Derived Algebraic Geometry
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3 Higher categories I

Reference: Higher Topos Theory, Lurie.

Infinity categories should be categories in a world where sets are replaced by homotopy types. So we might expect an infinity category to be a category object in the category of homotopy types.[1] Something essentially like this (simplicial categories) will appear. However, just as simplicial sets has been the standard language of homotopy theory since the 50's, *quasi-categories* has been the dominant language for ∞-categories for the last decade. As such most people just call them ∞-categories.

Here is the map of this lecture and the next. We will start on the right side and work towards the left.

$$\left\{ \begin{array}{c} \text{Simplicial} \\ \text{Model} \\ \text{Categories} \end{array} \right\} \to \left\{ \begin{array}{c} \text{Simplicial} \\ \text{Categories} \end{array} \right\} \to \left\{ \ \infty\text{-categories} \ \right\} \subseteq \mathcal{S}\mathrm{et}_\Delta$$

In this lecture we will meet ∞-categories and simplicial categories. In the next lecture we will discuss simplicial model categories, and see some examples of interest.

[1] That is, a homotopy type of objects C_0, a homotopy type of morphisms C_1, source and target morphisms $C_1 \rightrightarrows C_0$, an identity morphisms $C_0 \to C_1$, and composition $C_1 \times_{C_0} C_1 \to C_1$ satisfying some kind of associativity.

3.1 Quasi-categories

To begin with we show how a "usual" category can be encoded in a simplicial set.

Definition 1. Let C be a small category. Considering the ordered sets $[n]$ as categories[2] the assignment

$$N : [n] \mapsto \mathrm{Fun}([n], C)$$

sending $[n]$ to the set of functors $[n] \to C$ defines a simplicial set. This is called the *nerve* of C.

Remark 2. Explicitly,

1. $N(C)_0$ is the set of objects of C,
2. $N(C)_1$ is the set of (all) morphisms in C,
3. The two morphisms $N(C)_1 \rightrightarrows N(C)_0$ induced by the two morphisms $[0] \rightrightarrows [1]$ send morphisms in $N(C)_1$ to their source and target.

$$(X \xrightarrow{f} Y) \qquad \mapsto \qquad X, Y$$

[2]So, for $0 \leq i, j \leq n$ there is exactly one morphism $i \to j$ if $i \leq j$, and no morphisms otherwise.

4. The morphism $N(C)_0 \to N(C)_1$ induced by $[1] \to [0]$ sends each object to its identity morphism.
$$X \quad \mapsto \quad (X \xrightarrow{\mathrm{id}_X} X)$$
5. $N(C)_2$ is the set of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$.
6. The three maps $d_0, d_1, d_2 : N(C)_2 \rightrightarrows N(C)_1$ induced by the three monomorphisms $[1] \rightrightarrows [2]$ send $\xrightarrow{f}\xrightarrow{g}$ to g, $g \circ f$, and f respectively.
$$\begin{array}{ccc} & Y & \\ {}^{f}\nearrow & & \searrow^{g} \\ X & \xrightarrow[g\circ f]{} & Z \end{array} \quad \mapsto \quad (Y \xrightarrow{g} Z), \quad (X \xrightarrow{g\circ f} Y), \quad (X \xrightarrow{f} Y)$$
7. More generally, $N(C)_n$ is the set of sequences of n composable morphisms $\xrightarrow{f_1} \cdots \xrightarrow{f_n}$ and the various maps $N(C)_n \to N(C)_m$ come from various combinations of composition and inserting identities.

Note that we can completely recover C from $N(C)$. In fact we have a lot of degenerate information.

Exercise 3. Suppose that C is a simplicial set such that:
1. Each $\Lambda^2_1 \to C$ extends to a unique $\Delta^2 \to C$, and
2. Each $\Lambda^3_1 \to C$ extends to some $\Delta^3 \to C$.

Show that C canonically determines a category whose set of objects is C_0 and set of morphisms is C_1.

