Pro étale topology

§ Étale cohomology

We began with the question:

<u>Question</u> Given a smooth projective variety $X/\mathbb{F}_q$, how many $\mathbb{F}_{q^n}$-points does X have for each n?
That is, calculate

$$Z(X,t) = \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}\right) \in \mathbb{Q}[[t]]$$

$$\mathbb{Q}(t) \subseteq \mathbb{Q}((t))$$

<u>Theorem</u> (Weil conjectures)
If X is a (connected) smooth projective variety of dimension d over $\mathbb{F}_q$.

1) (Rationality) $Z(X,t)$ is a rational function of t.
i.e., its in $\mathbb{Q}(t)$

2) (Functional equation) There is an integer e such that

$$Z(X, q^{-d}t^{-1}) = \pm q^{ed/2} t^e Z(X,t)$$

3) (Riemann Hypothesis) We can write

$$Z(X,t) = \frac{P_1(t)\, P_3(t) \cdots P_{2d-1}(t)}{P_0(t)\, P_2(t) \cdots P_{2d}(t)}$$

with $P_i(t) \in \mathbb{Z}[t]$, such that the roots of $P_i(t)$ have absolute value $q^{-i/2}$. Moreover, $P_0(t) = 1-t$ and $P_{2d}(t) = 1-q^d t$.

(Betti numbers)

4) If X comes from a smooth projective variety over $\mathbb{Z}_{(p)} \subset \mathbb{C}$

$$\deg P_i(t) = \dim_{\mathbb{Q}} H^i_{sing}(X(\mathbb{C}), \mathbb{Q})$$

The strategy was to develop a cohomology theory

$$H^\bullet : (\text{Varieties}/k)^{op} \to \text{graded } \mathbb{Q}\text{-vector spaces}$$

for arbitrary varieties over any field k, which satisfies the following when X is smooth and projective.

1) (Finiteness) $\dim_{\mathbb{Q}} H^\bullet(X)$ is finite, and $H^i(X) = 0$ for $i \notin \{0, 1, 2, \dots, 2\dim X\}$

2) (Poincaré Duality) There is a canonical isomorphism $H^{2\dim X}(X) \cong \mathbb{Q}$ and a natural perfect pairing

$$H^i(X) \times H^{2d-i}(X) \to \mathbb{Q}$$

(i.e., $H^i(X) \cong \hom_{\mathbb{Q}}(H^{2d-i}(X), \mathbb{Q})$).

3) (Lefschetz Trace Formula)

$$|X(\mathbb{F}_{q^m})| = \sum_{i=0}^{2\dim X} (-1) \operatorname{Tr}(\Phi_i^m)$$

where $X_{\overline{\mathbb{F}}_q} := X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$, $\Phi : X_{\overline{\mathbb{F}}_q} \to X_{\overline{\mathbb{F}}_q}$ is the Frobenius,

and $\Phi_i^m : H^i(X_{\overline{\mathbb{F}}_q}) \to H^i(X_{\overline{\mathbb{F}}_q})$ is the induced morphism.

4) (Compatibility) If $k = \mathbb{C}$, then $H^\bullet(X)$ is isomorphic to singular cohomology. Then,

(Lefschetz Trace Formula) $\Rightarrow$ (Rationality)

(Poincaré Duality) $\Rightarrow$ (Functional Equation)

(Compatibility) $\Rightarrow$ (Betti numbers)

Eigenvalues $\alpha_{i,j}$ of $\Phi_i|_{H^i(X_{\overline{\mathbb{F}}_q})}$ have $|\alpha_{i,j}| = q^{-i/2}$ $\Rightarrow$ (Riemann Hypothesis)

Remarks

1. (Serre) Due to the existance of super-singular elliptic curves, there cannot be any cohomology theory with the above properties taking values in $\mathbb{Q}$-vector spaces.

$H^1(E, \mathbb{Q})$ $\mathbb{Q}$-algebra

$E \times E \xrightarrow{\text{mult.}} E$ Super singular $\Rightarrow$ not split

$\mathbb{Q} \to H^1$

2. For curves, étale cohomology with $\mathbb{Z}/\ell^n$-coefficients has Poincaré Duality and

$$\operatorname{rank}_{\mathbb{Z}/\ell^n} H^i_{et}(X_{\overline{\mathbb{F}}_q}, \mathbb{Z}/\ell^n) = \dim_{\mathbb{Q}} H^i_{sing}(X(\mathbb{C}), \mathbb{Q})$$

So we define

$$H^i_{et}(X, \mathbb{Q}_\ell) := \left(\varprojlim_{n \geq 1} H^i_{et}(X, \mathbb{Z}/\ell^n) \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

$$\in \varprojlim \mathbb{Z}/\ell^n = \mathbb{Z}_\ell$$

Successes

Theorem The $\mathbb{Q}_\ell$-vector spaces $H^i_{et}(X, \mathbb{Q}_\ell)$ satisfy (Finiteness), (Poincaré Duality), (Lefschetz), (Riemann Hypothesis).

