サンプルトスト

① $\begin{pmatrix} -5 & -4 & 2 & -6 \\ 8 & -2 & 0 & 5 \\ 3 & 6 & 3 & -8 \end{pmatrix}$ の転置行列は

$${}^t\begin{pmatrix} -5 & -4 & 2 & -6 \\ 8 & -2 & 0 & 5 \\ 3 & 6 & 3 & -8 \end{pmatrix} = \begin{pmatrix} -5 & 8 & 3 \\ -4 & -2 & 6 \\ 2 & 0 & 3 \\ -6 & 5 & -8 \end{pmatrix}$$

②

a) $\mathrm{sgn}(3\ 2\ 1) = (-1)^3 = \boxed{-1}$

$3>2$
$3>1$
$2>1$

b) $\mathrm{sgn}(4\ 1\ 2\ 5\ 3) = (-1)^4 = \boxed{1}$

$4>1$
$4>2$
$4<5$
$4>3$
$1<2$
$1<5$
$1<3$
$2<5$
$2<3$
$5>3$

c) $\mathrm{sgn}(6\ 2\ 5\ 4\ 7\ 1\ 3) = (-1)^{13} = \boxed{-1}$

$6>2$
$6>5$
$6>4$
$6<7$
$6>1$
$6>3$
$2<5$
$2<4$
$2<7$
$2>1$
$2<3$
$5>4$
$5<7$
$5>1$
$5>3$
$4<7$
$4>1$
$4>3$
$7>1$
$7>3$
$1<3$

③ $$\begin{array}{cc} & \begin{matrix} 1 & & & \end{matrix} \\ 2 & \begin{pmatrix} -1 & -5 & -2 & 2 \\ -1 & -1 & 4 & -3 \\ 1 & 3 & -3 & 0 \\ -2 & 0 & -3 & 0 \end{pmatrix} \end{array}$$ の第(2,1)小行列は $$\begin{pmatrix} -5 & -2 & 2 \\ 3 & -3 & 0 \\ 0 & -3 & 0 \end{pmatrix}$$

第(2,1)小行列式は

第3列による展開で

$$\begin{vmatrix} -5 & -2 & 2 \\ 3 & -3 & 0 \\ 0 & -3 & 0 \end{vmatrix} = (-1)^{1+3}\cdot 2 \cdot \begin{vmatrix} 3 & -3 \\ 0 & -3 \end{vmatrix} + (-1)^{2+3}\cdot 0 \cdot \begin{vmatrix} -5 & -2 \\ 0 & -3 \end{vmatrix} + (-1)^{3+3}\cdot 0 \cdot \begin{vmatrix} -5 & -2 \\ 3 & -3 \end{vmatrix}$$

$$= 2\cdot\left((3)\cdot(-3) - (-3)\cdot 0\right) + 0 + 0$$

$$= \boxed{-18}$$

第(2,1)余因子 $= (-1)^{2+1}(-18) = \boxed{18}$

④

$$\begin{cases} x + y - 2z - 2w = -1 \\ -2x - 4y + 6z + 4w = -2 \\ 4x + 8y - 12z - 8w = a \end{cases}$$

の拡大係数行列は

$$\begin{pmatrix} 1 & 1 & -2 & -2 & -1 \\ -2 & -4 & 6 & 4 & -2 \\ 4 & 8 & -12 & -8 & a \end{pmatrix}$$

$$\begin{array}{l} \\ r_2 + 2r_1 \\ r_3 + 2r_2 \end{array} \begin{pmatrix} 1 & 1 & -2 & -2 & -1 \\ 0 & -2 & 2 & 0 & -4 \\ 0 & 0 & 0 & 0 & a-4 \end{pmatrix}$$

$$\begin{array}{l} r_1 + \frac{1}{2} r_2 \\ -\frac{1}{2} r_2 \\ \\ \end{array} \begin{pmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & a-4 \end{pmatrix}$$

$\boxed{a \neq 4}$ の場合, $\boxed{\mathrm{rank} = 3}$, $\boxed{\text{解なし}}$

$\boxed{a = 4}$ の場合. $\boxed{\mathrm{rank} = 2}$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} s + 2t - 3 \\ s + 2 \\ s \\ t \end{pmatrix}$$

確認:

$$\begin{cases} (s+2t-3) + (s+2) - 2s - 2t = -1 & \checkmark \\ -2(s+2t-3) - 4(s+2) + 6s + 4t = -2 & \checkmark \\ 4(s+2t-3) + 8(s+2) - 12s - 8t = 4 & \checkmark \end{cases}$$