Exercise 4 (HTT, Proposition 1.1.2.2)**.** (Difficult) Show that a simplicial set K is of the form $N(C)$ if and only if for every $0 < i < n$ and each diagram
$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$
there exists a *unique* dotted arrow making a commutative triangle.

Definition 5. An ∞-*category* is a simplicial set K such that for every $0 < i < n$ and each diagram
$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$
there exists a (not necessarily unique) dashed arrow making a commutative triangle.

A *functor* between ∞-categories is a morphism of simplicial sets. That is, the category of ∞-categories is a full subcategory of the category of simplicial sets
$$\mathcal{C}\mathrm{at}_\infty \subset \mathcal{S}\mathrm{et}_\Delta.$$

Elements of K_0 are called *objects* and elements of K_1 are called 1-morphisms, or often just *morphisms*. Given two morphisms $f, g \in K_1$ such that $d_1 f = d_0 g$ (equivalently, a diagram $\Lambda_1^2 \to K$), for any $\sigma : \Delta^2 \to K$ completing the diagram, $d_1\sigma \in K_1$ will be called a *composition* of g and f. For any object $X \in K_0$, the morphism $s_0 X \in K_1$ is called the *identity morphism* of X, and written id_X.

Example 6. The nerve $N(C)$ of any small category C is an ∞-category. So we get a functor (of 1-categories)

$$N : \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty.$$

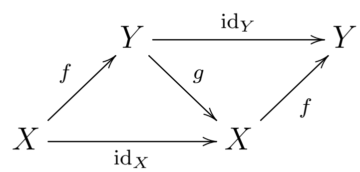
Example 7. Any Kan complex is an ∞-category. That is, we have fully faithful inclusions

$$\mathcal{S}\mathrm{et}_\Delta \supset \mathcal{C}\mathrm{at}_\infty \supset \{ \text{ Kan complexes } \}.$$

In particular, for any topological space X, the simplicial set $\mathrm{Sing}\, X$ is an ∞-category. In fact, Kan complexes are precisely the ∞-*groupoids* (see below).

Exercise 8.

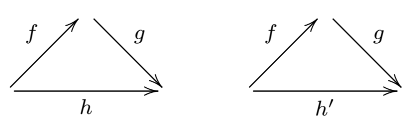
1. Show that every Kan complex is an ∞-category.
2. Show that if K is a Kan complex, then every morphism in K is invertible up to homotopy in the sense that:
   - For every $X \xrightarrow{f} Y$ in K_1 we can find two 2-cells in K_2 fitting into a diagram of the form



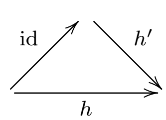
3. (Harder) Show that if K is an ∞-category satisfying the above property, then K is a Kan complex. Hint.[3]

Note that in general, in $\mathrm{Sing}\, X$ composition is not unique, but any two choices of composition are homotopic. This is a general feature of ∞-categories.

Exercise 9. Show that any in an ∞-category C, any two compositions are "homotopic" in the sense that if there exist two 2-cells in C_2 of the form



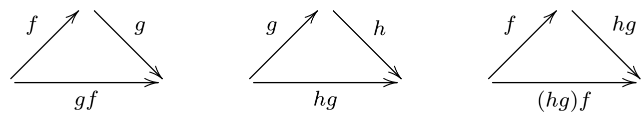
then there exists a 2-cell of the form



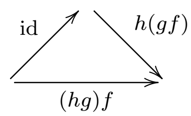
[3] Start with the case $\Lambda_0^2 \to C$ and work up to Λ_0^n by induction. Use opposite categories to deduce Λ_n^n from Λ_0^n.

Similarly, in Sing X composition is not associative on the nose, but only up to homotopy.

