Derived Algebraic Geometry

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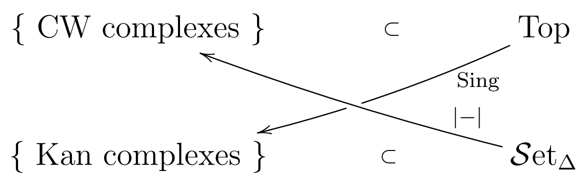
2 Homotopy types

References:

[Goerss, Jardine, Simplicial Homotopy Theory]

[Lurie, "Keredon", `https://kerodon.net/tag/00SY`]

One of the principles of derived geometry is that one should expand the category of sets to include not necessarily discrete "homotopy types". In this lecture we develop some models for a category of homotopy types. More concretely, we explain the following picture



whose rows should be compared with the inclusion

$$\mathrm{Ch}_R^{free} \subseteq \mathrm{Ch}_R$$

we saw last week.

In both $\mathcal{S}\mathrm{et}_\Delta$ and Top there is a notion of when a morphism is a *weak equivalence*. One way of defining the term "homotopy type" one option would be: a homotopy type is a weak equivalence class of objects in $\mathcal{S}\mathrm{et}_\Delta$ or Top. That is, a connected component of the graph whose verticies are objects of $\mathcal{S}\mathrm{et}_\Delta$ (or Top) and edges are weak equivalences.

2.1 Topological spaces

Definition 1. Let $f, g : X \to Y$ be two continuous morphisms between topological spaces. A *homotopy* from f to g is a continuous morphism $h : X \times [0,1] \to Y$ such that $h(-,0) = f(-)$ and $h(-,1) = g(-)$. In this case we write $f \sim g$.

$$***conepicture***$$

Exercise 2.

1. Show that any two continuous morphisms $X \rightrightarrows \mathbb{R}$ are homotopic. Give an example of two continuous morphisms $X \rightrightarrows Y$ which are not homotopic.[1]

[1] Hint: Try $X = \{0\}$ and $Y = \{\pm 1\}$

2. Show that $\sim$ is an equivalence relation on the set of continuous morphisms $\hom_{\mathrm{Top}}(X, Y)$ between two topological spaces.
3. Show that $\sim$ is preserved by pre- and post-composition. That is, if $f \sim g$ then $fa \sim ga$ and $bf \sim bg$ for any continuous $W \stackrel{a}{\to} X$, $X \stackrel{f}{\to} Y$, $X \stackrel{g}{\to} Y$, $Y \stackrel{b}{\to} Z$.

Definition 3. Two topological spaces X, Y are said to be *homotopy equivalent* if there exist continuous morphisms $f : X \rightleftarrows Y : g$ such that $\mathrm{id}_X \sim gf$ and $fg \sim \mathrm{id}_Y$.

If X is homotopy equivalent to a singleton $\{*\}$ then we say X is *contractible*.

Remark 4. By Exercise 2 we can make a new category $h\mathrm{Top}$ with the same objects as Top, and homotopy classes of morphisms. Then X and Y are homotopy equivalent if and only if they become isomorphic in $h\mathrm{Top}$. Similarly, a space is contractible if and only if it isomorphic to $\{*\}$ in $h\mathrm{Top}$.

Exercise 5.

1. Show that $\mathbb{R}^n$ is contractible.
2. Give an example of two topological spaces which are not homeomorphic, but which are homotopy equivalent.
3. Given an example of two topological spaces which are not homotopy equivalent.

Remark 6. There is also a pointed notion of homotopy. A *pointed space* is a pair (X, x_0) with X a topological space and $x_0 \in X$ a point. A *morphism* of pointed spaces $(X, x_0) \to (Y, y_0)$ is any continuous map $f : X \to Y$ such that $f(x_0) = y_0$. We write Top_* for the category of pointed topological spaces. A *homotopy* between morphisms $f, g : (X, x_0) \rightrightarrows (Y, y_0)$ of pointed spaces is a homotopy $h : X \times [0, 1] \to Y$ from $X \stackrel{f}{\to} Y$ to $X \stackrel{g}{\to} Y$ such that $h(x_0, t) = y_0$ for all $t \in [0, 1]$. Exercise 2 can also be done in the pointed setting.

