Today: 5月31日 → EC 7
6月 7日 → EC 8
6月 14日 → EC 9
6月 21日 → EC 10

start proetale topology

Q1

Q2

No lecture.
Question and Answer session.

Q1 Report:

Submit at least one solution to an exercise from each lecture.
I.e. 6 solutions.
You can submit more if you want.

(lecture 7 and lecture 8 are "cancelled" due to lack of time)

Deadline 〆切: 6月14日

Submit pdf by email: shanekelly64@gmail.com

$$\overset{\text{scheme}}{V} = \overset{\text{opens}}{U_1 \cup U_2}$$

$$\operatorname{div}(V) = \ker\left(\operatorname{div}(U_1) \oplus \operatorname{div}(U_2) \to \operatorname{div}(U_1 \cap U_2)\right)$$

$$\|$$

X set $\qquad X = \overset{\text{subsets}}{Y_1 \cup Y_2}$

$V^{(1)}$ $\|$

$$\mathbb{Z}(X) \xrightarrow{\sim} \ker\left(\mathbb{Z}(Y_1) + \mathbb{Z}(Y_2) \to \mathbb{Z}(Y_1 \cap Y_2)\right)$$

$x \in X^{(1)} \qquad i_x : x \to X$

$$i_{x*} : \mathrm{Shv}_{\acute{e}t}(x) \to \mathrm{Shv}_{\acute{e}t}(X)$$

$$\underset{\substack{\text{constant} \\ \text{sheaf}}}{\underline{\mathbb{Z}}} \in \mathrm{Shv}_{\acute{e}t}(x)$$

$V \in \acute{E}t(X)$, $f : V \to X$

$$(i_{x*}\underline{\mathbb{Z}})(V) = \underline{\mathbb{Z}}(x \times_X V)$$

$= \text{free ab. gp. on } f^{-1}(x)$

$$\boxed{\text{etale morphisms preserve codimension}}$$

$v \in V^{(1)} \iff f(v) \in X^{(1)}$

$$\bigoplus_{X^{(1)}} i_{x*}\mathbb{Z} = \operatorname{div}$$

$\mathbb{Z}$ constant presheaf $\quad V \mapsto \mathbb{Z}$

$V = V_1 \amalg V_2 \qquad \{V_1 \to V,\ V_2 \to V\}$

covering

$$0 \to \mathbb{Z}(V) \to \mathbb{Z}(V_1) \oplus \mathbb{Z}(V_2) \to \mathbb{Z}(V_1 \cap V_2)$$
$$= \mathbb{Z}(\emptyset)$$
$$= 0$$

$$0 \to \mathbb{Z}(\emptyset) \to 0$$
$$= \prod_{\emptyset}$$

$\underline{\mathbb{Z}} : V \longmapsto \prod_{\substack{\text{connected} \\ \text{components}}} \mathbb{Z}$ is an étale sheaf.

$$\underline{\mathbb{Z}} := \hom_{Sch/S}\left(-, \coprod_{\mathbb{Z}} S\right)$$

iso. on connected schemes

$$\mathbb{Z} \xrightarrow{\Phi} \underline{\mathbb{Z}}$$

$$\hom_{Sch/S}(X, S) = \{*\}$$

Zariski locally schemes are connected (use stalks)

$$\underset{\cap}{\Phi} \longmapsto \overset{\text{iso.}}{a_{zar}\Phi} \longmapsto \overset{\text{iso}}{a_{\acute{e}t}\Phi}$$

$$PSh \to Sh_{zar} \to Sh_{\acute{e}t}$$

$V = \mathbb{A}^1_{\mathbb{C}}$

$$g_*(\mathcal{O}_{M,H})(\mathbb{A}^1_{\mathbb{C}}) = \mathbb{C}(t) = \mathrm{Frac}(\mathbb{C}[t])$$

$$= \left\{ \frac{\prod (t-a_i)}{\prod (t-b_j)} \;\middle|\; a_i, b_j \in \mathbb{C} \right\}$$

$$(\mathbb{A}^1_{\mathbb{C}})^{(1)} \cong \mathbb{C}$$

$$\mathrm{Spec}(\mathbb{C}[t]) \ni (t-a) \longleftarrow a$$

local ring $\quad \mathbb{C}[t]_{(t-a)} = \left\{ \frac{\prod (t-a_i)}{\prod (t-b_j)} \;\middle|\; a_i, b_j \in \mathbb{C} \right\}$

