Derived Algebraic Geometry
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4 Higher (co)limits

In these lectures we present the theory of (co)limits in ∞-categories. We see that for ∞-categories coming from simplicial model categories, (co)limits can be calculated via derived (co)limits and weighted (co)limits. We list some important properties of (co)limits in ∞-categories. We finish with special cases of interest.

4.1 Discussion

Recall that we want to replace sets with homotopy types. In particular, we want to consider any contractible spaces, such as $\Delta^1_{\mathrm{top}} \cong [0,1] \subseteq \mathbb{R}^1$ equivalent to a single point. Now consider the following two diagrams in Top

$$\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & * \end{array} \qquad\qquad \begin{array}{ccc} & & \{1\} \\ & & \downarrow \\ \{0\} & \longrightarrow & \Delta^1_{\mathrm{top}} \end{array}$$

These two diagrams are homotopy equivalent in a very strong sense, but their pullbacks in Top are different

$$* \times_* * = * \quad \neq \quad \varnothing = \{0\} \times_{[0,1]} \{1\}.$$

There are a couple of ways we can explain this problem.

1. (Local point of view). A classical pullback $X \times_Y Z$ is, by definition, an object which functorially represents commutative squares.

$$\hom(W, X \times_Y Z) \cong \left\{ \begin{array}{c} \text{commutative} \\ \text{squares} \end{array} \quad \begin{array}{ccc} W & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array} \right\}$$

   But in our new world, the left side is replaced with $\mathrm{Map}(W, X \times_Y Z)$ and the right hand side is also a homotopy type. Indeed, a commutative square in the ∞-category associated to Top is not a property $f \circ i = g \circ h$ of morphisms, but the datum of a homotopy $f \circ i \sim g \circ h$ between compositions.

$$\begin{array}{ccc} W & \overset{h}{\dashrightarrow} & Y \\ {\scriptstyle i}\downarrow & \nearrow\!\!\!\nearrow & \downarrow{\scriptstyle g} \\ X & \underset{f}{\longrightarrow} & Z \end{array}$$

So instead of trying to build a fibre product in Top using the set

$$X \times_Y Z = \{(x \in X, z \in Z) | f(x) = g(z) \in Y\}$$

we should at least start with the set

$$X \times^h_Y Z = \left\{ (x \in X, z \in Z, [0,1] \xrightarrow{\gamma} Y) \,\middle|\, \begin{array}{c} \gamma(0) = f(x); \\ \gamma(1) = g(z) \end{array} \right\}$$

Exercise 1. Let $X \to Y \leftarrow Z$ be morphisms in $\mathrm{Top}^{\mathrm{cg}}$. Give $X \times^h_Y Z \subseteq X \times Z \times \hom_{\mathrm{Top}}([0,1], Y)$ the subspace topology where $\hom_{\mathrm{Top}}([0,1], Z)$ has the compact-open topology[1] Show that there is an isomorphism of sets

$$\hom_{\mathrm{Top}^{\mathrm{cg}}}(W, X \times^h_Y Z) = \left\{ (W \xrightarrow{i} X, W \xrightarrow{h} Z, [0,1] \xrightarrow{\gamma} Y) \,\middle|\, \begin{array}{c} \gamma(0) = f \circ i, \\ \gamma(1) = g \circ h \end{array} \right\}$$

and that these isomorphisms are functorial in W.

2. (Global point of view). The fibre product functor (if it exists) is the right adjoint to the constant diagram functor

$$\text{const.} : C \rightleftarrows \mathrm{Fun}\,(\Lambda^2_2, C) : \varprojlim$$

The fibre product we are used to in Top doesn't preserve weak equivalences[2] in general (as we saw above). We saw such a problem already in the first lecture: the functor $- \otimes_R M : \mathrm{Ch}_R \to \mathrm{Ch}_R$ doesn't preserve quasi-isomorphisms in general. The solution was to find some nice subcategory Ch_R^{free} where $- \otimes_R M$ *does* preserve quasi-isomorphisms, and such that every chain complex is quasi-isomorphic to one in Ch_R^{free}. We could hope that the same thing happens here, namely, that there is a nice subcategory

$$\mathrm{Fun}(\Lambda^2_2, \mathrm{Top})^{\mathrm{cf}} \subseteq \mathrm{Fun}(\Lambda^2_2, \mathrm{Top})$$

where $\varprojlim$ *does* preserve weak equivalences, and such that every object of $\mathrm{Fun}(\Lambda^2_2, \mathrm{Top})$ is weakly equivalent to one in $\mathrm{Fun}(\Lambda^2_2, \mathrm{Top})^{\mathrm{cf}}$.