Definition 7. The set of path components of a topological space X is

$$\pi_0(X) = \hom_{\mathrm{Top}}(\{*\}, X)/\sim .$$

Let $S^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \ : \sum x_i^2 = 1\}$. Equipped with $e_0 := (1, 0, \ldots, 0) \in S^n$ it becomes a pointed space. For $n \geq 0$, the nth *homotopy group* of a pointed space (X, x_0) is the set of morphisms of pointed spaces up to (pointed) homotopy equivalence

$$\pi_n(X, x_0) = \hom_{\mathrm{Top}_*}((S^n, e_0), (X, x_0))/\sim .$$

Remark 8. The homotopy groups $\pi_n(X, x_0)$ are a way of formalising how many "holes" are in a topological space.

Remark 9. Note that $S^0 = \{\pm 1\}$ and $S^n = \varnothing$ for $n < 0$.

Example 10.

1. If X is contractible, then $\pi_j(X, x_0) = \{0\}$ for all $0 \leq j, n$.
2. If $x_0, x_1, \ldots, x_n \in \mathbb{R}^2$ are $n+1$ distinct points, then $\pi_1(\mathbb{R}^2 \setminus \{x_0, x_1, \ldots, x_n\})$ is the free group on n generators.

3.

$$\pi_j(S^n, e_0) = \begin{cases} \{*\} & j = 0 < n \\ \{0\} & 0 < j < n \\ \mathbb{Z} & 0 < j = n \\ \text{major open problem} & 1 < n \ll j \end{cases}$$

4. $\pi_j\Big(X \times Y, (x_0, y_0)\Big) \cong \pi_j(X, x_0) \times \pi_j(Y, y_0)$.

Definition 11 ([HTT, Def.1.1.3.4])**.** A continuous morphism $X \to Y$ of topological spaces is a *weak equivalence* if

1. $\pi_0(X) \to \pi_0(Y)$ is an isomorphism, and
2. $\pi_j(X, x_0) \to \pi_j(Y, fx_0)$ is an isomorphism for all $x_0 \in X$.

Exercise 12. Show that any homotopy equivalence is a weak equivalence.

Example 13. Not every topological space behaves well with respect to homotopy. For example, the topologists sine curve

$$X = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\} \sqcup \{(t, \sin \tfrac{1}{t}) \in \mathbb{R}^2 \mid t > 0\}$$

$$***ADDPICTURE***$$

has a single connected component (i.e., there are is no homeomorphism $X \cong X_0 \sqcup X_1$ with X_0, X_1 non-empty) but it has two path connected components. In other words, $\{(0,0), (\pi, 0)\} \to X$ is a weak equivalence, but not a homotopy equivalence.

Remark 14. There exists a nice category of topological spaces (called CW complexes) where every weak equivalence is a homotopy equivalence. Moreover, every topological space is weakly equivalent to a CW complex.

Up to homotopy, every CW complex can be built from a simplicial set.

2.2 Simplicial sets

We write Δ for the 1-category of finite linearly ordered sets. Every such set is isomorphic to one of the form $[n] = \{0 < 1 < \cdots < n\}$ (here $n \in \mathbb{Z}$). The 1-category of simplicial sets $\mathcal{S}\text{et}_\Delta$ is the category of functors $\Delta^{op} \to \mathcal{S}\text{et}$. Given such a functor $K : \Delta^{op} \to \mathcal{S}\text{et}$ we write $K_n := K([n])$. Elements of K_n are called *n-simplicies* of K.