Exercise 10. Show that composition in an ∞-category C is associative "up to homotopy" in the sense that if we have 2-cells in C_2 of the form

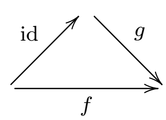


Then $(hg)f$ is a composition of gf and h. In particular, by Exercise 9, if $h(gf)$ is any other choice of composition of gf and h, then there is a 2-cell of the form:

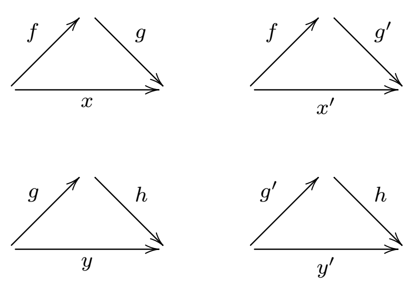


Exercise 11. Recall the nerve functor from Example 6. We will show that the nerve functor admits a left adjoint.

1. Let C be an ∞-category. Define a relation on 1-morphisms in C by saying $f \sim g$ if f is a composition of g and id. That is, if there exists a 2-cell in C_2 of the form



   Show that this is an equivalence relation.
2. Show that the above equivalence relation preserves composition. That is, suppose that $g \in C_1$ is equivalent to $g' \in C_1$, and suppose we have 2-cells of the following form.



   Show that $x \sim x$ and $y \sim y'$. (Use Exercise 9 if necessary).
3. Define hC to be the 1-category whose objects are vertices C_0, morphisms are edges C_1 modulo the above equivalence relation, and composition is induced by composition in C. Show that this is actually a 1-category. That is, show that it satisfies the identity and associativity axioms. (Use Exercise 10 for associativity).

4. Show that

$$h : \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}$$

defines a functor which is left adjoint to N. Hint.[4]

Definition 12. The 1-category hC defined above is called the *homotopy category* of C. A morphism $X \xrightarrow{f} Y \in C_1$ in an ∞-category is said to be an *equivalence* if it becomes an isomorphism in hC. If such an equivalence exists, we say X and Y are equivalent.

3.2 Mapping spaces

We wanted to replace sets with homotopy types, so for any two objects $x, y \in C_0$ in an ∞-category, we should have a homotopy type $\mathrm{Map}_C(x, y)$ of morphisms. Here are two models for this homotopy type.

Definition 13. Let C be an ∞-category, and $x, y \in C_0$ objects. Define

$$\hom^R_C(x, y)_J = \{z : \Delta^{J \sqcup [0]} \to C \mid z|_{\Delta^J} = x \text{ and } z|_{\Delta^0} = y\}$$

where $J \sqcup [0] = \{j_0 < \cdots < j_n\} \sqcup \{0\} = \{j_0 < j_1 < \cdots < j_n < 0\}$ and we use x for the constant morphism $\Delta^J \to \Delta^0 \xrightarrow{x} C$. Similarly, define

$$\hom^L_C(x, y)_J = \{z : \Delta^{[0] \sqcup J} \to C \mid z|_{\Delta^0} = x \text{ and } z|_{\Delta^j} = y\}$$

where $[0] \sqcup J = \{0\} \sqcup \{j_0 < \cdots < j_n\} \sqcup \{0\} = \{0 < j_0 < j_1 < \cdots < j_n\}$.

Exercise 14. Suppose C is an ∞-category and $x, y \in C_0$ are objects. Show that $\hom^R_C(x, y)$ and $\hom^L_C(x, y)$ are Kan complexes.

Exercise 15.

1. Let C be a 1-category. Show that $\hom^R_{NC}(x, y)_J = \hom_C(x, y)$ for all J.
2. Let X be a topological space and $x, y \in X$ two points. Let PX denote the set $\hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)$ equipped with the compact-open topology[5] and $PX(x, y) \subseteq \hom(\Delta^1, X)$ the subspace of maps $\gamma : \Delta^1_{\mathrm{top}} \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Define an isomorphism of simplicial sets $\hom^R_{\mathrm{Sing\,Top}}(x, y) \cong \mathrm{Sing}\, PX(x, y)$.