⑤ $\begin{pmatrix} 2 & 0 & 0 \\ -12 & 4 & 3 \\ 12 & -3 & -3 \end{pmatrix}$ の逆行列は $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & \frac{-4}{3} \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ -12 & 4 & 3 & 0 & 1 & 0 \\ 12 & -3 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \frac{1}{2}r_1 \\ r_2 + r_3 \\ r_3 - 6r_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & -3 & -3 & -6 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} \\ \\ r_3 + 3r_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -6 & 3 & 4 \end{array}\right)$$

$$\begin{array}{c} \\ \\ \frac{-1}{3}r_3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 & \frac{-4}{3} \end{array}\right)$$

確認：
$$\begin{pmatrix} 2 & 0 & 0 \\ -12 & 4 & 3 \\ 12 & -3 & -3 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & \frac{-4}{3} \end{pmatrix} =$$

$$\begin{pmatrix} 2\cdot\frac{1}{2} + 0\cdot 0 + 0\cdot 2 & 2\cdot 0 + 0\cdot 1 + 0\cdot(-1) & 2\cdot 0 + 0\cdot 1 + 0\cdot(\frac{-4}{3}) \\ -12\cdot\frac{1}{2} + 4\cdot 0 + 3\cdot 2 & -12\cdot 0 + 4\cdot 1 + 3\cdot(-1) & -12\cdot 0 + 4\cdot 1 + 3\cdot(\frac{-4}{3}) \\ 12\cdot\frac{1}{2} - 3\cdot 0 - 3\cdot 2 & 12\cdot 0 - 3\cdot 1 - 3\cdot(-1) & 12\cdot 0 - 3 + -3\cdot(\frac{-4}{3}) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ok

⑥

$$\begin{vmatrix} a-1 & 1 & 1 \\ -4 & a-5 & -2 \\ 2 & 1 & a-2 \end{vmatrix} \underset{r_3-r_1}{\overset{r_2+2r_3}{=}} \begin{vmatrix} a-1 & 1 & 1 \\ 0 & a-3 & 2a-6 \\ 3-a & 0 & a-3 \end{vmatrix}$$

$$= (a-3)(a-3)\begin{vmatrix} a-1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{vmatrix}$$

$$\overset{c_3+c_1}{=} (a-3)(a-3)\begin{array}{r c} & \begin{vmatrix} a-1 & 1 & a \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{vmatrix} \\ 3 \rightarrow & \\ & \uparrow 1 \end{array}$$

$$= (a-3)(a-3)\ (-1)^{3+1}\left((-1)\left(1\cdot 2 - a\cdot 1\right)\right)$$

$$= (a-3)^2(a-2)$$

$\boxed{a = 3,\ 2}$ のとき、$\begin{pmatrix} a-1 & 1 & 1 \\ -4 & a-5 & -2 \\ 2 & 1 & a-2 \end{pmatrix}$ は正則ではない

⑦ $\mathbf{a} = \begin{pmatrix} -1 \\ -4 \\ 2 \end{pmatrix}$ $\quad \mathbf{b} = \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$ $\quad \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ $\quad \mathbf{d} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$ とすると

クラメルの公式で

$$\begin{cases} -x & y & z & = & 0 \\ -4x & -5y & -2z & = & -2 \\ 2x & y & -2z & = & -2 \end{cases}$$

の解は

$$x = \frac{|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{\begin{vmatrix} 0 & 1 & 1 \\ -2 & -5 & -2 \\ -2 & 1 & -2 \end{vmatrix}}{(0-3)^2(0-2)} \overset{r_2-r_3}{=} \frac{-1}{18} \begin{array}{c} \begin{array}{ccc} 1 & & \end{array} \\ \begin{vmatrix} 0 & 1 & 1 \\ 0 & -6 & 0 \\ -2 & 1 & -2 \end{vmatrix} \end{array} \begin{array}{c} \\ \\ \leftarrow 3 \end{array}$$