Example 15 ($\operatorname{Sing} X$)**.** Define

$$\Delta^n_{\text{top}} := \left\{ (x_0, \dots, x_n) \mid 0 \le x_i \le 1; \sum_{i=0}^n x_i = 1 \right\}$$

to be the convex hull of the standard basis vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$. So Δ^0_{top} is a point, Δ^1_{top} is a line segment, Δ^2_{top} is a triangle, Δ^3_{top} is a tetrahedron, ...

$$***ADDPICTURE***$$

Any morphism $p : [n] \to [m]$ in Δ defines an $\mathbb{R}$-linear morphism $\mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$; $e_i \mapsto e_{p(i)}$, which restricts to a continuous morphism $\Delta^n_{\text{top}} \to \Delta^m_{\text{top}}$. In this way we get a functor

$$\Delta \to \text{Top}; \qquad [n] \mapsto \Delta^n_{\text{top}}$$

from Δ to the 1-category of topological spaces. For any other topological space X, the assignment

$$\operatorname{Sing} X : [n] \mapsto \hom_{\text{Top}}(\Delta^n_{\text{top}}, X)$$

defines a simplicial set. Explicitly,

1. $\operatorname{Sing}_0 X$ is the set of points of X,
2. $\operatorname{Sing}_1 X$ is the set of paths in X,
3. $\operatorname{Sing}_2 X$ is the set of triangles in X,
4. ...

Remark 16. The term "singular" refers to the fact that we might have restricted our attention to smooth manifolds X and smooth maps $\Delta^n_{\text{top}} \to X$. However, our maps are only required to be continuous, and we allow any topological space X (for now).

Remark 17. One can think of $\operatorname{Sing} X$ as a combinatorial model of X. The functor $\operatorname{Sing} : \text{Top} \to \mathcal{S}\text{et}_\Delta$ admits a left adjoint $|\cdot| : \mathcal{S}\text{et}_\Delta \to \text{Top}$, and the counit $|\operatorname{Sing} X| \to X$ is always a weak equivalence.

$$***PICTUREOFTRIANGULATEDSURFACE***$$

Remark 18. We describe the geometric realisation functor in more detail later, but for now, note that by Yoneda's lemma, we must have $|\Delta^n| = \Delta^n_{\text{top}}$ since

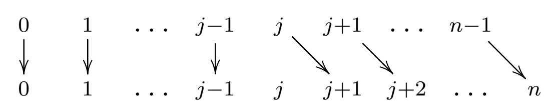
$$\hom_{\text{Top}}(|\Delta^n|, -) = \hom_{\mathcal{S}\text{et}_\Delta}(\Delta^n, \operatorname{Sing} -) = (\operatorname{Sing} -)_n = \hom_{\text{Top}}(\Delta^n_{\text{top}}, -).$$

2.3 Kan complexes

Example 19 (Δ^n)**.** For each n, the functor $\Delta^n := \hom_\Delta(-, [n]) : \Delta^{op} \to \mathcal{S}\text{et}$ defines a simplicial set. By Yoneda's Lemma, for any $K \in \mathcal{S}\text{et}_\Delta$,

$$\hom_{\mathcal{S}\text{et}_\Delta}(\Delta^n, K) \cong K_n.$$

Definition 20. For each n, j we have the *face* morphism $\delta_j : [n-1] \to [n]$ defined as the unique injection which does not have j in its image.



For any simplicial set $K : \Delta^{op} \to \mathcal{S}\text{et}$ we have a corresponding morphism

$$d_j : K_n \to K_{n-1}.$$

These (i.e., δ_j and d_j) are called *face* morphisms. For $\sigma \in K_n$ we call $d_j\sigma$ the *jth face of* σ.

Exercise 21. Show that every monomorphism in Δ is a composition of δ_j's.

Exercise 22. Consider the morphism $\Delta^n_{\text{top}} \to \Delta^{n+1}_{\text{top}}$ associated to δ_j. Draw this morphism for $0 \le j \le n \le 2$.

