Derived Algebraic Geometry
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The goal of this lecture is to define the ∞-category of spaces (resp. simplicial commutative rings).

$$\mathcal{S} \qquad \text{(resp. } \mathcal{SCR}\text{)}.$$

We will also briefly discuss the Yoneda embedding associated to an ∞-category $C \in \mathcal{C}\mathrm{at}_\infty$,

$$j : C \to \mathrm{Fun}(C^{op}, \mathcal{S}), \qquad \in \mathcal{C}\mathrm{at}_\infty.$$

We will do both of these via an adjunction comparing ∞-categories and simplicial categories

$$\mathfrak{C} : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{C}\mathrm{at}_\Delta : N.$$

3.4 Comparing ∞-categories and simplicial categories

Definition 1 (Cordier 1982, [HTT, §1.1.5])**.** Define $\mathfrak{C}[\Delta^n]$ to be the simplicial category whose objects are elements of $[n] = \{0 < \cdots < n\}$. For $0 \le i, j \le n$ the mapping space is the nerve of the partially ordered set

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = N\left\{ \{i, j\} \subseteq J \subseteq \{i, i{+}1, \dots, j\} \right\}$$

of subsets J containing i, j and contained in $\{i, i{+}1, \dots, j\}$. Composition is induced by

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) \times \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(j, k) \to \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, k)$$

union. Note that $\mathfrak{C}[\Delta^n]$ is functorial in n, cf.[HTT, Def.1.1.5.3], so we obtain a functor

$$\mathfrak{C}[\Delta^-] : \Delta \to \mathcal{C}\mathrm{at}_\Delta$$

The *nerve* of a simplicial category C is the simplicial set, [HTT, Def.1.1.5.5],

$$NC : [n] \mapsto \hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[\Delta^n], C).$$

Exercise 2. Show that $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = N[1]^{j-i-1}$. That is, show that $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ is the $(j{-}i{-}1)$-dimensional simplicial cube.

Remark 3. The zero simplices of $\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j)$ can be interpreted as all of the different ways of writing the morphism $i \to j$ in $N[n]$ as a composition

$$i \to k_1 \to \cdots \to k_n \to j,$$

and the higher simplicies can be interpreted as homotopies between these various compositions.

For example, a morphism of topological spaces

$$X_0 \overset{f_{01}}{\to} X_1$$

gives a 0-simplex in $\operatorname{Sing}\hom(X_0, X_1)$. Given another morphism $X_1 \overset{f_{12}}{\to} X_2$ and a homotopy

$$f_{02} \sim f_{12} \circ f_{01}$$

for some third morphism $X_0 \overset{f_{02}}{\to} X_2$ gives a 1-simplex in $\operatorname{Sing}\hom(X_0, X_2)$ whose faces are $f_{12} \circ f_{01}$ and f_{02}. Continuing in this way leads to two 2-simplices in $\operatorname{Sing}\hom(X_0, X_3)$ which fit together to make a simplicial square whose vertices are the various compositions.

$$\begin{array}{ccc} f_{03} & \longrightarrow & f_{13} \circ f_{01} \\ \downarrow & \searrow & \downarrow \\ f_{23} \circ f_{02} & \longrightarrow & f_{23} \circ f_{12} \circ f_{01} \end{array}$$

Here is the main comparison theorem.

Theorem 4 ([HTT, §2.2], [HTT, Prop.1.1.5.10, Thm.2.2.5.1])**.**

1. *The nerve functor admits a left adjoint*

$$\mathfrak{C} : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{C}\mathrm{at}_\Delta : N.$$

2. *The functor N sends fibrant simplicial categories*[1] *to ∞-categories.*
3. *Both $\mathfrak{C}$ and N both preserve and reflect categorical equivalences.*[2]
4. *Given $C \in \mathcal{C}\mathrm{at}_\infty$, $X, Y \in C_0$, and $* = R, L$, there are isomorphisms in $h\mathcal{S}\mathrm{et}_\Delta$*

$$\hom^*_C(X, Y) \cong \mathrm{Map}_{\mathfrak{C}[C]}(X, Y) \cong \mathrm{Map}^{nec}_C(X, Y).$$

Remark 5.

1. Since the functor $\mathfrak{C}$ is a left adjoint and we know its values on the representables Δ^n, its value on a general simplicial set K is a kind of geometric realisation $\mathfrak{C}[K] = \varinjlim_{([n],f)\in\int K} \mathfrak{C}[\Delta^n]$.[3] This description is usually useless since colimits (for example coequalisers) in $\mathcal{C}\mathrm{at}_\Delta$ are difficult to describe in general. Only in some simple cases (e.g. $\partial\Delta^n$, Λ^n_i) something can be said.
2. In [HTT, Thm.2.2.5.1] categorical equivalences of simplicial sets are *defined* as those morphisms sent to equivalences under $\mathfrak{C}[-]$. So this part of the above theorem is empty in some sense. However, as we saw above, for ∞-categories C, the mapping spaces in $\mathfrak{C}[C]$ can also be computed via other more accessible models.

[1] Recall, a simplicial category if *fibrant* if all Map are Kan complexes.

[2] That is, a morphism f in $\mathcal{C}\mathrm{at}_\infty$ (resp. $\mathcal{C}\mathrm{at}_\Delta$) is a categorical equivalence if and only if $\mathfrak{C}^{nec}(f)$ (resp. $N(f)$) is a categorical equivalence.

[3] For this, we also need to know that $\mathcal{C}\mathrm{at}_\Delta$ admits colimits. This follows from abstract nonsense because it sits in a monadic adjunction $\mathcal{G}\mathrm{r}_\Delta \rightleftarrows \mathcal{C}\mathrm{at}_\Delta$ with the category $\mathcal{G}\mathrm{r}_\Delta$ of simplicial graphs, i.e., graph objects $E \rightrightarrows V$ in $\mathcal{S}\mathrm{et}_\Delta$ such that V is a constant simplicial set. Cf. the Barr-Beck Theorem.

Recall that last time we saw that $\mathcal{S}\mathrm{et}_\Delta$ had a structure of simplicial category.