Exercise 2. Suppose that

$$\begin{array}{ccccc} X & \longrightarrow & Y & \xleftarrow{g} & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \xleftarrow[g']{} & Z' \end{array}$$

$\in \mathrm{Ch}_{\mathbb{Z}}$ is a commutative diagram of chain complexes of abelian groups such that the vertical morphisms are quasi-isomorphisms and the two morphisms g, g' are

[1] So for any $W \in \mathrm{Top}^{\mathrm{cg}}$ we have $\hom_{\mathrm{Top}}(W, \hom_{\mathrm{Top}}([0,1], Y)) = \hom_{\mathrm{Top}}(W \times [0,1], Y)$.

[2] We say that a natural transformation $\eta : X \to Y$ of diagrams $X, Y : I \to \mathrm{Top}$ is a weak equivalence if $X_i \to Y_i$ is a weak equivalence for all objects $i \in I$.

(termwise) surjective. Show that the induced morphism $X \times_Y Z \to X' \times_{Y'} Z'$ is a quasi-isomorphism. Hint.[3] Give an example where g, g' are not surjective and $X \times_Y Z \to X' \times_{Y'} Z'$ is not a quasi-isomorphism.

We will come back to these two points of view below. To begin with, we see what the theory of (co)limits looks like inside the $\mathcal{C}\mathrm{at}_\infty$.

4.2 (Co)limits in ∞-categories

Recall that for classical category theory, the notion of final (initial) objects is equivalent to that of (co)limits. Namely, a final object is the limit of the empty diagram $\varnothing \to C$, and a general limit $\varprojlim_{i \in I} X_i$ is a final object in the category $C_{/p}$ of cones, i.e., the over category of the diagram $p : I \to C; i \mapsto X_i$.

We start with final (initial) objects. In classical category theory, an object $*$ is final if the sets $\hom(X, *)$ are all singleton sets. Since we are replacing sets with homotopy types, "singleton" becomes "singleton up to homotopy", or in other words, "contractible".

Definition 3 ([HTT, Prop.1.2.12.4])**.** Let C be an ∞-category. An object $X \in C_0$ is *final* (resp. *initial*) if $\hom^R_C(Y, X)$ (resp. $\hom^L_C(X, Y)$) is contractible for all $Y \in C_0$.

Remark 4 ([HTT, Prop.1.2.12.9])**.** Let C be an ∞-category, and C' the full subcategory of C spanned by the final vertices of C. Then C' is either empty, or is a contractible Kan complex. That is, any two final objects are equivalent, and any two equivalences are equivalent, and any two equivalences of equivalences are equivalent, and...

Exercise 5.

1. Let C be a 1-category. Show that $X \in C_0$ is final (resp. initial) if and only if it is final (resp. initial) in the classical sense. I.e., there exists a unique morphism $Y \to X$ for every $Y \in C_0$.
2. Recall that an exercise in Lecture 3 was to construct an isomorphism of simplicial sets $\mathrm{Map}^R_{\mathrm{Sing}\, X}(x, y) \cong \mathrm{Sing}\, PX(x, y)$ associated to a topological space X where $PX(x, y) \subseteq \hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ is the subspace of paths from x to y. Using the facts[4] that:
   (a) for any topological space Y the natural transformation $|\mathrm{Sing}\, Y| \to Y$ is always a weak equivalence, and
   (b) a Kan complex is contractible if and only if all its homotopy groups are trivial,
   (c) there exist isomorphisms $\pi_n(PX(x, y), \gamma) \cong \pi_{n+1}(X, x)$ for all $n \geq 0$, $x, y \in X$, $\gamma \in PX(x, y)$,

   show that $\mathrm{Sing}\, X$ admits a final object if and only if X is weakly equivalent to a point $*$, in which case every object of $\mathrm{Sing}\, X$ is final.

[3] Note that $X \times_Y Z = \ker(X \oplus Z \to Y)$ (and similar for X', Y', Z') and use the Snake Lemma and the Five Lemma.

[4] The first two facts are theorems. The third is possible to prove directly.