Example 23 ($\partial\Delta^n$)**.** Consider the morphisms of simplicial sets $\delta_j : \Delta^{n-1} \to \Delta^n$. We define

$$\partial\Delta^n = \bigcup_{j=0}^{n} \delta_j(\Delta^{n-1})$$

as the union of these faces. Explicitly, $(\partial\Delta^n)_j \subseteq (\Delta^n)_j = \hom_\Delta([j],[n])$ is the set of morphisms $[j] \to [n]$ of linearly ordered sets which are not surjective.

Exercise 24. Show that $\partial\Delta^n_{\text{top}} = \cup_{j=0}^{n} \delta_j(\Delta^{n-1}_{\text{top}})$ is the boundary of $\Delta^n_{\text{top}} \subseteq \mathbb{R}^{n+1}$.

Exercise 25. Let K be a simplicial set.

1. Show that a morphism $f : \partial\Delta^n \to K$ of simplicial sets canonically determines a collection of simplicies $k_0, k_1, \dots, k_n \in K_{n-1}$ such that we have $\delta_i^* k_j = \delta_{j-1}^* k_i$ for $i < j$.
2. (Harder) Conversely, show that a collection of simplicies $k_0, k_1, \dots, k_n \in K_{n-1}$ such that we have $\delta_i^* k_j = \delta_{j-1}^* k_i$ for $i < j$ determines a morphism $f : \partial\Delta^n \to K$ of simplicial sets. Hint.[2]

Exercise 26. Let I be the category associated to the partially ordered set of the sub-linearly ordered sets of $[n]$ of size n and $n-1$. Show that $\partial\Delta^n_{\text{top}} = \varinjlim_{L \in I} \Delta^L_{\text{top}}$. Using the fact that $|-|$ preserves colimits and Remark 18 deduce that $|\partial\Delta^n| = \partial\Delta^n_{\text{top}}$.

Definition 27 (Λ^n_j)**.** For $0 \le j \le n$ we define the *jth horn* as the union

$$\Lambda^n_j = \bigcup_{i \ne j} \delta_i(\Delta^{n-1}).$$

Equivalently, $(\Lambda^n_j)_i \subseteq (\Delta^n)_i = \hom_\Delta([i],[n])$ is the set of those $[i] \to [n]$ whose image does *not contain* the subset $\{0, 1, \dots, j{-}1, j{+}1, \dots, n\}$.

Exercise 28. Define $\Lambda^{n+1}_{\text{top},j} = \bigcup_{i \ne j} \delta_i(\Delta^n_{\text{top}})$. Draw $\Lambda^n_{\text{top},j}$ for $0 \le j \le n \le 2$.

Exercise 29. Do the Λ^n_i analogue of Exercise 25.

Definition 30 (Kan fibration)**.** A morphism $f : X \to Y$ of simplicial sets is a *Kan fibration* if for every $0 \le j \le n$, and commutative square

$$\begin{array}{ccc} \Lambda^n_j & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

[2] I would do this as follows. Consider the partially ordered set I consisting of those sub-linearly ordered sets $\sigma \subseteq [n]$ such that $\sigma \cong [n{-}1]$ or $\sigma \cong [n{-}2]$. This determines a diagram $I \to \mathcal{S}\mathrm{et}_\Delta$; $\sigma \mapsto \Delta^\sigma$. Show that $\partial\Delta^n \cong \varinjlim_{\sigma \in I} \Delta^\sigma$, and therefore $\hom(\partial\Delta^n, K) = \lim_{\sigma \in I} \hom(\Delta^\sigma, K)$. Now use Yoneda $\hom(\Delta^\sigma, K) \cong K_{|\sigma|-1}$.

a dashed morphism exists making two triangles commutative. A simplicial set K is a *Kan complex* if the canonical morphism $K \to \Delta^0$ is a Kan fibration.

Remark 31. Note $\varnothing \to Y$ is a Kan fibration.

Remark 32. Kan fibrations are the simpicial version of *Serre fibrations* of topological spaces.[3] A Serre fibration $X \to Y$ of topological spaces has the nice property that any $x \in X$ gives rise to a long exact sequence

$$\cdots \to \pi_n(F, x) \to \pi_n(X, x) \to \pi_n(Y, fx) \to \pi_{n-1}(F, x) \to \dots$$

where $F = f^{-1}f(x)$.

