サンプルテスト

① $\begin{pmatrix} 0 & 7 & -5 & 7 \\ 5 & -7 & 1 & 1 \\ 5 & 4 & 8 & 8 \end{pmatrix}$ の転置行列は $\begin{pmatrix} 0 & 5 & 5 \\ 7 & -7 & 4 \\ -5 & 1 & 8 \\ 7 & 1 & 8 \end{pmatrix}$

② a) $\mathrm{sgn}(1\ 2\ 3) = (-1)^0 = \boxed{1}$

$1 < 2, 3$

$2 < 3$

b) $\mathrm{sgn}(2\ 4\ 5\ 1\ 3) = (-1)^5 = \boxed{-1}$

$2 > 1^{1}$

$4 > 1^{2}, 3^{3}$

$5 > 1^{4}, 3^{5}$

c) $\mathrm{sgn}(5\ 7\ 1\ 4\ 2\ 3\ 6) = (-1)^{11} = \boxed{-1}$

$5 > 1^{1}, 4^{2}, 2^{3}, 3^{4}$

$7 > 1^{5}, 4^{6}, 2^{7}, 3^{8}, 6^{9}$

$4 > 2^{10}, 3^{11}$

③ $\begin{pmatrix} \cancel{-5} & \cancel{4} & \cancel{-1} & \cancel{-3} \\ \cancel{-4} & 3 & 4 & 2 \\ \cancel{0} & 1 & -4 & 3 \\ \cancel{-5} & -1 & 4 & 1 \end{pmatrix}$ の第(1,1)小行列は $\boxed{\begin{pmatrix} 3 & 4 & 2 \\ 1 & -4 & 3 \\ -1 & 4 & 1 \end{pmatrix}}$

第(1,1)小行列式は $\begin{vmatrix} 3 & 4 & 2 \\ 1 & -4 & 3 \\ -1 & 4 & 1 \end{vmatrix} \underset{r_3+r_2}{\overset{r_1+r_2}{=}} \begin{vmatrix} 4 & 0 & 5 \\ 1 & -4 & 3 \\ 0 & 0 & 4 \end{vmatrix}$

$= (-1)^{3+3} \, 4 \begin{vmatrix} 4 & 0 \\ 1 & -4 \end{vmatrix}$

$= 1 \cdot 4 \cdot 4 \cdot (-4)$

$= \boxed{-64}$

第(1,1)余因子は $(-1)^{1+1}(-64) = \boxed{-64}$

④

$$\begin{cases} x \qquad\quad -3z - 3w = 1 \\ 0 + y + 2z + 2w = 0 \\ -3x + y + 11z + 11w = a \end{cases}$$

の拡大係数行列は

$$\begin{bmatrix} 1 & 0 & -3 & -3 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ -3 & 1 & 11 & 11 & a \end{bmatrix}$$

$$\begin{array}{r} \\ \\ r_3+3r_1 \end{array} \begin{bmatrix} 1 & 0 & -3 & -3 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & a+3 \end{bmatrix}$$

$$\begin{array}{r} \\ \\ r_3-r_2 \end{array} \begin{bmatrix} 1 & 0 & -3 & -3 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & a+3 \end{bmatrix}$$

$\boxed{a \neq -3}$ のとき、階数は $\boxed{3、\underline{解ない}}$.

$\boxed{a = -3}$ のとき、階数は $\boxed{2}$、解は

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 3s + 3t + 1 \\ -2s + -2t \\ s \\ t \end{pmatrix}}$$

確認:

$$(3s+3t+1) \qquad -3s \quad -3t = 1 \quad ✓$$

$$0 + (-2s-2t) + 2s + 2t = 0 \quad ✓$$

$$-3(3s+3t+1) + (-2s-2t) + 11s + 11t = -3 \quad ✓$$

⑤ $\begin{pmatrix} -11 & 2 & 4 \\ -5 & 1 & 2 \\ -3 & 0 & 1 \end{pmatrix}$ の逆行列は $\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & -2 \\ -3 & 6 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} -11 & 2 & 4 & 1 & 0 & 0 \\ -5 & 1 & 2 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$r_1 - 2r_2 \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & -2 & 0 \\ -5 & 1 & 2 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{r} -r_1 \\ r_2 - 5r_1 \\ r_3 - 3r_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 2 & -5 & 11 & 0 \\ 0 & 0 & 1 & -3 & 6 & 1 \end{array}\right)$$

$$r_2 - 2r_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -3 & 6 & 1 \end{array}\right)$$