前のページ

$$= \frac{-1}{18}(-1)^{1+3}(-2)\left(1\cdot 0 - 1\cdot(-6)\right)$$

$$= \frac{12}{18} = \boxed{\frac{2}{3}}$$

$$y = \frac{|\mathbf{a}\ \mathbf{d}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{-1}{18} \begin{vmatrix} -1 & 0 & 1 \\ -4 & -2 & -2 \\ 2 & -2 & -2 \end{vmatrix} \overset{r_2-r_3}{=} \frac{-1}{18} \begin{vmatrix} -1 & 0 & 1 \\ -6 & 0 & 0 \\ 2 & -2 & -2 \end{vmatrix}$$

$$= \frac{-1}{18}(-1)^{1+2}(-6)\left(0\cdot(-2) - 1\cdot(-2)\right)$$

$$= \frac{-12}{18} = \boxed{\frac{-2}{3}}$$

$$z = \frac{|\mathbf{a}\ \mathbf{b}\ \mathbf{d}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{-1}{18} \begin{vmatrix} -1 & 1 & 0 \\ -4 & -5 & -2 \\ 2 & 1 & -2 \end{vmatrix} \overset{r_2-r_3}{=} \frac{-1}{18} \begin{vmatrix} -1 & 1 & 0 \\ -6 & -6 & 0 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= \frac{-1}{18}(-6) \begin{array}{c} \begin{vmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & -2 \end{vmatrix} \\ \begin{array}{ccc} & & \uparrow 3 \end{array} \end{array} \begin{array}{c} \\ \\ \leftarrow 3 \\ \\ \end{array} = \frac{1}{3}(-1)^{3+3}(-2)\left((-1)(1) - 1\cdot 1\right)$$

$$= \boxed{\frac{4}{3}}$$

確認：

$$-\frac{2}{3} \qquad -\frac{2}{3} + \frac{4}{3} = 0 \quad ✓$$

$$-4\left(\frac{2}{3}\right) \qquad -5\left(\frac{-2}{3}\right) \qquad -2\cdot\frac{4}{3} = -2 \quad ✓$$

$$2\left(\frac{2}{3}\right) \qquad \frac{-2}{3} \qquad -2\cdot\frac{4}{3} = -2 \quad ✓$$

$$-\frac{8}{3} \qquad +\frac{10}{3} \qquad -\frac{8}{3} = \frac{-6}{3} = -2$$

$$\frac{4}{3} \qquad \frac{-2}{3} \qquad \frac{-8}{3} = \frac{-6}{3} = -2$$

⑧

$$\begin{vmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 2 & \cdot \\
\cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -4 & \cdot & \cdot & 1 & \cdot \\
\cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -5 \\
-3 & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{vmatrix} = 0$$

（第1行に0しかがない）

⑨

$$\begin{vmatrix} -3 & 3 & & & \\ 1 & -3 & 3 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 3 \\ & & & 1 & -3 \end{vmatrix} = ?$$

9×9

3重対角行列で $\boxed{D_n = aD_{n-1} - bcD_{n-2}}$

$\boxed{D_0 = 1}$ $\boxed{D_{-1} = 0}$ をつかう。

ここで $a = -3$, $b \cdot c = 3$

$$D_{-1} = 0$$

$$D_0 = 1$$

$$D_1 = -3D_0 - 3D_{-1} = -3$$

$$D_2 = -3D_1 - 3D_0 = -3(-3) - 3(1) = 6$$

$$D_3 = -3D_2 - 3D_1 = -3(6) - 3(-3) = -18 + 9 = -9$$

$$D_4 = -3D_3 - 3D_2 = -3(-9) - 3(6) = 27 - 18 = 9 = 3^2$$

$$D_5 = -3D_4 - 3D_3 = -3(9) - 3(-9) = 0$$

$$D_6 = -3D_5 - 3D_4 = -3(0) - 3(9) = -27 = -3^3$$

$$D_7 = -3D_6 - 3D_5 = -3(-27) - 3(0) = 81 = 3^4$$

$$D_8 = -3D_7 - 3D_6 = -3(3^4) - 3(-3^3) = 3^4(-3+1) = -2 \cdot 3^4$$

$$D_9 = -3D_8 - 3D_7 = -3(-2 \cdot 3^4) - 3(3^4) = 3^5(2-1) = 3^5$$

$$= 9 \cdot 9 \cdot 3$$

$$= 81 \cdot 3$$

$$= \boxed{243}$$

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$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 6 \\ 1^2 & 2^2 & 4^2 & 6^2 \\ 1^3 & 2^3 & 4^3 & 6^3 \end{vmatrix} = (6-4)(6-2)(6-1)(4-2)(4-1)(2-1)$$

$$= 2 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 1$$

$$= 8 \cdot 10 \cdot 3$$

$$= 80 \cdot 3$$

$$= \boxed{240}$$

$\boxed{\prod_{1 \le i < j \le n} (x_j - x_i) \text{ をつかった}}$