Exercise 33. Show that for any $0 \leq j \leq n$ there exists a continuous retraction[4] $\Delta^n_{\text{top}} \to \Lambda^n_{\text{top},j}$ to the inclusion $\Lambda^n_{\text{top},j} \subseteq \Delta^n_{\text{top}}$. Using the adjunction

$$|-| : \mathcal{S}\text{et}_\Delta \rightleftarrows \text{Top} : \text{Sing},$$

and Exercise **??**, show that for any topological space X, the simplicial set $\operatorname{Sing} X$ is a Kan complex.

Just as for topological spaces, we can define the notion of homotopy equivalence of simplicial sets.

Definition 34. If K, L are two simplicial sets, we get a new simplicial set $K \times L$ by setting

$$(K \times L)_n = K_n \times L_n.$$

Exercise 35.

1. Given $[m] \to [n]$ in Δ, describe the associated morphisms of sets $(K \times L)_n \to (K \times L)_m$.
2. Let X, Y be topological spaces and show that $\operatorname{Sing}(X \times Y) = (\operatorname{Sing} X) \times (\operatorname{Sing} Y)$.
3. Draw the topological spaces $\Delta^1_{\text{top}} \times \Delta^1_{\text{top}}$ and $\Delta^1_{\text{top}} \times \Delta^2_{\text{top}}$. Describe all non-degenerate simplices[5] in Δ^1, Δ^2, $\Delta^1 \times \Delta^1$ and $\Delta^1 \times \Delta^2$.

Definition 36. Let $f, g : K \rightrightarrows L$ be two morphisms of simplicial sets. A *homotopy* from f to g is a morphism

$$h : K \times \Delta^1 \to L$$

such that $h(-, 0) = f(-)$ and $h(-, 1) = g(-)$. Here, $h(-, 0)$ (resp. $h(-, 1)$) means the the composition $K \cong K \times \Delta^0 \to K \times \Delta^1 \to L$ where $\Delta^0 \to \Delta^1$ corresponds to $[0] \to [1]; 0 \mapsto 0$ (resp. $0 \mapsto 1$).

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[3] In fact, a morphism of topological spaces $X \to Y$ is a Serre fibration if and only if $\operatorname{Sing} X \to \operatorname{Sing} Y$ is a Kan fibration, [Keredon, https://kerodon.net/tag/021V]. That is, if and only if it satisfies the lifting criterion of Def.30 using $\Lambda^n_{\text{top},j} \to \Delta^n_{\text{top}}$ instead of $\Lambda^n_j \to \Delta^n$.

[4] That is, a continuous morphism such that the composition $\Lambda^n_{\text{top},j} \to \Delta^n_{\text{top}} \to \Lambda^n_{\text{top},j}$ is the identity.

[5] A simplex $\sigma \in K_n$ is called *non-degenerate* if it is not of the form $p^*\sigma$ for some surjection $p : [n] \to [n-1]$.

Exercise 37. Suppose that $f, g : X \rightrightarrows Y$ are two continuous morphisms of topological spaces which are homotopic. Show that $\operatorname{Sing} f, \operatorname{Sing} g : \operatorname{Sing} X \rightrightarrows \operatorname{Sing} Y$ are homotopic.

Definition 38. A morphism of simplicial sets $K \to L$ is a *(Quillen) weak equivalence* if $|K| \to |L|$ is a weak equivalence of topological spaces.

2.4 Colimits of topological spaces

For concreteness, let us recall the following.

Definition 39. Let I be a category and $X : I \to \mathrm{Top}$ a functor. The colimit of this diagram can be constructed explicitly as follows. The underline set of $\varinjlim_{i \in I} X_i$ is the colimit taken in the category of sets. That is, it is the quotient of the disjoint union $\sqcup_{i \in I} X_i$ by the equivalence relation *generated by* $x_i \in X_i$ is equivalent to $x_j \in X_j$ if there exists $u : i \to j$ in I such that $X_u(x_i) = x_j$.