Definition 16. A morphism $C \to D$ of ∞-categories is:

1. *fully faithful* if for every pair of objects $X, Y \in C_0$ the induced morphism $\hom^R_C(X, Y) \to \hom^R_D(FX, FY)$ is a weak equivalence of Kan complexes,
2. *essentially surjective* if $hC \to hD$ is essentially surjective,
3. a *categorical equivalence* if it is essentially surjective and fully faithful.

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[4] It suffices to show that $hN = \mathrm{id}$ and to give a natural transformation $\eta : \mathrm{id} \to Nh$ such that $h(\eta)$ is the identity natural transformation.

[5] Or indeed, any topology such that $\hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, \hom_{\mathrm{Top}}(\Delta^1_{\mathrm{top}}, X)) = \hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}} \times \Delta^1_{\mathrm{top}}, X)$.

Exercise 17. Let $F : C \to C'$ be a functor between 1-categories. Show that F is an equivalence of 1-categories if and only if $F : NC \to NC'$ is an equivalence of ∞-categories.

Clearly, we would like composition morphisms

$$\hom_C^R(X,Y) \times \hom_C^R(Y,Z) \stackrel{?}{\to} \hom_C^R(X,Z).$$

One way to obtain these is the generalise both $\hom^R$ and $\hom^L$ as follows.

Definition 18 (Dugger, Spivak)**.** A *necklace* is a simplicial set of the form

$$\Delta^{J_0} \coprod_{\Delta^0} \Delta^{J_1} \coprod_{\Delta^0} \dots \coprod_{\Delta^0} \Delta^{J_n}$$

where the maps $\Delta^{J_{i-1}} \stackrel{\max}{\leftarrow} \Delta^0 \stackrel{\min}{\to} \Delta^{J_i}$ correspond to including the maximum, resp. minimum elements of the linearly ordered sets J_{i-1} resp. J_i. The full subcategory of necklaces is written $\mathcal{N}\mathrm{ec} \subseteq \mathcal{S}\mathrm{et}_\Delta$. Given a simpicial set C we write $\mathcal{N}\mathrm{ec} \downarrow C$ for the category of morphisms $T \to C$ where T is a necklace, [6] and for verticies $X, Y \in C_0$, we write $(\mathcal{N}\mathrm{ec} \downarrow C)_{X,Y}$ for the full subcategory of $(\mathcal{N}\mathrm{ec} \downarrow C)$ of those $T \to C$ which send the initial vertex of T to X and the final vertex to Y. Finally, we define the simplicial set

$$\mathrm{Map}_C^{nec}(X,Y) = N(\mathcal{N}\mathrm{ec} \downarrow C)_{X,Y}$$

as the nerve of the category $(\mathcal{N}\mathrm{ec} \downarrow C)_{X,Y}$.

Exercise 19. Consider the operation

$$\vee : \mathcal{N}\mathrm{ec} \times \mathcal{N}\mathrm{ec} \to \mathcal{N}\mathrm{ec}$$

given by

$$\begin{aligned}&(\Delta^{J_0} \coprod_{\Delta^0} \dots \coprod_{\Delta^0} \Delta^{J_n}) \vee (\Delta^{J'_0} \coprod_{\Delta^0} \dots \coprod_{\Delta^0} \Delta^{J'_m})\\ =&\Delta^{J_0} \coprod_{\Delta^0} \dots \coprod_{\Delta^0} \Delta^{J_n} \coprod_{\Delta^0} \Delta^{J'_0} \coprod_{\Delta^0} \dots \coprod_{\Delta^0} \Delta^{J'_m}\end{aligned}$$

Show that this defines a morphism of categories

$$(\mathcal{N}\mathrm{ec} \downarrow C)_{X,Y} \times (\mathcal{N}\mathrm{ec} \downarrow C)_{Y,Z} \to (\mathcal{N}\mathrm{ec} \downarrow C)_{X,Z}$$

which induces morphisms of simplicial sets

$$\mathrm{Map}_C^{nec}(X,Y) \times \mathrm{Map}_C^{nec}(Y,Z) \to \mathrm{Map}_C^{nec}(X,Z).$$

Show that these morphisms satisfy identity and associativity properties (cf. Def.22).