$b_j \neq a$

$u := t-a$

$$\mathbb{C}[t]_{(t-a)} \to \widehat{\mathbb{C}[t]_{(t-a)}} \cong \mathbb{C}[[u]]$$

completion

$\mathrm{Spec}(\mathbb{Z}) \quad$ local rings $\quad \mathbb{Z}_{(p)} \to \mathbb{Z}_p = \widehat{\mathbb{Z}_{(p)}}$

$V \in \mathrm{Et}(\mathrm{Spec}(\mathbb{Z}))$

$V = \mathrm{Spec}(\mathcal{O}_K[S^{-1}]) \qquad \mathcal{O}_K \subset K$

$K / \mathbb{Q}$

$$\widehat{\mathcal{O}}_{V,v} \cong W(\mathbb{F}_q)$$

integral

k alg. closed $\qquad k \to A \qquad$ k-algebra

$$\mu_n(A) = \{a \in A \mid a^n = 1\}$$
$$\subseteq \{a \in \mathrm{Frac}(A) \mid a^n = 1\}$$
$$\cup\!\!\!/ \qquad //$$
$$\{a \in k \mid a^n = 1\}$$

$T^n - 1$

$X = \bigcup \mathrm{Spec}(A_i)$

$$\begin{array}{ccccccc}
0 \to \mu_n(X) & \to & \prod_i \mu_n(U_i) & \to & \prod_{i,j} \mu_n(U_i \cap U_j) \\
\| & & \| & & \\
0 \to \mu_n(X) & \to & \prod_i \mu_n(k) & \to & \prod_{i,j} \mu_n(k)
\end{array}$$

$U = \mathrm{Spec}(A)$ $\qquad$ sm. affine curve. $/ k$.

smooth compactification

$\{$closed Points of $X\}$ $\cong$ valuations of $k(U)$ with $v(k) = 0$

$$\begin{array}{ccc}
\mathbb{A}^d & \hookrightarrow & \mathbb{P}^d \\
\uparrow & & \uparrow \\
U & \hookrightarrow & \bar{U}
\end{array}$$

projective curve.

$(\bar{U})^{\sim}$ normalisation

$\|$

X

$$\begin{array}{ccc}
 & \hookrightarrow X & \\
U & & \cong \ \exists ! \qquad \dim U = 1 \\
 & \hookrightarrow X' &
\end{array}$$

$$
\begin{tikzcd}
 & \ker(n) \arrow[r, phantom, "\cong"] \arrow[d] & H^1 \arrow[d, hook] & 0 \arrow[d, hook] & \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d, "n"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "n"] & \mathbb{Z} \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[d, two heads] & \mathbb{Z} \arrow[r] \arrow[d, two heads] & 0 \\
 & 0 & H^2 \arrow[r, phantom, "\cong"] & \mathbb{Z}/n &
\end{tikzcd}
$$

$\exists$ A s.t. $A(k) \cong \mathrm{Pic}^0(X)$

$$A \times A \xrightarrow{\text{mult.}} A$$

$X / \mathbb{F}_q$

$$H^n_{et}(X, \mathbb{Q}_\ell) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_m H^n_{et}(X, \mathbb{Z}/\ell^m)$$

$\neg \exists$ coh. th. for varieties $/ \mathbb{F}_q$ with $\mathbb{Q}$-coefficients

supersingular elliptic curve. E

$$H^1(E, \mathbb{Q})$$

Serre counter example

$U \to V$ étale

$$H^n_c(V) \to H^n_c(U)$$

$U \to V$ proper

$$H^n_c(U) \to H^{n+1}_c(V)$$

$$\begin{array}{ccc} U & \to & V \\ \downarrow j & & \downarrow j \\ \bar{U} & \to & \bar{V} \end{array}$$

$$H_c := \operatorname{colimit} H(X, j_! F)$$

$$\begin{array}{ccc} U & \hookrightarrow & X' \\ & & \downarrow \\ U & \hookrightarrow & X \end{array}$$

$\dim U = 1$

$$\cong H(\text{initial}, j_! F)$$