確認:

$$\begin{pmatrix} -11 & 2 & 4 \\ -5 & 1 & 2 \\ -3 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & -2 \\ -3 & 6 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -11(-1) + 2\cdot 1 + 4\cdot(-3) & -11\cdot 2 + 2(-1) + 4\cdot 6 & 0 + 2\cdot(-2) + 4 \\ -5(-1) + 1\cdot 1 + 2\cdot(-3) & -5\cdot 2 + 1\cdot(-1) + 2\cdot 6 & 0 + 1\cdot(-2) + 2 \\ -3(-1) + 0 + 1\cdot(-3) & -3\cdot 2 + 0 + 6 & 0 + 0 + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✓

⑥

$$\begin{vmatrix} a-5 & -2 & 2 \\ 2 & a-1 & -2 \\ -2 & -2 & a-1 \end{vmatrix} \overset{r_1+r_2}{\underset{r_3+r_2}{=}} \begin{vmatrix} a-3 & a-3 & 0 \\ 2 & a-1 & -2 \\ 0 & a-3 & a-3 \end{vmatrix}$$

$$= (a-3)^2 \begin{vmatrix} 1 & 1 & 0 \\ 2 & a-1 & -2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\overset{c_2-c_1}{=} (a-3)^2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & a-3 & -2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (a-3)^2 (-1)^{1+1} \begin{vmatrix} a-3 & -2 \\ 1 & 1 \end{vmatrix}$$

$$= (a-3)^2 \left((a-3)\cdot 1 - (-2)\cdot 1 \right)$$

$$= (a-3)^2 (a-1)$$

$\boxed{a \neq 3,\ 1}$ のとき $\begin{pmatrix} a-5 & -2 & 2 \\ 2 & a-1 & -2 \\ -2 & -2 & a-1 \end{pmatrix}$ の行列式 0 でなくて、正則である。

⑤ $\mathbf{a} = \begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ $\mathbf{d} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

とすると、クラメルの公式によって

$$\begin{cases} -5x & -2y & +2z & = -1 \\ 2x & -y & -2z & = -1 \\ -2x & -2y & -z & = 2 \end{cases}$$

の解は

$$x = \frac{|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{\begin{vmatrix} -1 & -2 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -1 \end{vmatrix}}{(-3)^2(-1)} \underset{c_3+2c_1}{\overset{c_2-2c_1}{=}} \frac{-1}{9} \begin{vmatrix} -1 & 0 & 0 \\ -1 & 1 & -4 \\ 2 & -6 & 3 \end{vmatrix}$$

$$= \frac{-1}{9}(-1)^{1+1}(-1)\begin{vmatrix} 1 & -4 \\ -6 & 3 \end{vmatrix} = \frac{1}{9}\Big(1\cdot 3 - (-4)(-6)\Big) = \frac{-21}{9} = \boxed{\frac{-7}{3}}$$

$$y = \frac{|\mathbf{a}\ \mathbf{d}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{-1}{9}\begin{vmatrix} -5 & -1 & 2 \\ 2 & -1 & -2 \\ -2 & 2 & -1 \end{vmatrix} \underset{r_3+2r_2}{\overset{r_1-r_2}{=}} \frac{-1}{9}\begin{vmatrix} -7 & 0 & 4 \\ 2 & -1 & -2 \\ 2 & 0 & -5 \end{vmatrix}$$

$$= \frac{-1}{9}(-1)^{2+2}(-1)\begin{vmatrix} -7 & 4 \\ 2 & -5 \end{vmatrix} = \frac{1}{9}\Big((-7)(-5) - 4\cdot 2\Big) = \frac{35-8}{9} = \frac{27}{9} = \boxed{3}$$

$$z = \frac{|\mathbf{a}\ \mathbf{b}\ \mathbf{d}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{-1}{9}\begin{vmatrix} -5 & -2 & -1 \\ 2 & -1 & -1 \\ -2 & -2 & 2 \end{vmatrix} = \frac{-1}{9}(-1)^2\begin{vmatrix} 5 & 2 & 1 \\ 2 & -1 & -1 \\ 2 & 2 & -2 \end{vmatrix}$$

