in $\mathrm{Fun}(I,\mathcal{M})$ is injectively fibrant if and only if X_0 is fibrant and all $X_{n+1}\to X_n$ are fibrations. Consequently, for such objects, the limit $\varprojlim(\ldots{\to}X_2{\to}X_1{\to}X_0)$ in $\mathcal{M}_0$ is a sent to an equaliser in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc}: N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Example 35 (Finite Posets)**.** Suppose I is a finite partially ordered set and $\mathcal{M}$ is a combinatorial simplicial model category. An object

$$p: I \to \mathcal{M}$$

in $\mathrm{Fun}(I,\mathcal{M})$ is injectively fibrant if and only if for each i the canonical map $p(i)\to \varprojlim_{i\lneq j} p(j)$ is a fibration. Consequently, for such objects, the limit $\varprojlim(p)$ in $\mathcal{M}_0$ is a sent to a limit in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc}: N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Exercise 36. Prove one of the above fibrancy claims. Hint.[6]

[6] For the finite poset case use induction.

4.6 Appendix II: Examples of weighted limits

Example 37 (Products)**.** As we mentioned above, if I is discrete (that is, all morphisms are identity morphisms) then the injective and projective model structures on $\mathrm{Fun}(I, \mathcal{M})$ agree. So the constant diagram $* : I \to \mathcal{S}\mathrm{et}_\Delta$ is already projectively cofibrant. So in this case we just recover the above claim that for any collection $\{X_i\}_{i \in I}$ of fibrant objects in $\mathcal{M}$, the product

$$\prod_{i \in I} X_i$$

in $\mathcal{M}_0$ is a product in $N\mathcal{M}^{\mathrm{cf}}$.

Example 38 (Pullbacks)**.** Suppose $I{=}\Lambda^2_2$. We proved in Example 14 that the diagram $(\{0\}{\to}\Delta^1{\leftarrow}\{1\})$ is projective cofibrant. So for a general combinatorial simplicial model category $\mathcal{M}$, and diagram $(X \to Y \leftarrow Z)$, if all three X, Y, Z are fibrant then

$$X \times_Y Y^{\Delta^1} \times_Y Z$$

is sent to a fibre product in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc} : N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Example 39 (Equalisers)**.** Suppose $I{=}(0 \rightrightarrows 1)$. Then $(\{*\} \underset{1}{\overset{0}{\rightrightarrows}} \Delta^1)$ is projectively cofibrant. So for a combinatorial simplicial model category $\mathcal{M}$ and diagram $(X \rightrightarrows Y)$, if X and Y are fibrant then

$$X \times_{(Y \times Y)} (Y^{\Delta^1})$$

is sent to an equaliser in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc} : N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Example 40 (Towers)**.** Suppose $I{=}N\mathbb{N}^{op}$. Define

$$T = \cdots \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \Delta^1 \sqcup_{\Delta^0} \Delta^1$$

where each 1 on the left of a $\sqcup$ is glued to the 0 on the right. Note this is much smaller than the simplicial set $N\mathbb{N}^{op}$. The latter has countably many non-degenerate simplicies of dimension > 1. Let $\Sigma : T \to T$ denote the inclusion which sends the nth Δ^1 to the $(n+1)$th Δ^1. Then

$$(\ldots \xrightarrow{\Sigma} T \xrightarrow{\Sigma} T \xrightarrow{\Sigma} T) : N\mathbb{N}^{op} \to \mathcal{S}\mathrm{et}_\Delta$$

is projectively cofibrant. So for a combinatorial simplicial model category $\mathcal{M}$ and diagram $(\ldots{\to}X_2{\to}X_1{\to}X_0)$ if all X_n are fibrant then

$$\cdots \times_{X_2^T} X_2^T \times_{X_1^T} X_1^T \times_{X_0^T} X_0^T.$$

is sent to an equaliser in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc} : N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$. Here, the fibre products are associated to $X_{n+1} \to X_n$ on the left side and $\Sigma : T \to T$ on the right side.

Proposition 41 ([] Hirschhorn, Bousfield-Kan)**.** *For any small category I, the diagram $W : i \mapsto N(I/i) \in \mathcal{S}\mathrm{et}_\Delta$ is projectively cofibrant. So for a combinatorial simplicial model category $\mathcal{M}$ and diagram $p : I \to \mathcal{M}_0$ such that all $p(i)$ are fibrant, the weighted limit*

$$\varprojlim{}^W p; \qquad\qquad W : i \mapsto N(I/i) \in \mathcal{S}\mathrm{et}_\Delta$$

is sent to a limit in $N\mathcal{M}^{\mathrm{cf}}$ under the localisation functor $\mathrm{loc} : N\mathcal{M}_0 \to N\mathcal{M}^{\mathrm{cf}}$.

Remark 42. In the pre-[HTT] literature (Hischhorn, Bousfield-Kan, Dugger, ...) the weighted limit in Prop.41 is called the *homotopy limit*. Since [HTT] perverted the universally accepted definition of "descent" I assume it has also destroyed the term "homotopy limit". So I will use weighted limit instead.

Remark 43. Note that the general procedure of Prop.41 sometimes gives nice results: for discrete categories $N(I/-)$ is the constant diagram $*$, for Λ^2_2 (resp. $(0 \rightrightarrows 1)$) we get $(\{0\}{\to}\Lambda^2_2{\leftarrow}\{1\})$ (resp. $(\{0\} \overset{1}{\underset{0}{\rightrightarrows}} \Lambda^2_2)$) which almost what we used above. But for $I = N\mathbb{N}^{op}$ we get something much larger.