[6] Morphisms are commutative triangles of simplicial sets $T' \to T \to C$.

Theorem 20 ([HTT, Cor.4.2.1.8], [Dugger, Spivak, Rigidification, Thm.5.2]). *Let C be an ∞-category. Then for $* = R, L$, there is a zig-zag of weak equivalences of Kan complexes*

$$\hom_C^*(X, Y) \hookrightarrow \leftarrow \rightarrow \mathrm{Map}_C^{nec}(X, Y).$$

Remark 21. Explicitly, the zig-zag is

$$\hom_C^*(X, Y) \overset{a}{\hookrightarrow} \mathrm{Map}_{\mathfrak{C}[C]}(X, Y) \overset{b}{\leftarrow} \mathrm{Map}_C^{hoc}(X, Y) \overset{c}{\rightarrow} \mathrm{Map}_C^{nec}(X, Y).$$

where b and c are compatible with the composition morphisms, the complex $\mathrm{Map}_{\mathfrak{C}[C]}(X, Y)$ and a is defined in [HTT] (see also below), and $\mathrm{Map}_C^{hoc}(X, Y)$ and b, c are defined in [DS].

3.3 Simplicial categories

References:
[1982 Max Kelly, Basic Concepts of Enriched Category Theory]
[2003 Hirschorn, Model categories and their localisations, Def.9.1.2]
[2012 Lurie, Higher Topos Theory]

Definition 22 ([HTT, Def.1.1.4.1]). A *simplicial category* C is a category enriched over $\mathcal{S}\mathrm{et}_\Delta$. Explicitly, it is the data of:

1. A collection of objects $Ob\ C$.
2. For every pair of objects $X, Y \in Ob\ C$, a simplicial set $\mathrm{Map}_C(X, Y)$.
3. For every triple of objects $W, X, Y \in Ob\ C$ a morphism of simplicial sets

$$- \circ - : \mathrm{Map}_C(W, X) \times \mathrm{Map}_C(X, Y) \to \mathrm{Map}_C(W, Y).$$

These data are required to satisfy:

(Id.) Every object has an identity morphism. That is, for every $X \in Ob\ C$ there is a vertex $\mathrm{id}_X \in \mathrm{Map}(X, X)_0$ such that

$$\Delta^0 \times \mathrm{Map}(X, Y) \xrightarrow{\{\mathrm{id}_X\} \times \mathrm{id}_{\mathrm{Map}(X,Y)}} \mathrm{Map}(X, X) \times \mathrm{Map}(X, Y) \xrightarrow{\circ} \mathrm{Map}(X, Y)$$

is the canonical identification $\Delta^0 \times \mathrm{Map}(X, Y) \cong \mathrm{Map}(X, Y)$, and similarly for $\mathrm{Map}(W, X) \times \mathrm{Map}(X, X) \to \mathrm{Map}(W, X)$.

(Assoc.) The composition is associative. That is the following diagram of simplicial sets commutes for any objects W, X, Y, Z.

$$\begin{array}{ccc} \mathrm{Map}_C(W, X) \times \mathrm{Map}_C(X, Y) \times \mathrm{Map}_C(Y, Z) & \longrightarrow & \mathrm{Map}_C(W, Y) \times \mathrm{Map}_C(Y, Z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_C(W, X) \times \mathrm{Map}_C(X, Z) & \longrightarrow & \mathrm{Map}_C(W, Z) \end{array}$$

A simplicial category is called *fibrant* if all $\mathrm{Map}_C(X, Y)$ are Kan complexes.

Example 23. In the previous section we saw that every simplicial set K has an associated simplicial category, $\mathfrak{C}^{nec}(K)$, whose objects are zero simplicies of K and mapping spaces are $\mathrm{Map}_{\mathfrak{C}^{nec}(K)}(X,Y) = \mathrm{Map}^{nec}_K(X,Y)$.

