サンプルポスト

① $\begin{pmatrix} -9 & -8 & -4 & 3 \\ -1 & -1 & -6 & -1 \\ -5 & 5 & -1 & -9 \end{pmatrix}$ の転置行列は $\begin{pmatrix} -9 & -1 & -5 \\ -8 & -1 & 5 \\ -4 & -6 & -1 \\ 3 & -1 & -9 \end{pmatrix}$ である。

② a) $\operatorname{sgn}(3\ 2\ 1) = (-1)^3 = -1$

$3>2$

$3>1$

$2>1$

b) $\operatorname{sgn}(4\ 5\ 1\ 3\ 2) = (-1)^7 = -1$

$4>1,3,2$

$5>1,3,2$

$3>2$

c) $\operatorname{sgn}(6\ 5\ 2\ 7\ 4\ 1\ 3) = (-1)^{15} = -1$

$6>5,2,4,1,3$

$5>2,4,1,3$

$2>1$

$7>4,1,3$

$4>1,3$

③ $\begin{pmatrix} 1 & 1 & 1 & 4 \\ -5 & 2 & -3 & 2 \\ -2 & -5 & 0 & -1 \\ 2 & -4 & 3 & -3 \end{pmatrix}$ の第(3,3)小行列は $\boxed{\begin{pmatrix} 1 & 1 & 4 \\ -5 & 2 & 2 \\ 2 & -4 & -3 \end{pmatrix}}$ である。

第(3,3)小行列式は $\begin{vmatrix} 1 & 1 & 4 \\ -5 & 2 & 2 \\ 2 & -4 & -3 \end{vmatrix} \overset{\substack{C_1-C_2 \\ C_3-4C_2}}{=} \begin{vmatrix} 0 & 1 & 0 \\ -7 & 2 & -6 \\ 6 & -4 & 13 \end{vmatrix}$

$$= (-1)^{1+2} \begin{vmatrix} -7 & -6 \\ 6 & 13 \end{vmatrix}$$

$$= (-1)\left((-7)(13) - (-6)(6)\right)$$

$$= -(-91 + 36)$$

$$= \boxed{55}$$

第(3,3)余因子は $(-1)^{3+3}55 = \boxed{55}$

④

$$\begin{cases} x & & & +2w & = -2 \\ 2x & +y & -z & +6w & = -5 \\ -4x & -2y & +2z & -12w & = a \end{cases}$$

の拡大係数行列は

$$\begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 2 & 1 & -1 & 6 & -5 \\ -4 & -2 & 2 & -12 & a \end{bmatrix}$$

$$\begin{array}{c} \\ r_2-2r_1 \\ r_3+4r_1 \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -2 & 2 & -4 & a-8 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ r_3+2r_2 \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & a-10 \end{bmatrix}$$

$a \neq 10$ のとき、階数は 3 で、解なし

$a = 10$ のとき、階数は 2 で解は

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2t-2 \\ s-2t-1 \\ s \\ t \end{pmatrix}$$

確認：

$$\begin{aligned} (-2t-2) & & & +2t & = -2 \quad ✓ \\ 2(-2t-2) + (s-2t-1) & -s & & +6t & = -5 \quad ✓ \\ -4(-2t-2)-2(s-2t-1) & & +2s & -12t & = 10 \quad ✓ \end{aligned}$$

⑤ $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 0 & 1 & -1 \end{pmatrix}$ の逆行列は $\boxed{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}}$

$$\left(\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 1 & 4 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} r_1 - 2r_3 \\ r_2 - r_1 \\ \end{array} \left(\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & -2 \\ 0 & 2 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$r_2 \leftrightarrow r_3 \left(\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\begin{array}{c} \\ \\ r_3 - 2r_2 \end{array} \left(\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right)$$

$$\begin{array}{c} r_1 - r_3 \\ r_2 + r_3 \\ \end{array} \left(\begin{array}{cccccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -2 \end{array}\right)$$

