サンプルテスト

① $\begin{pmatrix} -4 & -8 & 3 & 2 \\ 2 & 6 & 1 & 6 \\ -4 & -4 & 2 & -9 \end{pmatrix}$ の転置行列は

$$ {}^t\begin{pmatrix} -4 & -8 & 3 & 2 \\ 2 & 6 & 1 & 6 \\ -4 & -4 & 2 & -9 \end{pmatrix} = \begin{pmatrix} -4 & 2 & -4 \\ -8 & 6 & -4 \\ 3 & 1 & 2 \\ 2 & 6 & -9 \end{pmatrix} $$

② a) $\mathrm{sgn}(1\ 2\ 3) = (-1)^0 = \boxed{1}$

$1<2$

$1<3$

$2<3$

b) $\mathrm{sgn}(3\ 4\ 2\ 1\ 5) = (-1)^5 = \boxed{-1}$

$3<4$

$3>2$

$3>1$

$3<5$

$4>2$

$4>1$

$4<5$

$2>1$

$2<5$

$1<5$

c) $\operatorname{sgn}(7\ 6\ 4\ 3\ 2\ 1\ 5) = (-1)^{6+5+3+2+1} = (-1)^{17} = \boxed{-1}$

$7 > 6$
$7 > 4$
$7 > 3$
$7 > 2$
$7 > 1$
$7 > 5$

6

$6 > 4$
$6 > 3$
$6 > 2$
$6 > 1$
$6 > 5$

5

$4 > 3$
$4 > 2$
$4 > 1$
$4 < 5$

3

$3 > 2$
$3 > 1$
$3 < 5$

2

$2 > 1$
$2 < 5$

1

$1 < 5$

③ $\begin{pmatrix} -5 & 3 & \cancel{1} & -5 \\ \cancel{2} & \cancel{4} & \cancel{-5} & \cancel{-3} \\ 2 & 3 & \cancel{0} & -1 \\ 0 & -2 & \cancel{-2} & -1 \end{pmatrix}$ の第(2,3)小行列は $\begin{pmatrix} -5 & 3 & -5 \\ 2 & 3 & -1 \\ 0 & -2 & -1 \end{pmatrix}$

第(2,3)小行列式は $(-5)\cdot 3\cdot(-1) + 3\cdot(-1)\cdot 0 + (-5)\cdot 2\cdot(-2)$
$- (-5)\cdot(-1)\cdot(-2) - 3\cdot 2\cdot(-1) - (-5)\cdot 3\cdot 0$

$= 15 + 0 + 20$
$- (-10) - (-6) - 0$

$= 35 + 16 = 51$

第(2,3)余因子は $(-1)^{2+3}\cdot 51 = -51$

④
$$\begin{cases} -2x + 2y + 8z - 4w = -10 \\ 4x - 3y - 15z + 7w = 17 \\ 0 - y - z + w = a \end{cases}$$
の拡大係数行列は

$$\begin{bmatrix} -2 & 2 & 8 & -4 & -10 \\ 4 & -3 & -15 & 7 & 17 \\ 0 & -1 & -1 & 1 & a \end{bmatrix}$$

$$\begin{array}{c} \\ r_2 + 2r_1 \\ \\ \end{array} \begin{bmatrix} -2 & 2 & 8 & -4 & -10 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & -1 & -1 & 1 & a \end{bmatrix}$$

$$\begin{array}{c} -\frac{1}{2} r_1 \\ \\ r_3 + r_1 \end{array} \begin{bmatrix} 1 & -1 & -4 & 2 & 5 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & a-3 \end{bmatrix}$$

$$\begin{array}{c} r_1 + r_2 \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & a-3 \end{bmatrix}$$

$a \neq 3$ のとき 階数は 3 で、解なし

$a = 3$ のとき 階数は 2 で、解は

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3s - t + 2 \\ -s + t - 3 \\ s \\ t \end{pmatrix}$$

確認：

$$\begin{array}{lllllll}
-2(3s-t+2) & +2(-s+t-3) & +8s & -4t & = & -10 \\
4(3s-t+2) & -3(-s+t-3) & -15s & +7t & = & 17 \\
0 & -(-s+t-3) & -s & +t & = & 3
\end{array}$$

⑤

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ -3 & 1 & -5 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} \\ r_2-2r_1 \\ r_3+3r_1 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 3 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} \\ -r_2 \\ r_3+r_2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\begin{array}{l} r_1-2r_3 \\ \\ \\ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & -2 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 4 \\ -3 & 1 & -5 \end{pmatrix}$ の逆行列は $\begin{pmatrix} -1 & -2 & -2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

確認：

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 4 \\ -3 & 1 & -5 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot(-1)+0\cdot 2+2\cdot 1 & 1\cdot(-2)+0\cdot(-1)+2\cdot 1 & 1\cdot(-2)+0\cdot 0+2\cdot 1 \\ 2\cdot(-1)+(-1)\cdot 2+4\cdot 1 & 2\cdot(-2)+(-1)(-1)+4\cdot 1 & 2\cdot(-2)+(-1)\cdot 0+4\cdot 1 \\ -3\cdot(-1)+1\cdot 2+(-5)\cdot 1 & -3\cdot(-2)+1\cdot(-1)+(-5)\cdot 1 & -3\cdot(-2)+1\cdot 0+(-5)\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

⑥ $\begin{pmatrix} a-1 & 1 & 1 \\ -1 & a-3 & -1 \\ -1 & -1 & a-3 \end{pmatrix}$ が正則 $\iff$ 行列式 $\neq 0$