Now we want to define categories over and under a diagram. Lurie first does this using a generalisation $\star : \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ of the constructions $\Delta^{J\sqcup[0]}$ and $\Delta^{[0]\sqcup J}$ which appear in the definition of $\hom^R$ and $\hom^L$.

Note that Δ is equipped with an operation

$$\sqcup : \Delta \times \Delta \to \Delta$$

that sends finite linearly ordered sets $I = \{i_0 < \cdots < i_n\}$ and $I' = \{i'_0 < \cdots < i'_{n'}\}$ to $I \sqcup I' := \{i_0 < \cdots < i_n < i'_0 < \cdots < i'_{n'}\}$.

Definition 6 ([HTT, Def.1.2.8.1])**.** Let K, L be simplicial sets. For any linearly ordered set J we define

$$(K \star L)_J := \coprod_{J = I \sqcup I'} K_I \times L_{I'}$$

In the case I or I' is empty, we set $K_\varnothing = \{*\} = L_\varnothing$ to be a single element set. Given a morphism $p : J \to J'$ of linearly ordered sets and a decomposition $J' = I \sqcup I'$, there is an induced decomposition $J = p^{-1}I \sqcup p^{-1}I'$, and an induced morphism

$$K_I \times L_{I'} \to K_{p^{-1}I} \times L_{p^{-1}I'}.$$

These fit together to define morphisms

$$p^* : (K \star L)_{J'} \to (K \star L)_J$$

giving $K \star L$ the structure of a simplicial set.

Exercise 7.

1. Show that $K \star \varnothing = K = \varnothing \star K$ for any $K \in \mathcal{S}\mathrm{et}_\Delta$.
2. Show that $\Delta^0 \star \Delta^n \cong \Delta^{n+1} \cong \Delta^n \star \Delta^0$. More generally, show that
$$\Delta^{i-1} \star \Delta^{j-1} \cong \Delta^{(i+j)-1}.$$
3. Suppose that P, Q are partially ordered sets. Consider the coarsest partial order on $P \amalg Q$ such that $P, Q \to P \amalg Q$ are both morphisms of partially ordered sets, and such that $p \leq q$ for all $(p, q) \in P \times Q$. Show that $N(P \sqcup Q) = N(P) \star N(Q)$. Deduce that there are pushout squares in $\mathcal{S}\mathrm{et}_\Delta$

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{d_1} & \Delta^{n+2} \\ {\scriptstyle d_1}\downarrow & & \downarrow \\ \Delta^{n+2} & \longrightarrow & \Lambda^2_0 \star \Delta^n \end{array} \qquad \begin{array}{ccc} \Delta^{n+1} & \xrightarrow{d_{n+1}} & \Delta^{n+2} \\ {\scriptstyle d_{n+1}}\downarrow & & \downarrow \\ \Delta^{n+2} & \longrightarrow & \Delta^n \star \Lambda^2_2 \end{array}$$

$$\begin{array}{ccc} \coprod_{i\in I} \Delta^{n-1} & \xrightarrow{d_n} & \coprod_{i\in I} \Delta^n \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \Delta^{n-1} & \longrightarrow & \Delta^n \star (\coprod_{i\in I} \Delta^0) \end{array} \qquad \begin{array}{ccc} \coprod_{i\in I} \Delta^{n-1} & \xrightarrow{d_0} & \coprod_{i\in I} \Delta^n \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ \Delta^{n-1} & \longrightarrow & (\coprod_{i\in I} \Delta^0) \star \Delta^n \end{array}$$

4. Let C, D be a 1-categories. Define a new category $C \star D$ by taking the disjoint union $C \coprod D$ and adding one morphism from c to d for every pair $(c, d) \in Ob\ C \times Ob\ D$. Show that there is a unique composition law making $C \coprod D \to C \star D$ a functor. Show that $NC \star ND = N(C \star D)$. Deduce that there are pushout squares in $\mathcal{S}\mathrm{et}_\Delta$

$$\begin{array}{ccc} \left(\Delta^{\{0\}} \coprod \Delta^{\{1,\dots,n\}}\right)^{\amalg 2} & \longrightarrow & (\Delta^n)^{\amalg 2} \\ \downarrow & & \downarrow \\ \Delta^{\{0\}} \coprod \Delta^{\{1,\dots,n\}} & \longrightarrow & NK \star \Delta^{n-2} \end{array} \qquad \begin{array}{ccc} \left(\Delta^{\{0,\dots,n-1\}} \coprod \Delta^{\{n\}}\right)^{\amalg 2} & \longrightarrow & (\Delta^n)^{\amalg 2} \\ \downarrow & & \downarrow \\ \Delta^{\{0,\dots,n-1\}} \coprod \Delta^{\{n\}} & \longrightarrow & \Delta^{n-2} \star NK \end{array}$$

where K is the category $0 \rightrightarrows 1$.