Definition 6. The *∞-category of spaces* is the nerve of the simplicial category of Kan complexes.

$$\mathcal{S} := N(\mathcal{G}\mathrm{pd}_\infty).$$

Remark 7 ([HTT, §1.2.15])**.** Here we run into Russell's paradox, the set of all sets cannot be a set. There are various ways to resolve this. One way is to choose a Grothendieck universe, or equivalently, a strongly inaccessible cardinal κ. This is a cardinal such that the category $\mathcal{S}\mathrm{et}_\kappa$ of sets of cardinality $< \kappa$ satisfies: if $f : X \to Y$ is a morphism of sets such that $Y \in \mathcal{S}\mathrm{et}_\kappa$ and all $f^{-1}(y) \in \mathcal{S}\mathrm{et}_\kappa$ then $X \in \mathcal{S}\mathrm{et}_\kappa$ and $\{Z \subseteq Y\} \in \mathcal{S}\mathrm{et}_\kappa$. Then we define $\mathcal{S}\mathrm{et}_\Delta$ to be the category of simplicial sets in $\mathcal{S}\mathrm{et}_\kappa$, i.e., $(\mathcal{S}\mathrm{et}_\kappa)_\Delta$. In this way it's not a member of itself.

Remark 8. As in Remark 3, we have:

1. $\mathcal{S}_0$ is the set of (small) ∞-groupoids.
2. $\mathcal{S}_1$ is the set of morphisms between ∞-groupoids.
3. $\mathcal{S}_2$ is the set of simplicial homotopies[4] from a morphism to a composition (in $\mathcal{S}\mathrm{et}_\Delta$) of morphisms.
4. $\mathcal{S}_3$ is the set of simplicial squares in $\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(X, Y)$ whose corners are various compositions, edges are various homotopies, and 2-simplicies are homotopies between homotopies.
5. etc.

3.5 Yoneda embedding

We don't really need the Yoneda embedding just yet, but we use it as motivation for Model Categories.

One of the many uses of simplicial categories is to define the Yoneda embedding in the ∞-category context, and prove that it is fully faithful.

For simplicial categories C (and enriched categories in general), one can put a structure of simplicial category on the category of simplicial functors $F : C \to \mathcal{S}\mathrm{et}_\Delta$ by setting[5]

$$\mathrm{Map}(F, F') = \mathrm{eq}\left(\prod_{X \in Ob\ C} \mathrm{Map}(F(X), F'(X)) \rightrightarrows \prod_{X,Y \in Ob\ C} \mathrm{Map}(\mathrm{Map}_C(X, Y), \mathrm{Map}(F(X), F'(Y)) \right).$$

Then it is a formal consequence that

$$C \to \mathrm{Fun}(C, \mathcal{S}\mathrm{et}_\Delta);$$
$$X \mapsto \mathrm{Map}(-, X)$$

[4] I.e., a morphism $X \times \Delta^1 \to Y$ in $\mathcal{S}\mathrm{et}_\Delta$.

[5] Cf. [1982 Max Kelly, Basic Concepts of Enriched Category Theory, §2.2]

is a fully faithful simplicial functor of simplicial categories [1982 Max Kelly, Basic Concepts of Enriched Category Theory, §2.4].

Exercise 9. Show that if all Map sets then the above equaliser is the usual definition of a natural transformation.

For ∞-categories, the situation is a little more complicated. Given a $C \in \mathcal{C}\mathrm{at}_\infty$, a first attempt at Yoneda would be the morphism $C \to N\,\hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[C]^{op}, \mathcal{S}\mathrm{et}_\Delta)$ adjoint to the $\mathfrak{C}[C] \to \hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[C]^{op}, \mathcal{S}\mathrm{et}_\Delta)$ from above. Unfortunately, $N\,\hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[C]^{op}, \mathcal{S}\mathrm{et}_\Delta)$ is not equivalent to $\mathrm{Map}(C^{op}, N(\mathcal{G}\mathrm{pd}_\infty))$. However, it turns out that $\mathrm{Map}(C^{op}, N(\mathcal{G}\mathrm{pd}_\infty))$ is equivalent to the nerve of a subcategory of $\hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[C]^{op}, \mathcal{S}\mathrm{et}_\Delta)$, the subcategory $\hom_{\mathcal{C}\mathrm{at}_\Delta}(\mathfrak{C}[C]^{op}, \mathcal{S}\mathrm{et}_\Delta)^{cf}$ of cofibrant-fibrant objects,[6] [HTT, Prop.5.1.1.1]. We will explain this below. For now, we construct the Yoneda embedding.

Construction 10 ([HTT, §5.1.3])**.** Let C be an ∞-category. The assignment

$$\underbrace{(X,Y)}_{\in C_0^{op} \times C_0} \mapsto \mathrm{Sing}\,|\,\mathrm{Map}_{\mathfrak{C}[C]}(X,Y)|$$

defines a simplicial functor

$$\mathfrak{C}[C]^{op} \times \mathfrak{C}[C] \to \mathcal{G}\mathrm{pd}_\infty \qquad\qquad \in \mathcal{C}\mathrm{at}_\Delta$$

which has three associated functors

$$\begin{aligned} \mathfrak{C}[C^{op} \times C] &\to \mathcal{G}\mathrm{pd}_\infty && \in \mathcal{C}\mathrm{at}_\Delta \\ C^{op} \times C &\to N(\mathcal{G}\mathrm{pd}_\infty) && \in \mathcal{C}\mathrm{at}_\infty \\ C &\to \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(C^{op}, N(\mathcal{G}\mathrm{pd}_\infty)) && \in \mathcal{C}\mathrm{at}_\infty. \end{aligned}$$

The first one comes from composition with the canonical functor $\mathfrak{C}[C^{op} \times C] \to \mathfrak{C}[C]^{op} \times \mathfrak{C}[C]$ of simplicial categories, and the latter two functors are obtained by adjunction.