We equip $\sqcup_{i \in I} X_i / \sim$ with the finest topology such that the canonical morphisms $\iota_i : X_i \to \sqcup_{i \in I} X_i / \sim$ are continuous. Explicitly, a subset $U \subseteq \sqcup_{i \in I} X_i / \sim$ is open if and only if $\iota_i^{-1}(U)$ is open for all i.

Exercise 40. Suppose that $Z_1, Z_2 \subseteq X$ are two subspaces of a topological space X. Show that if Z_1, Z_2 are both closed, then $Z_1 \cup Z_2$ is homeomorphic to $Z_1 \sqcup_{Z_1 \cup Z_2} Z_2$. Give an example of subspaces $Z_1, Z_2 \subseteq X$ (not closed) such that $Z_1 \cup Z_2$ is not homeomorphic to $Z_1 \sqcup_{Z_1 \cup Z_2} Z_2$.

2.5 Geometric realisation

In this section we consider three different descriptions of the geometric realisation.

Consider the geometric realsation $|\cdot| : \mathcal{S}\mathrm{et}_\Delta \to \mathrm{Top}$. We would like $|\cdot|$ to be left adjoint to Sing. This forces the following properties:

1. $|\Delta^n|$ has to corepresent the functor $\operatorname{Sing}(-)_n = \hom_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, \operatorname{Sing} -)$. We know that this functor is isomorphic to $\hom_{\mathrm{Top}}(\Delta^n_{\mathrm{top}}, -)$, so we must have
$$|\Delta^n| = \Delta^n_{\mathrm{top}}.$$
2. $|-|$ has to preserve colimits (since it's a left adjoint). Since every simplicial set is a colimit of Δ^ns, this completely determines $|-|$. This leads to three descriptions.

As a colimit of representables. Let $\int K$ be the category whose objects are pairs $([n], k)$ where $n \in \mathbb{N}$ and $k \in K_n$; i.e., elements of $\amalg_{\mathbb{N}} K_n$. A morphism $([n], k) \to ([n'], k')$ is a morphism $\sigma : [n] \to [n']$ such that $K_{n'} \to K_n$ sends k' to k. Then $K = \varinjlim_{([n],k) \in \int K} \Delta^n$ (see Exercise 41) so we must have

$$|K| = \varinjlim_{([n],f) \in \int K} \Delta^n_{\mathrm{top}}.$$

Exercise 41. Let $F : C^{op} \to \mathcal{S}\mathrm{et}$ be a presheaf of sets. Show that

$$F = \varinjlim_{\substack{c \in Ob\ C, \\ \hom(-,c) \xrightarrow{s} F}} \hom_C(-, c)$$

where the colimit is indexed by the category $\int F$ described above (recall that $F(c) \cong \hom(\hom(-, c), F)$ by Yoneda's Lemma).

As a homotopy colimit. In general, we can write colimits in terms of coequalisers and coproducts, $\varinjlim_{i \in I} X_i = \mathrm{coeq}(\sqcup_{i \xrightarrow{u} j \in I} X_i \rightrightarrows \sqcup_{i \in I} X_i)$. If we do this for the above colimit, we get $K = \mathrm{coeq}\left(\coprod_{[n] \xrightarrow{\sigma} [m]} \coprod_{k \in K_m} \Delta^n \rightrightarrows \coprod_{n \in \Delta} \coprod_{k \in K_n} \Delta^n\right)$ where one morphism sends the kth copy of Δ^n to the $\sigma^* k$th copy of Δ^n, and the other morphism is the canonical $\sigma : \Delta^n \to \Delta^m$ from the kth copy to the kth copy. So we must have $|K| = \mathrm{coeq}\left(\coprod_{[n] \xrightarrow{\sigma} [m]} \coprod_{k \in K_m} \Delta^n_{\text{top}} \rightrightarrows \coprod_{n \in \Delta} \coprod_{k \in K_n} \Delta^n_{\text{top}}\right)$ which can be written as