行列式 $= (a-1)(a-3)(a-3) + 1\cdot(-1)\cdot(-1) + 1(-1)(-1)$

$- (a-1)(-1)(-1) - 1(-1)(a-3) - 1\cdot(a-3)\cdot(-1)$

$= (a-1)(a^2-6a+9) + 1 + 1$
$\quad -a+1 \quad + a-3 \quad + a-3$

$= a^3 - 7a^2 + 15a - 9 + a - 3$

$= a^3 - 7a^2 + 16a - 12$
$= (a-2)^2(a-3)$

上の行列は正則 $\iff$ $\boxed{a \neq 2,\ 3}$

⑦
$$\begin{cases} -x + y + z = 2 \\ -x - 3y - z = -1 \\ -x - y - 3z = 2 \end{cases}$$
の解を求めよ。

$$\mathbf{a} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

とするとクラメルの公式によって

$$x = \frac{|\mathbf{d}\ \mathbf{b}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{\begin{vmatrix} 2 & 1 & 1 \\ -1 & -3 & -1 \\ 2 & -1 & -3 \end{vmatrix}}{-12} = \frac{18 + (-2) + 1 - 2 - 3 - (-6)}{-12}$$
$$= \frac{18}{-12} = \frac{-3}{2}$$

$$\begin{vmatrix} a-1 & 1 & 1 \\ -1 & a-3 & -1 \\ -1 & -1 & a-3 \end{vmatrix} = (a-2)^2(a-3)$$
$$\Rightarrow |\mathbf{a}\ \mathbf{b}\ \mathbf{c}| = (-2)^2 \cdot (-3) = -12$$

$$y = \frac{|\mathbf{a}\ \mathbf{d}\ \mathbf{c}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{-1}{12} \begin{vmatrix} -1 & 2 & 1 \\ -1 & -1 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \frac{-1}{12}\left(-3 + 2 + (-2) - 2 - 6 - 1 \right)$$
$$= \frac{-1}{12}(-12) = 1$$

$$z = \frac{|\mathbf{a}\ \mathbf{b}\ \mathbf{d}|}{|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|} = \frac{-1}{12} \begin{vmatrix} -1 & 1 & 2 \\ -1 & -3 & -1 \\ -1 & -1 & 2 \end{vmatrix} = \frac{-1}{12}\left(6 + 1 + 2 - (-1) - (-2) - 6 \right)$$
$$= \frac{-1}{12} \cdot 6 = \frac{-1}{2}$$

確認：

$$-\left(\frac{-3}{2}\right) + 1 + \left(\frac{-1}{2}\right) = 2 \quad ✓$$

$$-\left(\frac{-3}{2}\right) - 3 - \left(\frac{-1}{2}\right) = -1 \quad ✓$$

$$-\left(\frac{-3}{2}\right) - 1 - 3\left(\frac{-1}{2}\right) = 2 \quad ✓$$

⑧

ここで $\cdot = 0$

$$\begin{vmatrix}
-3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1^{(2,5)} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1^{(2,4)} & \cdot & \cdot & -4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1^{2,4} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1^{(2,4)} & -1 & \cdot \\
-3 & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & -3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1^{(5,4)} & \cdot
\end{vmatrix}$$

$$= (-1)^{2+5}(-1)^{2+4}(-1)^{2+4}(-1)^{2+4}(-1)^{5+4}\begin{vmatrix} -3 & 1 & \cdot & \cdot \\ -3 & -3 & \cdot & \cdot \\ \cdot & \cdot & 2 & -1 \\ \cdot & \cdot & 4 & -3 \end{vmatrix}$$

$$= (-1)(1)(1)(1)(-1)\begin{vmatrix} -3 & 1 \\ -3 & -3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -1 \\ 4 & -3 \end{vmatrix}$$

$$= \big((-3)(-3) - (-3)\cdot(1)\big)\big(2\cdot(-3) - (-1)\cdot 4\big)$$

$$= (9+3)(-6+4)$$

$$= 12 \cdot (-2)$$

$$= \boxed{-24}$$

⑨
$$\underbrace{\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{vmatrix}}_{7\times 7} = 8$$

3重対角行列で $\boxed{D_n = aD_{n-1} - bcD_{n-2}}$

$\boxed{D_0 = 1}$ $\boxed{D_{-1} = 0}$ をつかう。

ここで $a = 2$, $b \cdot c = 1$

$$
\begin{aligned}
D_{-1} &= 0 \\
D_0 &= 1 \\
D_1 &= 2D_0 - D_{-1} &&= 2 \\
D_2 &= 2D_1 - D_0 &&= 4 - 1 = 3 \\
D_3 &= 2D_2 - D_1 &&= 6 - 2 = 4 \\
D_4 &= 2D_3 - D_2 &&= 8 - 3 = 5 \\
D_5 &= 2D_4 - D_3 &&= 10 - 4 = 6 \\
D_6 &= 2D_5 - D_4 &&= 12 - 5 = 7 \\
D_7 &= 2D_6 - D_5 &&= 14 - 6 = \boxed{8}
\end{aligned}
$$

⑩
$$
\begin{vmatrix} 1 & 1 & 1 & 1 \\ 6 & 3 & 7 & 8 \\ 6^2 & 3^2 & 7^2 & 8^2 \\ 6^3 & 3^3 & 7^3 & 8^3 \end{vmatrix} = (3-6)(7-6)(8-6)(7-3)(8-3)(8-7)
$$
$$
= (-3)(1)(2)(4)(5)(1)
$$
$$
= \boxed{-120}
$$

$\displaystyle\prod_{1\le i<j\le n}(x_j - x_i)$ をつかった