5. Let X be a topological space and consider $\Delta^1_{\mathrm{top}} \cong [0, 1] \subseteq \mathbb{R}$. Define

$$\mathrm{Cone}\, X := (X \times [0, 1]) \sqcup_{X \times \{1\}} \{1\}.$$

That is, $\mathrm{Cone}\, X$ is the topological space obtained from $X \times [0, 1]$ by identifying all points of the form $(x, 1)$. Show that there is a canonical morphism

$$(\mathrm{Sing}\, X) \star \Delta^0 \to \mathrm{Sing}(\mathrm{Cone}\, X)$$

sending $\mathrm{Sing}\, X$ to $\mathrm{Sing}(X \times \{(1, 0)\}$ and Δ^0 to $(0, 1)$.

Definition 8 (Joyal, [HTT, Prop.1.2.9.2])**.** Let $p : K \to S$ be a morphism of simplicial sets, define

$$(S_{/p})_n = \{f : \Delta^n \star K \to S \ : \ f|_K = p\}.$$

Similarly, define

$$(S_{p/})_n = \{f : K \star \Delta^n \to S \ : \ f|_K = p\}.$$

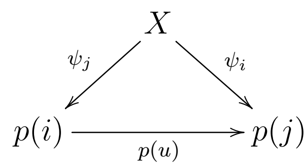
Note, these are both functorial in $[n] \in \Delta$, so define simplicial sets $S_{/p}$ and $S_{p/}$. Moreover, there are canonical projection morphisms $S_{/p} \to S$ and $S_{p/} \to S$.

Exercise 9. Cf. Example 7. Let S be a simplicial set.

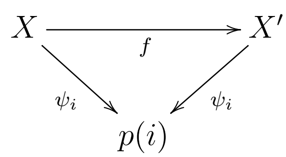
1. Given a vertex $s : \Delta^0 \to S$, show that $(S_{/s})_n$ can be identified with the set of n+1-simplicies $\sigma \in S_{n+1}$ whose top vertex is s. That is, such that $s = \underbrace{d_0 \dots d_0}_{n+1 \text{ times}} \sigma$.
2. Given a set of vertices $s : \coprod_{i \in I} \Delta^0 \to S$ show that $(S_{/s})_n$ can be identified with the set of sets of n+1-simplices $\{\sigma_i\}_{i \in I}$ such that the top vertex of σ_i is s_i, and the lower n-simplex of each σ_i is the same, that is, $d_n \sigma_i = d_n \sigma_j$ for all i, j.
3. Let $p : \Lambda^2_2 \to S$ be a morphism of simplicial sets. Show that $(S_{/p})_n$ can be identified with the set of pairs of n+2-simplicies $(\sigma, \tau) \in S^2_{n+2}$ whose $(n+1)$th faces agree, that is, such that $d_{n+1}\sigma = d_{n+1}\tau$.
4. Let K be the nerve of the category $0 \rightrightarrows 1$ and $p : K \to S$ a morphism of simplicial sets. Show that $(S_{/p})_n$ can be identified with the set of pairs of $(n+2)$-simplicies $(\sigma, \tau) \in S^2_{n+2}$ whose $(n+2)$th faces and final vertex agree, that is, such that $d_{n+2}\sigma = d_{n+2}\tau$ and $\underbrace{d_0 \dots d_0}_{n+2 \text{ times}} \sigma = \underbrace{d_0 \dots d_0}_{n+2 \text{ times}} \tau$.

Exercise 10.

1. Let X be a topological space, and give $\hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ the compact-open topology. Let $x \in X$ be a point and consider the subspace $X_{x/} \subseteq \hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ of those $\gamma : \Delta^1_{\mathrm{top}} \to X$ such that $\gamma((1,0)) = x$. Show that $\mathrm{Sing}(X_{x/}) = (\mathrm{Sing}\, X)_{x/}$.
2. Let $p : I \to C$ be a functor between 1-categories. Show that $C_{/p}$ is the 1-category of cones over p. That is, the category whose objects are collections of morphisms $(\psi_i : X \to p(i))_{i \in ObI}$ such that the triangles



commute for each $i \xrightarrow{u} j$ and whose morphisms $(X, \psi) \to (X', \psi')$ are those morphisms $f : X \to X'$ such that the triangles



commute for each i.