Much stronger form of Poincaré Duality:

Theorem For any separated finite type morphism between noetherian $\mathbb{Z}[\frac{1}{\ell}]$-schemes $f: Y \to X$, and object $E \in D(X_{et}, \mathbb{Z}/\ell^n)$ there are adjunctions

$$f^* : D(X_{et}, \mathbb{Z}/\ell^n) \rightleftarrows D(Y_{et}, \mathbb{Z}/\ell^n) : f_*$$

$$f_! : D(Y_{et}, \mathbb{Z}/\ell^n) \rightleftarrows D(X_{et}, \mathbb{Z}/\ell^n) : f^!$$

$$- \otimes E : D(X_{et}, \mathbb{Z}/\ell^n) \rightleftarrows D(X_{et}, \mathbb{Z}/\ell^n) : \underline{\hom}(E, -)$$

satisfying a number of properties such as proper base change, smooth base change, etc...

$j : U \overset{\text{open}}{\hookrightarrow} X$ $j_!$: extension by zero

$\rightsquigarrow H^i_c(X, \mathbb{Z}/\ell^n)$

$i : Z \overset{\text{closed}}{\hookrightarrow} X$, $i^!$

To have these functors for $\mathbb{Z}_\ell$-coefficients takes some work.

Def. For a scheme X, define $\mathrm{Shv}_{et}(X)^{\mathbb{N}}$ to be the category of $\mathbb{N}$-indexed projective systems in $\mathrm{Shv}_{et}(X)$. i.e., $(\dots \to F_2 \to F_1)$. The derived category of $\mathrm{Shv}_{et}(X)^{\mathbb{N}}$ is denoted by $D(X^{\mathbb{N}}_{et})$.

Write

$$D(X_{et}, (\mathbb{Z}_\ell)_\bullet) \subseteq D(X^{\mathbb{N}}_{et})$$ for the full subcategory of those objects $(\dots \to K_2 \to K_1)$ such that $K_m \in D(X_{et}, \mathbb{Z}/\ell^m)$ and $K_m \otimes_{\mathbb{Z}/\ell^m} \mathbb{Z}/\ell^{m-1} \to K_{m-1}$ is a quasi-isomorphism.

Theorem (Ekedahl) The functors $f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{hom}}$ can be extended to $D(X_{et}, (\mathbb{Z}_\ell)_\bullet)$ in a sensible way.

Theorem Let X be a connected scheme, $\bar{x} \in X$ a geometric point, FEt_X the category of finite étale X-schemes, and consider the functor

$$F : \mathrm{FEt}_X \to \mathrm{Set} \qquad Y \mapsto |Y_{\bar{x}}|$$

The étale fundamental group of X is the profinite group

$$\pi_1^{et}(X, \bar{x}) := \mathrm{Aut}(F)$$

and F induces an equivalence of categories

$$\mathrm{FEt}_X \simeq \mathrm{Fin.}\text{-}\pi_1^{et}\text{-}\mathrm{Set}$$

category of finite sets equipped with a continuous π_1^{et}-action.

(FinSet) — finite sets

Linear version: $\mathrm{Loc}_X(R)$ = sheaves F of R-modules s.t. $\exists$ $\{f_i : U_i \to X\}$ each $f_i^* F$ is iso. to constant sheaf R^n $\exists\, n$.

(S $\exists$ set S)

__Prop.__

$$\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mathrm{Loc}_X(\mathbb{Z}/\ell^n) \simeq \left\{ \begin{array}{c} \text{continuous fin. dim.} \\ \mathbb{Q}_\ell\text{-linear representations} \\ \text{of } \pi_1^{et}(X) \end{array} \right\}$$

$$(\cdots \to F_3 \to F_2 \to F_1) \leftrightsquigarrow \pi_1^{et} \xrightarrow{\text{cont.}} GL_n(\mathbb{Q}_\ell)$$

$F_3 \curvearrowleft \mathbb{Z}/\ell^3$, $F_2 \curvearrowleft \mathbb{Z}/\ell^2$, $F_1 \curvearrowleft \mathbb{Z}/\ell$

defined up to multiplication by $\ell \to$

__Shortcomings__

__Problem__ ① The definition $H^i_{et}(X, \mathbb{Q}_\ell) := \left(\varprojlim H^i_{et}(X, \mathbb{Z}/\ell^n)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is ad hoc, not pleasant to work with.

② The categories $D(X_{et}, (\mathbb{Z}_\ell)_\cdot)$ are horrible to work with.

③ $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim \mathrm{Loc}_X(\mathbb{Z}/\ell^n)$ is not nice.