Definition 11. The functor

$$j : C \to \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}\Big(C^{op}, N(\mathcal{G}\mathrm{pd}_\infty)\Big)$$

constructed above is the *Yoneda embedding.*

Exercise 12. Let C be an ∞-category and let $X, Y \in C_0$ be objects. Show that $j(Y)(X)$ is precisely $\mathrm{Sing}\,|\,\mathrm{Map}_{\mathfrak{C}[C]}(X,Y)|$.

Proposition 13 ([HTT, Prop.5.1.3.1])**.** *The Yoneda embedding*

$$j : C \to \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(C^{op}, N(\mathcal{G}\mathrm{pd}_\infty))$$

is fully faithful.

[6]With respect to the projective model structure.

Remark 14.

1. The proof makes heavy use of the fact that the adjunction $\mathfrak{C} : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{C}\mathrm{at}_\Delta : N$ is a *Quillen equivalence* (when $\mathcal{S}\mathrm{et}_\Delta$ is given the Joyal model structure).
2. As we mentioned above, another point of the proof that $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{S}\mathrm{et}_\Delta)$ is equivalent to (the nerve of) a subcategory of $\mathrm{Fun}(\mathfrak{C}[C], \mathcal{S}\mathrm{et}_\Delta)$ consisting of "nice" objects (analogous to $\mathrm{Ch}_R^{free} \subseteq \mathrm{Ch}_R$): The subcategory of cofibrant-fibrant objects.

We will explain these two points (Quillen adjunctions, and cofibrant-fibrant objects) below.

3.6 Localisation

Our second motivation for Model Categories is to calculating localisations of ∞-categories.

We've seen a number of settings (chain complexes, Kan complexes, ∞-categories, simplicial categories) where we have a category of objects that we would like to consider up to equivalence. Here is a way of formalising this.

Exercise 15.

1. Given an ∞-category C, define $(KC)_n$ to be the set of $\sigma \in C_n$ such that for all $u : [1] \to [n]$ in Δ, the morphism $u^*\sigma \in C_1$ is an equivalence. Show that these form a sub-∞-category of C.
2. Let $F : C \to C'$ be a morphism of ∞-categories. Show that if $f \in C_1$ is an equivalence, then so is $F(f)$.
3. Show that for any ∞-groupoid G, we have $\hom(G, KC) = \hom(G, C)$.
4. Let $\mathcal{G}\mathrm{pd}_\infty \subseteq \mathcal{C}\mathrm{at}_\infty$ denote the full subcategory of ∞-groupoids. Deduce that K is right adjoint to the canonical inclusion

$$\mathcal{C}\mathrm{at}_\infty \supset \mathcal{G}\mathrm{pd}_\infty \quad (K : \mathcal{C}\mathrm{at}_\infty \to \mathcal{G}\mathrm{pd}_\infty)$$

Definition 16. Let $W \subseteq C$ be a subcategory of an ∞-category. Write $\mathrm{Map}(C, D)^W$ for the pullback / preimage

$$\begin{array}{ccc} \mathrm{Map}(C, D)^W & \subseteq & \mathrm{Map}(C, D) \\ \downarrow & & \downarrow \\ \mathrm{Map}(W, KD) & \subseteq & \mathrm{Map}(W, D) \end{array}$$

That is, the space of functors which send all morphisms of W to equivalences in D. A morphism

$$L : C \to C[W^{-1}]$$

of ∞-categories is said to be a *localisation at* W if for every $D \in \mathcal{C}\mathrm{at}_\infty$, composition with L induces an equivalence of ∞-groupoids

$$K\,\mathrm{Map}(C[W^{-1}], D) \to K\,\mathrm{Map}(C, D)^W.$$

Exercise 17. Using the following two facts:

1. For any simplicial set K, the canonical morphism $K \to \mathrm{Sing}\,|K|$ is an inclusion and a weak equivalence.
2. If $K \to K'$ is an inclusion and a weak equivalence of simplicial sets, and X is a Kan complex, then $\mathrm{Map}(K', X) \to \mathrm{Map}(K, X)$ is a weak equivalence of Kan complexes.

Show that for any ∞-category C, the canonical morphism $C \to \mathrm{Sing}\,|C|$ is a localisation at C. That is, show

$$C[C^{-1}] = \mathrm{Sing}\,|C|.$$

Example 18. It is a theorem that the localisation of $\mathcal{S}\mathrm{et}_\Delta$ at the class of (Quillen) weak equivalences is the category of spaces

$$\mathcal{S}\mathrm{et}_\Delta[W^{-1}] = N(\mathcal{G}\mathrm{pd}_\infty).$$

Remark 19. The localisation $C[W^{-1}]$ does exist in general. There are various representatives for it. A "traditional" one is via the *Hammock localisation* of Dwyer and Kan, although as a calculational tool this is not very useful. We often have a more useful representative coming from the theory of model categories, generalising Example 18.

3.7 Simplicial model categories

Finally we reach the promised material on Model Categories.

Definition 20 ([Hirschhorn, Def.7.1.3, Def.9.1.6)**.** , [HTT, Def.A.3.1.5.]] A *simplicial model category* is a simplicial category $\mathcal{M}$ equipped with three subcategories $\mathcal{C}, \mathcal{W}, \mathcal{F} \subseteq \mathcal{M}_0$ of the underlying classical category, whose morphisms are respectively called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying seven axioms which we will introduce as we need them.

One of the axioms is:

(M1) $\mathcal{M}_0$ admits all limits and colimits.

In particular, $\mathcal{M}_0$ has an initial object $\varnothing$ and a terminal object $*$. An object C is called *cofibrant* if $\varnothing \to C$ is a cofibration. An object F is called *fibrant* if $F \to *$ is a fibration. Another axiom is:

(M5) Every morphism $X \to Y$ in $\mathcal{M}_0$ has two functorial factorisations: as compositions

$$X \xrightarrow{i} Y' \xrightarrow{q} Y$$
$$X \xrightarrow{j} X' \xrightarrow{p} Y$$

where $i \in \mathcal{C}$, $q \in \mathcal{W} \cap \mathcal{F}$, and $j \in \mathcal{C} \cap \mathcal{W}$, $p \in \mathcal{F}$.