確認：$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 4 & -2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ -1 & 1 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1\cdot 2 + 2(-1) + (-1)(-1) & -1 + 2 - 1 & 2(-1) + (-1)(-2) \\ 1(2) + 4(-1) + (-2)(-1) & -1 + 4 - 2 & 4\cdot(-1) + (-2)(-2) \\ 0(2) + 1(-1) + (-1)(-1) & 0 + 1 - 1 & 1\cdot(-1) + (-1)(-2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ✓

⑥

$$\begin{vmatrix} a+3 & -1 & 2 \\ -2 & a+2 & -2 \\ -2 & 1 & a-1 \end{vmatrix} \overset{r_3-r_2}{=} \begin{vmatrix} a+3 & -1 & 2 \\ -2 & a+2 & -2 \\ 0 & -a-1 & a+1 \end{vmatrix}$$

$$= (a+1) \begin{vmatrix} a+3 & -1 & 2 \\ -2 & a+2 & -2 \\ 0 & -1 & 1 \end{vmatrix}$$

$$\overset{c_2+c_3}{=} (a+1) \begin{vmatrix} a+3 & 1 & 2 \\ -2 & a & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (a+1)(-1)^{3+3} \begin{vmatrix} a+3 & 1 \\ -2 & a \end{vmatrix}$$

$$= (a+1)\left((a+3)a - (1)(-2)\right)$$

$$= a+1\,(a^2+3a+2)$$

$$= (a+1)^2(a+2)$$

$\boxed{a=-1,-2}$ のとき $\begin{pmatrix} a+3 & -1 & 2 \\ -2 & a+2 & -2 \\ -2 & 1 & a-1 \end{pmatrix}$ の行列式は0で $\boxed{\text{正則でない。}}$

$\boxed{a \neq -1,-2}$ のとき $\begin{pmatrix} a+3 & -1 & 2 \\ -2 & a+2 & -2 \\ -2 & 1 & a-1 \end{pmatrix}$ の行列式は0でない、

$\boxed{\text{正則である。}}$

⑦ $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$ $\quad \mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\quad \mathbf{c} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ $\quad \mathbf{d} = \begin{pmatrix} -3 \\ -3 \\ 2 \end{pmatrix}$

とするとクラメルの公式で

$$\begin{cases} 3x - y + 2z = -3 \\ -2x + 2y - 2z = -3 \\ -2x + y - z = 2 \end{cases}$$

の解は

$$x = \frac{|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{1}{2}\begin{vmatrix} -3 & -1 & 2 \\ -3 & 2 & -2 \\ 2 & 1 & -1 \end{vmatrix} \underset{r_2-2r_3}{\overset{r_1+r_2}{=}} \frac{1}{2}\begin{vmatrix} 0 & 1 & 0 \\ -7 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix}$$

$(0+1)^2(0+2)$

$$= \frac{1}{2}(-1)(1\cdot(-7)\cdot(-1))$$

$$= \boxed{\frac{-7}{2}}$$

$$y = \frac{|\mathbf{a}\ \mathbf{d}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{1}{2}\begin{vmatrix} 3 & -3 & 2 \\ -2 & -3 & -2 \\ -2 & 2 & -1 \end{vmatrix} \underset{r_2-2r_3}{\overset{r_1+r_2}{=}} \frac{1}{2}\begin{vmatrix} 1 & -6 & 0 \\ 2 & -7 & 0 \\ -2 & 2 & -1 \end{vmatrix}$$

$$\overset{r_2-2r_1}{=} \frac{1}{2}\begin{vmatrix} 1 & -6 & 0 \\ 0 & 5 & 0 \\ -2 & 2 & -1 \end{vmatrix}$$

$$= \frac{1}{2}(-1)^{3+3}(-1)\begin{vmatrix} 1 & -6 \\ 0 & 5 \end{vmatrix} = \boxed{-\frac{5}{2}}$$

$$z = \frac{|\mathbf{a}\ \mathbf{b}\ \mathbf{d}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{1}{2}\begin{vmatrix} 3 & -1 & -3 \\ -2 & 2 & -3 \\ -2 & 1 & 2 \end{vmatrix} \underset{c_3-2c_2}{\overset{c_1+2c_2}{=}} \frac{1}{2}\begin{array}{cc} \begin{vmatrix} 1 & -1 & -1 \\ 2 & 2 & -7 \\ 0 & 1 & 0 \end{vmatrix} & \leftarrow 3 \\ \quad\quad\uparrow 2 & \end{array}$$

$$= \frac{1}{2}(-1)^{3+2}(1)\begin{vmatrix} 1 & -1 \\ 2 & -7 \end{vmatrix} = \frac{-1}{2}(-7+2) = \boxed{\frac{5}{2}}$$