$$\underset{r_3+2r_2}{\overset{r_1+2r_2}{=}} \frac{-1}{9}\begin{vmatrix} 9 & 0 & -1 \\ 2 & -1 & -1 \\ 6 & 0 & -4 \end{vmatrix} = \frac{-1}{9}(-1)^{2+2}(-1)\begin{vmatrix} 9 & -1 \\ 6 & -4 \end{vmatrix}$$

$$= \frac{1}{9}\Big((9)\cdot(-4) - (-1)(6)\Big)$$

$$= \frac{(-36+6)}{9} = \frac{-30}{9} = \boxed{\frac{-10}{3}}$$

確認：

$$-5\left(\frac{-7}{3}\right) \quad -2\cdot 3 \quad +2\left(\frac{-10}{3}\right) = \frac{35}{3} - \frac{18}{3} - \frac{20}{3} = \frac{-3}{3} = -1 \checkmark$$

$$2\left(\frac{-7}{3}\right) \quad -3 \quad -2\left(\frac{-10}{3}\right) = \frac{-14}{3} - \frac{9}{3} + \frac{20}{3} = \frac{-23+20}{3} = -1 \checkmark$$

$$-2\left(\frac{-7}{3}\right) \quad -2\cdot 3 \quad -\left(\frac{-10}{3}\right) = \frac{14}{3} - \frac{18}{3} + \frac{10}{3} = \frac{-4+10}{3} = 2 \checkmark$$

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$$\begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & -5 & \cdot \\ \cdot & \cdot & -2 & \cdot & -3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 4 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \underbrace{(-1)^{2+1}(-1)^{7+1}(-1)^{1+7}(-1)^{2+4}(-1)^{4+2}}_{(-1)\ (1)\ (1)\ (1)\ (1)} \begin{vmatrix} \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & -2 & -5 \\ -2 & -3 & \cdot & \cdot \\ 4 & -1 & \cdot & \cdot \end{vmatrix}$$

$$= (-1)\begin{vmatrix} -1 & \cdot \\ -2 & -5 \end{vmatrix}\begin{vmatrix} -2 & -3 \\ 4 & -1 \end{vmatrix}$$

$$= -5\cdot(2+12) = \boxed{-70}$$

⑨
$$\begin{vmatrix} 2 & 1 & & & & \\ 3 & 2 & 1 & & & \\ & 3 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 3 & 2 & \end{vmatrix} = \boxed{56}$$

7×7

3重対角行列で $\boxed{D_n = aD_{n-1} - bcD_{n-2}}$

$\boxed{D_0 = 1}$ $\boxed{D_{-1} = 0}$ をつかう。

ここで $a = 2$, $b \cdot c = 3$

$D_{-1} = 0$

$D_0 = 1$

$D_1 = 2D_0 - 3D_{-1} = 2$

$D_2 = 2D_1 - 3D_0 = 2\cdot 2 - 3\cdot 1 = 1$

$D_3 = 2D_2 - 3D_1 = 2\cdot 1 - 3\cdot 2 = -4$

$D_4 = 2D_3 - 3D_2 = 2\cdot(-4) - 3\cdot 1 = -11$

$D_5 = 2D_4 - 3D_3 = 2\cdot(-11) - 3\cdot(-4) = -22 + 12 = -10$

$D_6 = 2D_5 - 3D_4 = 2\cdot(-10) - 3(-11) = -20 + 33 = 13$

$D_7 = 2D_6 - 3D_5 = 2(13) - 3(-10) = 26 + 30 = \boxed{56}$

⑩

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 7 & 4 & 3 \\ 5^2 & 7^2 & 4^2 & 3^2 \\ 5^3 & 7^3 & 4^3 & 3^3 \end{vmatrix} = (3-4)(3-7)(3-5)(4-7)(4-5)(7-5)$$

$$= (-1)(-4)(-2)(-3)(-1)(2)$$

$$= 4 \cdot 6 \cdot (-2)$$

$$= \boxed{-48}$$

$\prod_{1 \le i < j \le n} (x_j - x_i)$ をつかった