Consequently, for every object X, we can find a diagram of the form

$$\begin{array}{ccccc}
\varnothing & = & \varnothing & & \\
{\scriptstyle \in\mathcal{C}}\downarrow & & \downarrow{\scriptstyle \in\mathcal{C}} & & \\
RQX & \xleftarrow{\in\mathcal{W}\cap\mathcal{C}} & QX & \xrightarrow{\in\mathcal{W}\cap\mathcal{F}} & X \\
{\scriptstyle \in\mathcal{F}}\downarrow & & & & \\
* & & & &
\end{array}$$

That is, there is a span $RQX \overset{\in\mathcal{W}}{\leftarrow} QX \overset{\in\mathcal{W}}{\rightarrow} X$ of weak equivalences such that RQX is both cofibrant and fibrant. The full simplicial sub-category of fibrant-cofibrant objects is denoted by $\mathcal{M}^\circ$ on [HTT, pg.808] but we will write $\mathcal{M}^{\mathrm{cf}}$, following Dwyer-Kan (Calculating simplicial localisations), Hirschhorn [Model categories and their localizations], Quillen [Homotopical algebra] (the latter two put cf at the bottom).

Our main interest in simplicial model categories are as a tool to construct "large" ∞-categories.

Theorem 21 ([])**.** *Suppose that $\mathcal{M}$ is a simplicial model category. Then via the functor $RQ : \mathcal{M}_0 \to \mathcal{M}^{\mathrm{cf}}$ mentioned above, we have*

$$N(\mathcal{M}_0)[W^{-1}] = N(\mathcal{M}^{\mathrm{cf}}).$$

Remark 22. We would like to say that $N(\mathcal{M}_0) \to N(\mathcal{M}^{\mathrm{cf}})$ induces a commutative triangle

$$\begin{array}{ccc}
 & \left\{\begin{array}{c}\text{simplicial}\\ \text{model}\\ \text{categories}\end{array}\right\} & \\
{\scriptstyle (-)^{\mathrm{cf}}}\swarrow & & \searrow{\scriptstyle N(-)_0[\mathcal{W}^{-1}]} \\
\mathcal{C}\mathrm{at}_\Delta & \xrightarrow[N]{\qquad\qquad} & \mathcal{C}\mathrm{at}_\infty
\end{array}$$

But $(-)^{cf}$ is not functorial in $\mathcal{M}$, and the localisations $(-)[W^{-1}]$ are only well-defined up to categorical equivalence.

3.8 Examples

We cover the following examples here.

1. Topological spaces.
2. Simplicial sets with the Quillen model structure.
3. Simplicial modules.
4. Chain complexes.
5. Simplicial algebras.
6. Diagram categories.
7. Simplicial sheaves on the small Zariski site.

Remark 23. The fact that the below definitions actually define structures of simplicial model categories are theorems.

Example 24 (Topological spaces, [Hirschhorn, Thm.7.10.10, Exm.9.1.15])**.** The category of compactly generated topological spaces[7] is a simplicial model category.

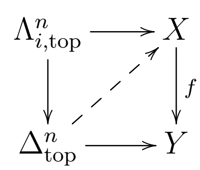
- *Mapping spaces.* The mapping space $\mathrm{Map}_{\mathrm{Top}^{\mathrm{cg}}}(X, Y)$ has n-simplicies the set

$$\mathrm{Map}_{\mathrm{Top}^{\mathrm{cg}}}(X, Y)_n = \hom_{\mathrm{Top}}(X \times \Delta^n_{\mathrm{top}}, Y).$$

  Composition is defined using the diagonal $\Delta^n_{\mathrm{top}} \to \Delta^n_{\mathrm{top}} \times \Delta^n_{\mathrm{top}}$.
- *Weak equivalences.* A weak equivalences are weak equivalences. That is, morphisms which induce isomorphisms on all homotopy groups.
- *Cofibrations.* The cofibrations are those morphisms which are a retraction of a transfinite composition of morphisms of the form

$$X \to \Delta^n_{\mathrm{top}} \amalg_{\partial\Delta^n_{\mathrm{top}}} X.$$

- *Fibrations.* Fibrations are Serre fibrations. That is, morphisms f such that for all $0 \leq i \leq n$, and all commutative squares, there exists a diagonal morphism making two commutative triangles.



- *Fibrant-cofibrant objects.* Every object is fibrant, so objects of $\mathrm{Top}^{\mathrm{cf}}$ are retracts of CW complexes.

Example 25 (Simplicial sets)**.** The category $\mathcal{S}\mathrm{et}_\Delta$ of simplicial sets with the *Quillen model structure* is a simplicial model category.

- *Mapping spaces.* Mapping spaces are the $\mathrm{Map}(-,-)$ we introduced previously.
- *Weak equivalences.* A weak equivalences are weak equivalences. That is, morphisms whose geometric realisation is a weak equivalences.
- *Cofibrations.* The cofibrations are monomorphisms.
- *Fibrations.* Fibrations are Kan fibrations. That is, morphisms satisfying the simplicial version of the Serre fibration property described above.
- *Fibrant-cofibrant objects.* Every simplicial set is cofibrant, so objects of $\mathcal{S}\mathrm{et}^{\mathrm{cf}}_\Delta$ are precisely the Kan complexes.

[7] A topological space is *compactly generated* if it is a colimit (in Top) of compact Hausdorff spaces. The main interest in compactly generated topological spaces is that every compact Hausdorff space is compactly generated, and the category of compactly generated topological spaces has a well behaved internal hom. [Escardó, Lawson, Simpson, "Comparing Cartesian closed categories of (core) compactly generated spaces" Lem.3.2(v), Def.3.3(iii), Thm.3.6].

Definition 26. Let $\mathcal{M}$ be a simplicial model category. The simplicial nerve

$$N(\mathcal{M}^{cf})$$

is the ∞-category associated to $\mathcal{M}$.

Remark 27. One may ask what is the point of having non-cofibrant-fibrant objects at all. This is essentially the same as asking, why do we ever consider complexes of sheaves which are not injective. The point is that many naturally occurring operations (such as (co)limit) do not preserve cofibrancy / fibrancy, and often it is more natural to describe a non-cofibrant / non-fibrant object.