確認：

$$3\left(\frac{-7}{2}\right) - \left(\frac{-5}{2}\right) + 2\left(\frac{5}{2}\right) = \frac{1}{2}(-21+5+10) = \frac{-6}{2} = -3 \quad ✓$$

$$-2\left(\frac{-7}{2}\right) + 2\left(\frac{-5}{2}\right) - 2\left(\frac{5}{2}\right) = 7-5-5 = -3 \quad ✓$$

$$-2\left(\frac{-7}{2}\right) + \left(\frac{-5}{2}\right) - \left(\frac{5}{2}\right) = 7 + \frac{-5}{2} - \frac{5}{2} = 7 - \frac{10}{2} = 7-5 = 2 \quad ✓$$

⑧

$$\begin{vmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
-4 & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -4 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -4 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot
\end{vmatrix}$$

$(1,5)$ $(5,2)$ $(4,5)$ $(7,9)$ $(7,2)$

$$= \underbrace{\underbrace{(-1)^{7+9}(-1)^{7+2}(-1)^{5+2}(-1)^{1+5}(-1)^{4+5}}_{(1)(-1)(-1)(1)(-1)}}_{(-1)} \begin{vmatrix} 4 & \cdot & \cdot & -1 \\ -4 & \cdot & \cdot & 3 \\ \cdot & -4 & 1 & \cdot \\ \cdot & -1 & 1 & \cdot \end{vmatrix} \overset{c_2 \leftrightarrow c_4}{=} (-1)(-1) \begin{vmatrix} 4 & -1 & & \\ -4 & 3 & & \\ & & 1 & -4 \\ & & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & -1 \\ -4 & 3 \end{vmatrix} \begin{vmatrix} 1 & -4 \\ 1 & -1 \end{vmatrix}$$

$$\overset{(r_2+r_1)\,(r_2-r_1)}{=} \begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} \begin{vmatrix} 1 & -4 \\ 0 & 3 \end{vmatrix}$$

$$= (4\cdot 2 - 0)(1\cdot 3 - 0)$$

$$= 8 \cdot 3 = \boxed{24}$$

⑨

$$\begin{vmatrix} -2 & 0 & & & \\ -2 & -2 & 0 & & \\ & -2 & \ddots & & \\ & & & & 0 \\ & & & -2 & -2 \end{vmatrix} = \boxed{-128}$$

7×7

3重対角行列で $\boxed{D_n = aD_{n-1} - bcD_{n-2}}$ $\boxed{D_0 = 1}$ $\boxed{D_{-1} = 0}$ をつかう。

ここで $a = -2$, $b \cdot c = 0$

$$D_{-1} = 0$$
$$D_0 = 1$$
$$D_1 = -2D_0 - 0 = -2$$
$$D_2 = -2D_1 - 0 = 4$$
$$\vdots$$
$$D_n = (-2)^n, \qquad D_7 = (-2)^7 = -128$$

⑩

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 4 & 5 & 6 & 2 \\ 4^2 & 5^2 & 6^2 & 2^2 \\ 4^3 & 5^3 & 6^3 & 2^3 \end{vmatrix} = (2-6)(2-5)(2-4)(6-5)(6-4)(5-4)$$

$$= (-4)(-3)(-2)(1)(2)(1)$$
$$= 12 \cdot (-4)$$
$$= \boxed{-48}$$

$\boxed{\prod_{1 \leq i < j \leq n} (x_j - x_i) \text{ をつかった}}$