Definition 11 ([HTT, Def.1.2.13.4]). Let C be an ∞-category and $p : K \to C$ a morphism of simplicial sets. A *colimit* for p is an initial object of $C_{p/}$ and a *limit* for p is a final object in $C_{/p}$.

Remark 12 ([HTT, 1.2.13.5]). Note that an object in $C_{p/}$ is a map $K \star \Delta^0 \to C$. Restricting to Δ^0, we obtain an object $\Delta^0 \to C$ of C. One says that $K \star \Delta^0 \to C$ is a *colimit diagram* if it is a colimit of p, and abuse terminology be referring to $\Delta^0 \to C$ as a colimit of p. We use the notation

$$\varprojlim(p), (\text{ resp. } \varinjlim(p)).$$

By Remark 4 there is a contractible space of choices for a limit (resp. colimit) diagram.

4.3 Calculating (co)limits I. Derived (co)limits

To make the exposition lighter we only discuss limits. For the colimit statements insert $(-)^{op}$ everywhere.

We begin with the global point of view. In classical category theory, if a category C has all limits then the limit functor is the right adjoint to the constant diagram functor

$$\text{const.} : C \rightleftarrows \mathrm{Fun}\,(I, C) : \varprojlim$$

For ∞-categories, one can define adjunctions in the familiar way, just replacing hom sets with mapping spaces.

Definition 13 ([HTT, Def.5.2.2.1, Prop.5.2.2.8])**.** Let $C, D \in \mathcal{C}\mathrm{at}_\infty$. An *adjunction* between C and D is a pair of functors $f : C \rightleftarrows D : g$ for which there exists a morphism $u : \mathrm{id} \to g \circ f$ in $\mathrm{Fun}(C, C)$ such that for every pair of objects $c \in C$, $d \in D$, the composition

$$\mathrm{Map}_D((f(c), d) \xrightarrow{g} \mathrm{Map}_C(g(f(c)), g(d)) \xrightarrow{u} \mathrm{Map}_C(c, g(d))$$

is a weak equivalence.

Exercise 14 ([HTT, Prop.5.2.2.9])**.** Recall the homotopy / nerve adjunction.

$$h : \mathcal{C}\mathrm{at}_\infty \rightleftarrows \mathcal{C}\mathrm{at} : N$$

Show that both h and N send adjunctions to adjunctions.

As in the classical case, if an adjoint exists, it is unique. In ∞-category land, uniqueness is up to homotopy unique up to homotopy unique up to...and this uniqueness is expressed as an equivalence.

Proposition 15 ([HTT, Prop.5.2.1.3, Rem.5.2.2.2, Prop.5.2.6.2])**.** *Consider $C, D \in \mathcal{C}\mathrm{at}_\infty$ and let $\mathrm{Fun}^L(C, D) \subseteq \mathrm{Fun}(C, D)$ (resp. $\mathrm{Fun}^R(D, C) \subseteq \mathrm{Fun}(D, C)$) denote the full subcategory whose objects are those functors which are left adjoints (resp. right adjoints). Then there is a canonical equivalence*

$$\mathrm{Fun}^L(C, D) = \mathrm{Fun}^R(D, C)^{op}$$

such that left adjoints correspond to their right adjoints.

As in the classical case, if limits exist, then $\varprojlim$ is functorial, and adjoint to the constant diagram functor.

Proposition 16 ([HTT, Def.4.3.2.2, Prop.4.3.2.17])**.** *Let $I, C \in \mathcal{C}\mathrm{at}_\infty$ and suppose that C admits all limits indexed by I. Then the constant diagram functor $C \to \mathrm{Fun}(I, C)$ admits a right adjoint Φ with the property that $\Phi(p) = \varprojlim(p)$ for all $p \in \mathrm{Fun}(I, C)$. In this situation we just write $\varprojlim$ for Φ.*

$$\mathrm{const.} : C \rightleftarrows \mathrm{Fun}(I, C) : \varprojlim$$

Warning 17. In general,

$$h\,\mathrm{Fun}(I, C) \neq \mathrm{Fun}(hI, hC).$$

The adjunction of Prop.16 of ∞-categories induce an adjunction of classical categories

$$hC \rightleftarrows h\,\mathrm{Fun}(I, C) : h\varprojlim$$

but there is no reason for $h\varprojlim$ to induce a limit functor on hC. That is an ∞-category can admit limits without its homotopy category admitting limits.