For example, the simplicial spheres are very naturally described as $\Delta^n/\partial\Delta^n$, but these are very far from being Kan complexes, and so are not in $\mathcal{S}$ according to the above definition!

Example 28 (Simplicial modules)**.** Let R be a ring. The category $R\text{-Mod}_\Delta$ of simplicial R-modules, i.e., functors $\Delta \to R\text{-Mod}$ with values in the category R-Mod of R-modules inherits the structure of a simplicial model category from $\mathcal{S}\text{et}_\Delta$ via the free R-module / forgetful adjunction[8]

$$R : \mathcal{S}\text{et}_\Delta \rightleftarrows R\text{-Mod}_\Delta : U.$$

The simplicial model structure on $R\text{-Mod}_\Delta$ is set up so that R preserves cofibrations, U preserves fibrations and preserves and detects weak equivalences, and U is a morphism of simplicial categories.

- *Mapping spaces.* Define

$$\text{Map}_{R\text{-Mod}_\Delta}(M, N)_n = \hom_{R\text{-Mod}_\Delta}(M \otimes_R R\Delta^n, N)$$

where $\otimes_R$ is defined termwise[9] Composition is inherited from the simplicial category structure of $\mathcal{S}\text{et}_\Delta$.
- *Weak equivalences (resp. fibrations).* Weak equivalences (resp. fibrations) are detected by U. Explicitly, $f : M \to N$ is a weak equivalence (resp. fibration) if and only if Uf is a weak equivalence (resp. fibration) in $\mathcal{S}\text{et}_\Delta$.
- *Cofibrations.* The cofibrations are those morphisms which are a retraction of a transfinite composition of morphisms of the form

$$X \to R\Delta^n \amalg_{R\partial\Delta^n} X.$$

It turns out that:

($\mathcal{W}$) $f : M \to N$ is a weak equivalence if and only if the associated[10] morphism of chain complexes

$$CM \to CN$$

is a quasi-isomorphism. Here, $(CM)_n = M_n$ with differentials $\sum(-1)^i d_i : (CM)_n \to (CM)_{n-1}$.

[8] U is probably for "U"nderlying set.

[9] I.e., $(A \otimes_R B)_n = A_n \otimes_R B_n$.

[10] We will discuss this later in the Dold-Kan correspondence. See also [May, Simplicial methods, Chap.V].

($\mathcal{F}$) $f : M \to N$ is a fibration if and only if

$$M_n \to (\pi_0 M \times_{\pi_0 N} N)_n$$

is surjective for all $n \geq 0$, where $\pi_0 M = \operatorname{coker}(M_1 \overset{d_0 - d_1}{\to} M_0)$ (and similar for $\pi_0 N$). Cf. [Stacks project, Tag 08P0].

($\mathcal{C}$) $f : M \to N$ is a cofibration if and only if each $M_n \to N_n$ is split surjective with projective cokernel. That is,

$$N_n = M_n \oplus P_n$$

for some projective module P_n. In particular, a simplicial module M is cofibrant if and only if each M_n is a projective module.

- *Fibrant-cofibrant objects.* Every object is fibrant, [May, Simplicial objects in algebraic topology, Thm.17.1], [Stacks project, Tag 08NZ], so fibrant-cofibrant objects are term-wise projective.

Warning 29. Some authors allow $\varnothing = \Lambda^0_0 \to \Delta^0 = *$ in the definition of Kan fibrations and some do not. This has the consequence that for some authors, $\pi_0 X \to \pi_0 Y$ must be surjective for Kan fibrations $X \to Y$, and for some authors there is no condition on π_0. We take the point of view that since $\varnothing \to \Delta^0$ is not a weak equivalence, it should not appear in the definition of a Kan fibration, so we do not require surjectivity on π_0.

Exercise 30. The sixth axiom for simplicial model categories is the following.

(M6) For every $X, Y \in Ob\ \mathcal{M}$ and $K \in \mathcal{S}\mathrm{et}_\Delta$ there are objects $X \otimes K$ and Y^K and isomorphisms

$$\mathrm{Map}_{\mathcal{M}}(X \otimes K, Y) \cong \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}_{\mathcal{M}}(X, Y^K)$$

which are functorial in X, Y, K.

Show that in R-Mod_Δ we must have $X \otimes K = X \otimes_R RK$ where the left $\otimes$ is the one from Axiom M6, the right one is induced by $- \otimes_R -$ on R-Mod. Hint.[11] Hint.[12] Similarly, show that we must have $Y^K = \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, UY)$ equipped with its canonical structure of simplicial R-module.

Example 31 (Chain complexes.[HA, Def.1.2.3.1])**.** Let R be a ring. A chain complex of R-modules is a sequence of morphisms of R-modules

$$M_\bullet = (\cdots \to M_2 \overset{d(2)}{\to} M_1 \overset{d(1)}{\to} M_0 \overset{d(0)}{\to} M_{-1} \to \dots)$$

[11] In general, if $\hom_C(A, -) \cong \hom_C(B, -)$ then $A \cong B$ via a unique isomorphism. So it suffices to show that $X \otimes_R RK$ corepresents the appropriate functor.