$$|K| = \mathrm{coeq}\left(\coprod_{[n] \xrightarrow{\sigma} [m]} K_m \times \Delta^n_{\text{top}} \rightrightarrows \coprod_{n \in \Delta} K_n \times \Delta^n_{\text{top}}\right)$$

if we think of each K_i as a discrete topological space. We will see in a few weeks that this is a model for the homotopy colimit

$$\mathrm{hocolim}_{[n] \in \Delta} K_n$$

in Top, where the K_n are considered as discrete spaces in Top.

As a tower of relative cells complexes. Finally, recall that one defines a simplex $\sigma \in K_n$ to be *degenerate* if $\sigma \in \bigcup_{\hom([n],[n-1])} \mathrm{im}(K_{n-1} \to K_n)$ and *non-degenerate* if it is not degenerate. Write $NK_n \subseteq K_n$ for the set of non-degenerate simplicies of dimension n, and for $n \geq -1$ define $\mathrm{sk}_{-1} K = \varnothing$ and let

$$\mathrm{sk}_n K = \bigcup_{\substack{0 \leq j \leq n \\ \sigma \in NK_n}} \mathrm{im}(\Delta^j \xrightarrow{\sigma} K) \subseteq K$$

be the smallest subsimplicial set containing all $\sigma \in NK_j; j \leq n$. Note that sk_n is functorial, and $\partial\Delta^n = \mathrm{sk}_{n-1} \Delta^n$. In particular, given any $\sigma \in NK_n$, we have a corresponding morphism $\partial\Delta^n \to \mathrm{sk}_{n-1} K$. In fact, one sees that there exist cocartesian squares

$$\begin{array}{ccc} \coprod_{NK_n} \partial\Delta^n & \longrightarrow & \mathrm{sk}_{n-1} K \\ \downarrow & & \downarrow \\ \coprod_{NK_n} \Delta^n & \longrightarrow & \mathrm{sk}_n K. \end{array}$$

By Remark 18 we have $|\Delta^n| = \Delta^n_{\text{top}}$ and by Exercise 26 we have $|\partial\Delta^n| = \partial\Delta^n_{\text{top}}$. Consequently, since $|-|$ preserves all colimits there are cocartesian squares

$$\begin{array}{ccc} \coprod_{NK_n} \partial\Delta^n_{\text{top}} & \longrightarrow & |\operatorname{sk}_{n-1} K| \\ \downarrow & & \downarrow \\ \coprod_{NK_n} \Delta^n_{\text{top}} & \longrightarrow & |\operatorname{sk}_n K|. \end{array}$$

Morevoer, $K = \varinjlim(\operatorname{sk}_0 K \to \operatorname{sk}_1 K \to \dots)$ so

$$|K| = \varinjlim\Big(|\operatorname{sk}_0 K| \to |\operatorname{sk}_1 K| \to \dots\Big).$$

In other words, we obtain $|K|$ be sequentially glueing cells Δ^n_{top} along their boundaries $\partial\Delta^n_{\text{top}} \to |\operatorname{sk}_{n-1} K|$.

$* * * *$ *picture* $* **$

Corollary 42. *For $K \in \mathcal{S}\text{et}_\Delta$ there is a bijection of* sets

$$|K| \cong K_0 \amalg \left(\coprod_{n>0} \coprod_{NK_n} (\Delta^n_{top})^\circ \right)$$

where $(\Delta^n_{top})^\circ$ means the interior of Δ^n_{top}. In particular, a simplex $k \in K_n$ is non-degenerate if and only if the induced continuous morphism $(\Delta^n_{top})^\circ \to |K|$ is injective, and degenerate if and only if it factors via a linear projection $(\Delta^n_{top})^\circ \to (\Delta^m_{top})^\circ \to |K|$ for some $m < n$ and some non-degenerate $\Delta^m \to K$.