Example 24. The simplicial category of simplicial sets is defined as follows. Objects are simplicial sets. Given two simplicial sets K, L the mapping space is defined by

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K,L)_n = \hom_{\mathcal{S}\mathrm{et}_\Delta}(K \times \Delta^n, L).$$

The simplicial set structure comes from functoriality in $[n] \in \Delta$. Composition is defined using the diagonal maps $\Delta^n \to \Delta^n \times \Delta^n$. Explicitly, the composition of two n-cells $f : K \times \Delta^n \to L$ and $g : L \times \Delta^n \to M$ is

$$K \times \Delta^n \xrightarrow{diag.} K \times \Delta^n \times \Delta^n \xrightarrow{f \times \mathrm{id}_{\Delta^n}} L \times \Delta^n \xrightarrow{g} M.$$

Exercise 25. Show that composition in the simplicial category $\mathcal{S}\mathrm{et}_\Delta$ satisfies the identity and associativity axioms.

Exercise 26 ([HTT, Prop.1.2.7.3], [Gabriel-Zisman, 3.1.3])**.** Let C be an ∞-category (resp. Kan complex). It turns out [HTT, Cor.2.3.2.4],[7] [Gabriel-Zisman, Prop.2.2] that C satisfies the stronger property:

(*) For every simplicial set K, every $0 < i < n$ (resp. $0 \le i \le n$), and every morphism $\Lambda^n_i \times K \to C$ there exists a factorisation

$$\begin{array}{ccc} \Lambda^n_i \times K & \longrightarrow & C \\ \downarrow & \nearrow & \\ \Delta^n \times K & & \end{array}$$

Using this property, show that for any $K \in \mathcal{S}\mathrm{et}_\Delta$, the simplicial set $\mathrm{Map}(K, C)$ is an ∞-category (resp. Kan complex).

Exercise 27. Give an example of $C, C' \in \mathcal{C}\mathrm{at}_\infty$ such that $\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(C, C')$ is not a Kan complex.

Like ∞-categories, simplicial categories also have associated 1-categories.

Exercise 28.

1. Let C be a simplicial category. For $X, Y \in Ob\ C$ define $\hom_C(X,Y) = \mathrm{Map}_C(X,Y)_0$. Show that this defines a 1-category. This category is sometimes denoted C_0. Be careful not to confuse this with the set of 0-simplicies of a simplicial set.
2. If K is a simplicial set, show that $\pi_0|K|$ is the set K_0 modulo the equivalence relation generated by K_1. Show that $\pi_0(|K| \times |L|) = \pi_0(|K \times L|)$ for simplicial sets K, L.

[7] This is a result of Joyal.

3. Let C be a fibrant simplicial category. For $X, Y \in Ob\, C$ define $\hom_{hC}(X, Y) = \pi_0|\operatorname{Map}_C(X, Y)|$. Show that this defines a 1-category.

Definition 29. A *morphism* $F : C \to D$ between two simplicial categories is defined in the obvious way. We have a map $Ob\ C \to Ob\ D$, for every pair $X, Y \in Ob\ C$ we have a morphism of simpicial sets $\operatorname{Map}_C(X, Y) \to \operatorname{Map}_D(FX, FY)$, and these morphisms are required to be compatible with composition and send identity morphisms to identity morphisms. The category of simplicial categories is denoted $\mathcal{C}\mathrm{at}_\Delta$.

Definition 30 ([HTT, Def.1.1.4.4])**.** A morphism $F : C \to C'$ of simplicial categories is an *equivalence* if

1. it is *fully faithful* in the sense that for every $X, Y \in Ob\, C$ the map $\operatorname{Map}_C(X, Y) \to \operatorname{Map}_{C'}(FX, FY)$ is a weak equivalence of simplicial sets, and
2. it is *essentially surjective* in the sense that $hC \to hC'$ is essentially surjective.