[12] Show that for a general for a set $K \in \mathcal{S}\mathrm{et}$ and R-modules $M, N \in R$-Mod we have $\hom_{R\text{-Mod}}(M \otimes_R RK, N) = \hom_{\mathcal{S}\mathrm{et}}(K, \hom_{R\text{-Mod}}(M, N))$, then upgrade this to a simplicial version.

such that $d(n-1) \circ d(n) = 0$ for every n. A morphism of chain complexes $M_\bullet \to N_\bullet$ is a sequence of morphisms $M_n \to N_n$ forming commutative squares. The tensor product of two chain complexes is

$$(M_\bullet \otimes N_\bullet)_n = \bigoplus_{i+j=n} (M_i \otimes N_j)$$

with differentials $d(m \otimes n) = (dm) \otimes n + (-1)^{|n|} m \otimes dn$ where $|n|$ means the degree of n (i.e., $n \in N_{|n|}$). There exists a canonical adjunction

$$N : R\text{-Mod}_\Delta \rightleftarrows \mathrm{Ch}_R : DK$$

where

$$(NM)_n = \bigcap_{i=1}^{n} \ker(d_i : M_n \to M_{n-1})$$

for $n \geq 0$ and $(NM)_n = 0$ for $0 > n$. The differentials of NM are the $d_0 : M_n \to M_{n-1}$ of M. See[HA, Construction 1.2.3.5][13] for details about DK. One can show that given a linearly ordered set J,

$$NR\Delta^J = \left(\dots \to 0 \to \bigwedge^{|J|} R^{\oplus J} \to \dots \to \bigwedge^{2} R^{\oplus J} \to \bigwedge R^{\oplus J} \to 0 \to \dots \right)$$

is the alternating algebra $\wedge^\bullet(R^{\oplus J})$ (without the right-most term) considered as a chain complex. Lets write

$$\Delta^n_{\mathrm{Ch}} := NR\Delta^J.$$

Note that this is functorial in J and defines a functor

$$\Delta \to \mathrm{Ch}_R; \qquad J \mapsto \Delta^J_{\mathrm{Ch}}.$$

As a left adjoint, NR preserves colimits, and it's not too difficult to use the Δ^n_{Ch} to describe $NR(K)$ for simple $K \in \mathcal{S}\mathrm{et}_\Delta$ (e.g., Λ^n_i, $\partial\Delta^n$, $N[1]^n$,...).

- *Mapping spaces.* The mapping space of two complexes is defined as

  $$\mathrm{Map}_{\mathrm{Ch}_R}(M, N)_n := \hom_{\mathrm{Ch}}(M \otimes \Delta^n_{\mathrm{Ch}}, N).$$

  Composition is a little fiddly. It uses the Alexander-Whitney map

  $$\Delta^n_{\mathrm{Ch}} \to \Delta^n_{\mathrm{Ch}} \otimes \Delta^n_{\mathrm{Ch}}$$

  defined using shuffles.
- *Weak equivalences.* A weak equivalences are quasi-isomorphisms. That is, morphisms $M_\bullet \to N_\bullet$ such that $H_n(M) \to H_n(N)$ is an isomorphism for all n, where

  $$H_n(M) = \frac{\ker(M_n \to M_{n-1})}{\mathrm{im}(M_{n+1} \to M_n)},$$

[13] Or any one of the million other references.

- *Fibrations.* Fibrations are morphisms $M_\bullet \to N_\bullet$ such that each
$$M_n \to N_n$$
is surjective.
- *Cofibrations.* Cofibrations are those morphisms which are a retraction of a transfinite composition of morphisms of the form
$$M \to D^n \amalg_{S^{n-1}} M$$
where
$$D^n = (\dots \to \underset{n}{R} \overset{\text{id}}{\to} \underset{n-1}{R} \to 0 \to \dots)$$
$$S^{n-1} = (\dots \to \underset{n}{0} \to \underset{n-1}{R} \to 0 \to \dots)$$
From this description one can see that if $M \to N$ is a cofibration, then each $M_n \to N_n$ is injective with projective cokernel. That is, $N_n = M_n \oplus P_n$ for some projective P_n. The converse is true if $\operatorname{coker}(M \to N)$ is bounded on the right, [Hover, Model Categories, Lem.2.3.6, Prop.2.3.9].
- *Fibrant-cofibrant objects.* Every object is fibrant, so objects of $\mathrm{Ch}_R^{\mathrm{cf}}$ are precisely the cofibrant ones. In particular, if $M \in \mathrm{Ch}_R^{\mathrm{cf}}$ then each M_n is projective. Conversely, if M is a bounded to the right complex of projectives, then $M \in \mathrm{Ch}_R^{\mathrm{cf}}$.

Exercise 32. For $n > 0$ define
$$D^n = (\dots \to \underset{n}{R} \overset{\text{id}}{\to} \underset{n-1}{R} \to 0 \to \dots)$$
$$S^{n-1} = (\dots \to \underset{n}{0} \to \underset{n-1}{R} \to 0 \to \dots)$$
where the degrees are written underneath.
1. Show that $\Delta^n_{\mathrm{Ch}} \cong NR(\Lambda^n_j) \oplus D^n$. Hint.[14]
2. Show that $NR(\partial\Delta^n) = NR(\Lambda^n_j) \oplus S^{n-1}$. Hint.[15]
3. Show that the canonical morphism $NR(\partial\Delta^n) \to \Delta^n_{\mathrm{Ch}}$ is isomorphic to the direct sum $NR(\Lambda^n_j) \oplus S^{n-1} \to NR(\Lambda^n_j) \oplus D^n$.

Example 33 (Simplicial commutative rings (cf. simplicial modules))**.** The category $\mathcal{R}\mathrm{ing}_\Delta$ of *simplicial commutative rings* is the category of functors $\Delta \to \mathcal{R}\mathrm{ing}$ where $\mathcal{R}\mathrm{ing}$ is the category of commutative rings with unit. Similar to the adjunction $\mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{A}\mathrm{b}_\Delta$ there is an adjunction
$$\mathbb{Z}[-] : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{R}\mathrm{ing}_\Delta : U$$
where $\mathbb{Z}[K]_n$ is the polynomial ring with one variable for each $k \in K_n$ (here $K \in \mathcal{S}\mathrm{et}_\Delta$).

[14] The canonical top element $e_0 \wedge \dots \wedge e_n$ of Δ^n_R defines a morphism $D^n \to \Delta^n_{\mathrm{Ch}}$ which admits a retraction with kernel $NR\Lambda^n_j$.

[15] This is induced by the previous step.