__Question__ Why can't we use sheaves with $\mathbb{Z}_\ell$-coefficients?

$\boxed{\text{Representability !}}$

Finite coefficients work well due to:

$$\mathrm{FEt}_X \cong \mathrm{Loc}_X(\mathrm{FinSet})$$

If we want to work with $\mathbb{Z}_\ell$-coefficients we should allow / add inverse limits in Et_X

Pro-étale schemes

Def. A morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of affine schemes is pro-étale if there exists a cofiltered system $(B_\lambda)_{\lambda \in \Lambda}$ of étale finite presentation A-algebras such that $B = \varinjlim B_\lambda$. The system (B_λ) is called a presentation for B.

Exercise 1.

1. Show that $\mathrm{Primes}(\varinjlim B_\lambda) = \varprojlim \mathrm{Primes}(B_\lambda)$
2. Show that for any $f \in B_\lambda$ with image $\bar{f} \in \varinjlim B_\lambda$, the set $D(\bar{f}) \subseteq \mathrm{Primes}(\varinjlim B_\lambda)$ of primes not containing $\bar{f}$ is the preimage of the set $D(f) \subseteq \mathrm{Primes}(B_\lambda)$ of primes not containing f.
3. Deduce that $\mathrm{Spe}(\varinjlim B_\lambda) = \varprojlim \mathrm{Spe}(B_\lambda)$

Exercise 2

Let k be an algebraically closed field. Using Ex 1 show that for every profinite set S, $\exists$ a pro-étale k-scheme $\mathrm{Spec}(B) \to \mathrm{Spec}(k)$ with $\mathrm{Spec}(B) \cong S$.

$S = \varprojlim S_i$

limit topology

finite, discrete topology

Exercise 3 Let k be a field, $k \subseteq k^{sep}$ a separable closure. (or any algebraic separable extension)
Show that $\mathrm{Spec}(k^{sep}) \to \mathrm{Spec}(k)$ is pro-étale.

Exercise 4 Suppose $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$, $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ are pro-étale.
Show that $\mathrm{Spec}(B) \times_{\mathrm{Spec}(A)} \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is pro-étale.

Exercise 5 Recall: If L/k is (finite) Galois then $\mathrm{Spec}(L \otimes_k L) \cong \coprod_{\mathrm{Gal}(L/k)} \mathrm{Spec}(L)$.

Recall: k^{sep}/k is the union of the finite Galois subextensions $k^{sep}/L/k$, and $\mathrm{Gal}(k^{sep}/k) \cong \varprojlim \mathrm{Gal}(L/k)$

Show that $\mathrm{Spec}(k^{sep} \otimes_k k^{sep}) \cong \mathrm{Gal}(k^{sep}/k)$ as <u>topological spaces</u>.

Theorem Let X be a connected noetherian scheme.

1. $$H^i_{pro\acute{e}t}(X, \mathbb{Q}_\ell) \cong H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$$

   sheaf cohomology of the constant sheaf $\mathbb{Q}_\ell$ $\qquad$ $\left(\varprojlim H^i_{\acute{e}t}(X, \mathbb{Z}/\ell^n)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$

2. The six functors $f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{hom}}$ work for the honest derived category of sheaves $D(X_{pro\acute{e}t}, \mathbb{Z}_\ell)$

3. If $X = \mathrm{Spec}(k)$ is the spectrum of a field, the subcategory of qcqs objects $X^{qcqs}_{pro\acute{e}t}$ is canonically isomorphic to the category of profinite continuous (not necessarily finite) $\mathrm{Gal}(k^{sep}/k)$-sets

   $$\mathrm{Spec}(k)^{qcqs}_{pro\acute{e}t} \simeq \text{Pro-Fin-Gal-Set}.$$

4. Honest $\mathbb{Q}_\ell$-local systems on X are equivalent to continuous representations of $\pi_1^{pro\acute{e}t}(X)$ on finite dimensional $\mathbb{Q}_\ell$-vector spaces.

Ex. 6 $\quad \mathrm{Spec}(A_{\mathfrak{p}}) \to \mathrm{Spec}(A)$ is pro-étale

$$\Big(\mathrm{Spec}(A_{\mathfrak{p}}^{sh}) \to \mathrm{Spec}(A) \text{ as well} \Big)$$

$$\mathrm{Spec}(A_{\mathfrak{p}}^{sh}) = \operatorname*{colim}_{\mathrm{Spec}\, k(\mathfrak{p})^{sep} \to U \xrightarrow{\text{étale}} X} \Gamma(U, \mathcal{O}_U)$$

$$\{ \cancel{\mathrm{Spec}(A_{\mathfrak{p}}^{sh}) \to \mathrm{Spec}(A)} \}_{\mathfrak{p} \in \mathrm{Spec}(A)} \leftarrow \text{not finite.}$$

Next week: $\quad \{ \mathrm{Spec}\, B_i \to \mathrm{Spec}\, A \}$

$$\forall \ \{ \mathrm{Spec}\, C_{ij} \to \mathrm{Spec}\, B_i \}$$

$$F \ldots$$

locally contractible topological spaces

locally compact top. spaces.

cdv. vs. cdh

v-topology vs. h-topology.

Affine Grassmanian