Example 18. Sometimes the homotopy category does have (co)limits, but they don't agree with the ∞-category (co)limits. For example, in the ∞-category $N(\mathrm{Ch}_{\mathbb{Q}})^{\mathrm{cf}}$ the pushout of $0 \leftarrow \mathbb{Q} \rightarrow 0$ is $\mathbb{Q}[1]$ (we will prove this below). But $h(N(\mathrm{Ch}_{\mathbb{Q}})^{\mathrm{cf}})$ is equivalent to the (1-)category of $\mathbb{Z}$-graded $\mathbb{Q}$-vector spaces, so the pushout of $0 \leftarrow \mathbb{Q} \rightarrow 0$ in $h(N(\mathrm{Ch}_{\mathbb{Q}})^{\mathrm{cf}}) \cong GrVec_{\mathbb{Q}}$ is zero.

Example 19. Sometimes homotopy categories lack (some) (co)limits. At the end of these notes we show that the (1-)category $h(N(\mathrm{Ch}_{\mathbb{Z}})^{\mathrm{cf}})$ does not have all pushouts. Of course the ∞-category $N(\mathrm{Ch}_{\mathbb{Z}})^{\mathrm{cf}}$ has all limits and colimits in the ∞-categoric sense.

On the other hand, we *do* have the following identification, applicable in many cases such as $(\mathcal{S}\mathrm{et}_{\Delta})_{\mathrm{Quillen}}$, $\mathcal{A}\mathrm{b}_{\Delta}$, Ch_R, $\mathcal{R}\mathrm{ing}_{\Delta}$.

Proposition 20 ([Prop.4.2.4.4, Rem.4.2.4.5])**.** *If $\mathcal{M}$ is a combinatorial*[5] *simplicial model category and I a small simplicial category there is an identification of ∞-categories*

$$N\,\mathrm{Fun}(I, \mathcal{M})^{\mathrm{cf}} \cong \mathrm{Fun}(NI, N\mathcal{M}^{\mathrm{cf}})$$

where $\mathrm{Fun}(I, \mathcal{M})$ *is equipped with either the injective or projective model structures.*

Remark 21. Prop.20 was used in the case $\mathcal{M} = (\mathcal{S}\mathrm{et}_{\Delta})_{\mathrm{Quillen}}$ to construct the Yoneda embedding. It's quite a strong statement. It says that any diagram of ∞-categories $NI \rightarrow N\mathcal{M}^{\mathrm{cf}}$ (where composition only has to be preserved up to homotopy) can be "rectified" to a diagram of 1-categories $I \rightarrow \mathcal{M}$ (where composition has to be preserved on the nose).

So we *can* hope to build limits in the ∞-cateory $N\mathcal{M}^{\mathrm{cf}}$ using limits in $\mathcal{M}$. We will do this next week.

Goal 22. Given a simplicial model category $\mathcal{M}$, build adjoints to the functor of ∞-categories

$$N\mathcal{M}^{\mathrm{cf}} \rightarrow N\,\mathrm{Fun}(I, \mathcal{M})^{\mathrm{cf}}.$$

induced by the constant diagram functor $\mathcal{M} \rightarrow \mathrm{Fun}(I, \mathcal{M})$.

4.4 A homotopy category lacking a pushout

Example 23 (Cf.[Strom, Modern classical homotopy theory, §20.1])**.**
Sometimes homotopy categories don't have (some) colimits. Consider $h(N(\mathrm{Ch}_{\mathbb{Z}})^{\mathrm{cf}})$. We will show that the diagram $0 \leftarrow \mathbb{Z} \rightarrow \mathbb{Z}/2$ (which of course has a colimit in the ∞-category $N(\mathrm{Ch}_{\mathbb{Z}})^{\mathrm{cf}}$) does not have a colimit in the homotopy category $h(N(\mathrm{Ch}_{\mathbb{Z}})^{\mathrm{cf}})$.

We begin with a lemma.

[5] Part of the definition of a combinatorial model category is the definition of a locally presentable category, and I don't want to talk about this. Basically, every model category you will see is combinatorial, so please ignore this for now.