- *Mapping spaces.* Define

$$\mathrm{Map}_{\mathcal{R}\mathrm{ing}_\Delta}(A,B)_n = \hom_{\mathcal{R}\mathrm{ing}_\Delta}(A[\Delta^n], B)$$

  where given a simplicial ring A and simplicial set K we write $A[K]_n = A_n[K_n]$. That is, the ring of polynomials with coefficients in the ring A_n and one variable for each element of K_n. Composition is inherited from the simplicial category structure of $\mathcal{S}\mathrm{et}_\Delta$.
- *Weak equivalences (resp. fibrations).* Weak equivalences (resp. fibrations) are detected by U. Explicitly, $f: A \to B$ is a weak equivalence (resp. fibration) if and only if Uf is a weak equivalence (resp. fibration) in $\mathcal{S}\mathrm{et}_\Delta$.
- *Cofibrations.* The cofibrations are those morphisms which are a retraction of a transfinite composition of morphisms of the form

$$A \to \mathbb{Z}[\Delta^n] \otimes_{\mathbb{Z}[\partial\Delta^n]} A.$$

  As in the case of simplicial modules:

  ($\mathcal{W}$) $f: A \to B$ is a weak equivalence if and only if the associated morphism of chain complexes

$$CA \to CB$$

  is a quasi-isomorphism. Here, $(CA)_n = A_n$ with differentials $\sum(-1)^i d_i : (CA)_n \to (CA)_{n-1}$.

  ($\mathcal{F}$) $f: A \to B$ is a fibration if and only if

$$A_n \to (\pi_0 A \times_{\pi_0 B} B)_n$$

  is surjective for all $n \geq 0$, where $\pi_0 A = \mathrm{coker}(A_1 \overset{d_0 - d_1}{\to} A_0)$ (and similar for $\pi_0 B$). Cf. [Stacks project, Tag 08P0].

  ($\mathcal{C}$) Let $f: A \to B \in \mathcal{R}\mathrm{ing}_\Delta$. If there exist sets I_k such that

$$B_n = A_n[\sqcup_{[n]\twoheadrightarrow[k]} I_k]$$

  then f is a cofbration. Conversely, every cofibration is a retract of a morphism of this form.[16] Here, given a ring R and a set X, we write $R[X]$ for the ring of polynomials with coefficients in R and one variable for each $x \in X$. The disjoint union is over surjections in Δ.
- *Fibrant-cofibrant objects.* Every object is fibrant,[17] so fibrant-cofibrant simplicial rings are retracts of simplical rings A such that $A_n = \mathbb{Z}[\sqcup_{[n]\twoheadrightarrow[k]} I_k]$ for some sets I_k.

Remark 34.

1. The above also works for simplicial algebras over some $R \in \mathcal{R}\mathrm{ing}$. In this case, we should replace the polynomial rings $A_n[\sqcup_{[n]\twoheadrightarrow[k]} I_k]$ with symmetric algebras $A_n \otimes (\bigotimes_{[n]\twoheadrightarrow[k]} Sym(P_k))$ for some set of projective modules P_k.

[16] [Goerss, Schemmerhorn, Model Categories and simplicial methods, Prop.4.21]
[17] [May, Simplicial objects in algebraic topology, Thm.17.1], [Stacks project, Tag 08NZ]

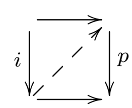
Exercise 35. Suppose that $K \in \mathcal{S}\mathrm{et}_\Delta$ and $A \in \mathcal{R}\mathrm{ing}_\Delta$. Set $B_n = A_n[K_n]$. Show that the B_n form a simplicial ring, and the canonical morphism $A \to B$ is a cofibration. Give an example of a cofibration which is not of this form.

Example 36 (Diagram categories)**.** Suppose that I is a small 1-category and consider the 1-category of diagrams $\mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)$. This category has two structures of simplicial model category: the projective, and injective model structures.

- *Mapping spaces.* In both structures mapping spaces are defined using the constant diagram functor $\gamma : \mathcal{S}\mathrm{et}_\Delta \to \mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)$. Given two diagrams $X, Y : I \to \mathcal{S}\mathrm{et}_\Delta$ we define
$$\mathrm{Map}(X, Y)_n = \hom(X \times \gamma\Delta^n, Y).$$
- *Weak equivalences.* In both structures, weak equivalences are determined object-wise. That is, a natural transformation $X \to Y$ is a weak equivalence if and only if $X_i \to Y_i$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ for each $i \in I$.
- *Cofibrations.* In the injective model structure, cofibrations are defined termwise. That is, a natural transformation $X \to Y$ is an injective cofibration if and only if $X_i \to Y_i$ is a cofibration in $\mathcal{S}\mathrm{et}_\Delta$ for each $i \in I$.
- *Fibrations.* In the projective model structure, fibrations are defined termwise. That is, a natural transformation $X \to Y$ is a projective fibration if and only if $X_i \to Y_i$ is a fibration in $\mathcal{S}\mathrm{et}_\Delta$ for each $i \in I$.
- *Fibrant-cofibrant objects.* The cofibrant-fibrant objects in both structures are in general a little difficult to describe. In special cases something can be said. We will see this more in the section on derived (co)limits.

Exercise 37. The third axiom of model categories is:

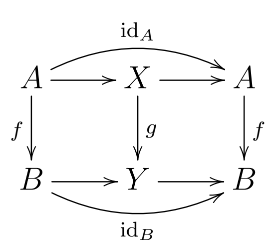
(M4) For every $i \in \mathcal{C}, p \in \mathcal{W} \cap \mathcal{F}$ (resp. $i \in \mathcal{C} \cap \mathcal{W}$, $p \in \mathcal{F}$) and every commutative square



there exists a diagonal morphism making two commutative triangles.

Show that the converse is true. If a morphism i (resp. p) satisfies the lifting property with respect to every $p \in \mathcal{W} \cap \mathcal{F}$ (resp. $i \in \mathcal{C} \cap \mathcal{W}$) then it is a cofibration (resp. fibration). To do this, use axioms (M3) and (M5) where (M3) is:

(M3) The class of morphisms in $\mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$) is closed under retracts. That is, if $g \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$) and there exists a commutative diagram of the form



then $f \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$).

Remark 38. By Exercise 37, the projective cofibrations and injective fibrations in $\mathrm{Fun}(I, \mathcal{S}\mathrm{et}_\Delta)$ are determined by the lifting property.

Remark 39. Example 36 works much more generally, cf.[HTT, Prop.A.3.3.2]. We can replace $\mathcal{S}\mathrm{et}_\Delta$ with any other "nice" model category, and there is also an enriched version, valid for functors between enriched categories. In particular, simplicial presheaves on a simplicial category. This is the model category we mentioned above to deal with the Yoneda embedding.

Example 40 (Zariski sheaves. Cf.[Brown, Gersten], [Joyal], [Dugger, Hollander, Isakson])**.**
Suppose that X is a topological space. Let X_{Zar} be the category associated to the partially ordered set of open subsets $U \subseteq X$. Consider the category $\mathrm{PSh}(X_{\mathrm{Zar}}, \mathcal{S}\mathrm{et}_\Delta)$ of functors $X_{\mathrm{Zar}}^{op} \to \mathcal{S}\mathrm{et}_\Delta$. We have seen above that this has an injective and projective model structure. In addition to these, there are "local" versions: the *local projective* model structure and the *local injective* model structure.

- *Mapping spaces.* Mapping spaces in $\mathrm{PSh}(X_{\mathrm{Zar}}, \mathcal{S}\mathrm{et}_\Delta)$ are those from $\mathrm{Fun}(X_{\mathrm{Zar}}^{op}, \mathcal{S}\mathrm{et}_\Delta)$ described above.
- *Weak equivalences.* A morphism $F \to G$ is a weak equivalence (in either of the local model structures) if for every $x \in X$, the morphism of stalks[18] $F_x \to G_x$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$.
- *Cofibrations.* The local injective (resp. local projective) cofibrations are the same as the injective (resp. projective) fibrations. That is, $\mathcal{C}_{inj} = \mathcal{C}_{loc.inj}$, and $\mathcal{C}_{inj} = \mathcal{C}_{loc.proj}$.
- *Fibrations.* The fibrations are defined by a lifting property.
- *Fibrant-cofibrant objects.* The projective cofibrant-fibrant objects, $\mathrm{PSh}(X_{\mathrm{Zar}}, \mathcal{S}\mathrm{et}_\Delta)^{cf}_{loc.proj}$, are those presheaves F such that all $F(U)$ are fibrant (i.e., Kan complexes) and for every covering $\{U_i \to U\}_{i \in I}$, the canonoical map

  $$F(X) \to \mathrm{holim}_{n \in \Delta} \prod_{i \in I^{n+1}} F(U_{i_0} \cap \cdots \cap U_{i_n})$$

  is a weak equivalence. If X is a Zariski topological space, a presheaf is fibrant if each $F(U)$ is fibrant and for each $U, V \subseteq X$ we have

  $$F(X) = F(U) \overset{h}{\times}_{F(U \cap V)} F(V).$$

Remark 41. We will discuss homotopy limits (such as holim and $\overset{h}{\times}$) and homotopy colimits next time.

3.9 Model category axioms

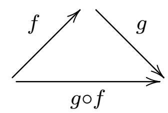
Here we collect all the model category axioms together.

[18]Recall that $F_x := \varinjlim_{x \in U} F(U)$.

Definition 42. A *simplicial model category* is a simplicial category $\mathcal{M}$ equipped with three subcategories $\mathcal{C}, \mathcal{W}, \mathcal{F} \subseteq \mathcal{M}_0$ of the underlying classical category, whose morphisms are respectively called *weak equivalences*, *cofibrations*, and *fibrations*, satisfying the following seven axioms.

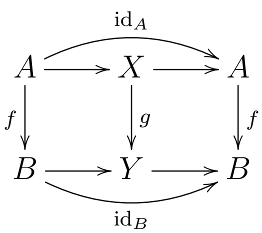
(M1) *Completeness.* $\mathcal{M}_0$ admits all limits and colimits.

(M2) *2-out-of-3.* The class $\mathcal{W}$ of weak equivalences satisfies the 2-out-of-3 property. That is, given a commutative triangle



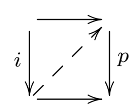
if two of f, g, $g \circ f$ are in $\mathcal{W}$, then so is the third.

(M3) *Retractions.* The class of morphisms in $\mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$) is closed under retracts. That is, if $g \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$) and there exists a commutative diagram of the form



then $f \in \mathcal{C}$ (resp. $\mathcal{F}, \mathcal{W}$).

(M4) *Lifting.* For every $i \in \mathcal{C}, p \in \mathcal{W} \cap \mathcal{F}$ (resp. $i \in \mathcal{C} \cap \mathcal{W}$, $p \in \mathcal{F}$) and every commutative square



there exists a diagonal morphism making two commutative triangles.

(M5) *Factorisation.* Every morphism $X \to Y$ in $\mathcal{M}$ has two functorial factorisations: as compositions

$$X \xrightarrow{i} Y' \xrightarrow{q} Y$$

$$X \xrightarrow{j} X' \xrightarrow{p} Y$$

where $i \in \mathcal{C}$, $q \in \mathcal{W} \cap \mathcal{F}$, $j \in \mathcal{C} \cap \mathcal{W}$, and $p \in \mathcal{F}$.

(M6) $\mathcal{S}\mathrm{et}_\Delta$*-action.* For every $X, Y \in Ob\ \mathcal{M}$ and $K \in \mathcal{S}\mathrm{et}_\Delta$ there are objects $X \otimes K$ and Y^K and isomorphisms

$$\mathrm{Map}_{\mathcal{M}}(X \otimes K, Y) \cong \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}_{\mathcal{M}}(X, Y^K)$$

which are functorial in X, Y, K.

(M7) *Corner axiom.* If $i : A \to B$ is in $\mathcal{C}$ and $p : X \to Y$ is in $\mathcal{F}$, then

$$\mathrm{Map}_{\mathcal{M}}(B, X) \to \mathrm{Map}_{\mathcal{M}}(A, X) \times_{\mathrm{Map}_{\mathcal{M}}(A,Y)} \mathrm{Map}_{\mathcal{M}}(B, Y)$$

is in $\mathcal{F}$. If either i or p are in $\mathcal{W}$ then so is the above map.