Lemma 24. *Suppose C is an ∞-category, $X : \Lambda^2_0 \star \Delta^0 \to C$ is a pushout diagram with pushout H, and suppose the induced diagram $\Lambda^2_0 \to hC$ admits a pushout P in the 1-category hC. Then P is a retract of H in hC.*

Proof. We have

$$\mathrm{Map}_C(H, Y) \overset{w.e.}{\cong} \mathrm{Map}_C(X_0, Y) \times_{\mathrm{Map}_C(X_2,Y)} \mathrm{Map}_C(X_1, Y)$$

$$\hom_{hC}(P, Y) = \hom_{hC}(X_0, Y) \times_{\hom_{hC}(X_2,Y)} \hom_{hC}(X_1, Y)$$

(the first fibre product is in the ∞-category of spaces $\mathcal{S}$, the second fibre product in the 1-category of sets). Setting $Y = H$ the second equation gives us a morphism $s : P \to H$, and setting $Y = P$, the first equation gives us a morphism $p : H \to P$. To see that $\mathrm{id}_P = p \circ s$ in hC is a diagram chase:

$$\begin{array}{ccccc} s & \hom_{hC}(P, H) & = & \hom_{hC}(X_0, H) \times_{\hom_{hC}(X_2,H)} \hom_{hC}(X_1, H) \\ \downarrow & \downarrow & & \downarrow \\ p \circ s & \hom_{hC}(P, P) & = & \hom_{hC}(X_0, P) \times_{\hom_{hC}(X_2,P)} \hom_{hC}(X_1, P) \end{array}$$

(and one needs to unwrap the definitions a bit). □

Set $C := N(\mathrm{Ch}_{\mathbb{Z}})^{\mathrm{cf}}$. With the above lemma in hand, suppose the diagram $0 \leftarrow \mathbb{Z} \to \mathbb{Z}/2$ has a pushout P in the homotopy category hC. The ∞-categorical pushout in the ∞-category C is[6] $\mathbb{Z}[1]$, so in the homotopy category, P is a retract of $\mathbb{Z}[1]$. Since[7] $\hom_{hC}(\mathbb{Z}[1], \mathbb{Z}[1]) = \mathbb{Z}$ we must have either $P = 0$ or $P = \mathbb{Z}[1]$. Direct calculation shows that both of these are impossible.[8]

$$\begin{aligned} \hom_{hC}(P, \mathbb{Z}[1]) &= \hom_{hC}(0, \mathbb{Z}[1]) \times_{\hom_{hC}(\mathbb{Z},\mathbb{Z}[1])} \hom_{hC}(\mathbb{Z}/2, \mathbb{Z}[1]) \\ &= \{0\} \times_{\{0\}} \mathbb{Z}/2 \\ &= \mathbb{Z}/2, \\ \hom_{hC}(\mathbb{Z}[1], \mathbb{Z}[1]) &= \mathbb{Z}, \\ \hom_{hC}(0, \mathbb{Z}[1]) &= 0. \end{aligned}$$

[6]To see this, one can choose the cofibrant-fibrant model $Q(\mathbb{Z}/2) := (\dots \to 0 \to \mathbb{Z} \overset{2}{\to} \mathbb{Z} \to 0 \to \dots)$ for $\mathbb{Z}/2$ and then the ∞-categorical pushout is the 1-categorical pushout of $0 \leftarrow \mathbb{Z} \to Q(\mathbb{Z}/2)$ in the model category $\mathrm{Ch}_{\mathbb{Z}}$. (This new diagram is not injectively cofibrant because only one morphism is a cofibration, but for diagrams indexed by Λ^2_0 as long as one morphism is cofibrant, pushout preserves weak equivalences, cf. Exercise 2. So this is enough to get the correct pushout.)

[7]If $P \overset{s}{\to} \mathbb{Z}[1]$ and $\mathbb{Z}[1] \overset{p}{\to} P$ are morphisms such that $\mathrm{id}_P = ps$ then we have $(sp)(sp) = sp$; that is, $sp \in \hom_{hC}(\mathbb{Z}[1], \mathbb{Z}[1]) = \mathbb{Z}$ is an idempotent. But the only two idempotents in $\mathbb{Z}$ are 0 and 1.

[8]One can use $Q(\mathbb{Z}/2)$ again for the calculation of $\hom_{hC}(\mathbb{Z}/2, \mathbb{Z}[1])$.