In this lecture "curve" means smooth connected dimension one variety over an algebraically closed field k.

In this lecture we calculate the étale cohomology with finite coefficients of curves.

1 Some topology

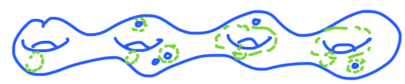
$U = \mathrm{Spec}\left(\mathbb{C}[z_1, \dots, z_n] / f_1, \dots f_m\right)$

smooth manifold $\dim_{\mathbb{R}} = 2$ $\quad U(\mathbb{C}) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ $\quad \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid f_i(z) = 0\}$

Suppose that $k = \mathbb{C}$, and U is a curve. Then the ~~associated~~ topological space $U(\mathbb{C})$ is homeomorphic to a sphere with g-handles attached M_g and some points removed

$$U(\mathbb{C}) \cong M_g \setminus \{x_1, \dots, x_m\}$$

— draw picture —



Consequently, we have the following.[1]

$H^r_{\mathsf{sing}}(U(\mathbb{C}), \mathbb{Q}) =$

$r \backslash m$	0	1	> 1
0	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$
1	$\mathbb{Q}^{2g}$	$\mathbb{Q}^{2g}$	$\mathbb{Q}^{2g+m-1}$
2	$\mathbb{Q}$	0	0
> 2	0	0	0

$H^r_{\mathsf{sing},c}(U(\mathbb{C}), \mathbb{Q}) =$

$r \backslash m$	0	1	> 1
0	$\mathbb{Q}$	0	0
1	$\mathbb{Q}^{2g}$	$\mathbb{Q}^{2g}$	$\mathbb{Q}^{2g+m-1}$
2	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$
> 2	0	0	0

Remark 1.

1. The symmetry here actually comes from a canonical pairing, known as Poincaré Duality (cf. Hatcher, "Algebraic Topology", Theorems 3.2 and 3.35).

2. If $U(\mathbb{C})$ was a non-orientable manifold, then we can still get a duality if instead of the constant sheaf $\mathbb{Q}$ we use an appropriate locally constant sheaf (cf. Hatcher, "Algebraic Topology", Theorem 3H.6). In the étale theory, the sheaf

$$\mu_n(V) = \{n\text{th roots of unity of } \Gamma(V, \mathcal{O}_V)\}$$

plays this rôle. Since we are using an algebraically closed field, μ_n is (non-canonically) isomorphic to the constant sheaf $\mathbb{Z}/n$, however, we still use μ_n because we want to keep track of how the automorphisms of k act on the cohomology.

[1] The first table can be calculated easily using Mayer-Vietoris sequences, and cohomology groups of spheres, and homotopy invariance, the second table is calculated easily using the closed / open complement long exact sequence for cohomology with compact support

3. Let us also point out that calculating the above tables is straightforward using very basic properties of singular cohomology (see the footnote). On the other hand, the calculation of étale cohomology of curves, even over an algebraically closed field, uses some serious algebraic results. For example, Hilbert's Theorem 90 and Tsen's Theorem are used in the proof of Theorem 4 to calculate $H^r_{\text{et}}(U, \mathbb{G}_m)$, and the theory of abelian varieties is used in the proof of Proposition 6 to calculate $H^r_{\text{et}}(X, \mu_n)$ when X is projective. We will take these input as given.

2 Some homological algebra

Lemma 2. *Suppose $\phi : \mathcal{A} \to \mathcal{B}$ is a left exact functor between abelian categories with enough injectives and*

$$0 \to F \to G \to H \to 0$$

is a short exact sequence in $\mathcal{A}$. Then there is a canonical long exact sequence

$$\cdots \to R^i\phi(F) \to R^i\phi(G) \to R^i\phi(H) \to R^{i+1}\phi(F) \to \dots$$

$[\dots \to 0 \to F \to G \to 0 \to \dots]$
$\quad -1 \quad 0$
$\downarrow q.i.$
$[\dots \to 0 \to 0 \to H \to 0 \to \dots]$
$\quad 0$

Proof. There is a canonical quasi-isomorphism $\text{Cone}(F{\to}G) \overset{q.i.}{\to} H$. On the other hand, F, G, H are functorially quasi-isomorphic to bounded below injective complexes $I^\bullet_F, I^\bullet_G, I^\bullet_H$. The Cone operation preserves quasi-isomorphisms,[2] so $\text{Cone}(I^\bullet_F{\to}I^\bullet_G) \overset{q.i.}{\to} I^\bullet_H$. The sequence

(iso. in $D(\mathcal{A})$)
$\cong K(\text{Inj})$

$$0 \to \underset{\simeq R\phi(G)}{\phi(I^\bullet_G)} \to \overset{R\phi(H)\,\cong}{\text{Cone}(\phi(I^\bullet_F){\to}\phi(I^\bullet_G))} \to \overset{R\phi(F)[1]\,\cong}{\phi(I^\bullet_F)[1]} \to 0$$

is a short exact sequence of chain complexes, and therefore by the Snake Lemma its cohomology fits into a long exact sequence. But the cohomology of these complexes are the right derived groups of ϕ of G, H, F respectively. $\square$

$0 \to F \to I^0 \to I^1 \to I^2 \to \dots$ in $D(\mathcal{A})$

Exercise 1. Prove the claim made in the proof that cone preserves quasi-isomorphisms. Hint: see the footnote.

$F \simeq [\dots \to 0 \to I^0 \to I^1 \to \dots]$
$G \simeq \dots$
$H \simeq \dots$

3 $\mathbb{G}_m$-coefficients

$R\Phi(F) = R\Phi(I^\cdot_F)$
$= \Phi(I^\cdot_F)$

Recall that $\mathbb{G}_m$ is the sheaf

$$\mathbb{G}_m : V \mapsto \Gamma(V, \mathcal{O}_V)^* \cong \hom(V, \operatorname{Spec} \mathbb{Z}[t, t^{-1}]).$$

We will leverage the cohomology of $\mathbb{G}_m$ to learn about the cohomology of μ_n. To calculate the cohomology of $\mathbb{G}_m$ we also use the étale sheaf which sends $V \in \text{Et}(X)$ to

Free abelian group generated by points of V of codimension one.

$$\text{Div} : V \mapsto \oplus_{V^{(1)}} \mathbb{Z} \cong (\oplus_{X^{(1)}} i_{x*}\mathbb{Z})\,(V).$$

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[2] That is, if $K_1 \overset{qi}{\to} K_2, L_1 \overset{qi}{\to} L_2$ are quasi-isomorphisms, and $K_1 \to L_1, K_2 \to L_2$ are morphisms making a commutative square, then $\text{Cone}(K_1{\to}L_1)$ is (canonically) quasi-isomorphic to $\text{Cone}(K_2{\to}L_2)$. This is easily checked using the two long exact sequences associated to $L_i \to \text{Cone}(K_i{\to}L_i) \to K_i[1]$.

(If $\dim V = 1$, then $V^{(1)} = V_{(0)}$ = closed points of V)

Here, $V^{(1)}, X^{(1)}$ are the sets of codimension one points, and $i_x : x \to X$ is the inclusion associated to $x \in X$.

Exercise 2. Prove the isomorphism $\Gamma(V, \mathcal{O}_V)^* \cong \hom_{\mathsf{Sch}}(V, \operatorname{Spec} \mathbb{Z}[t, t^{-1}])$ in the case V is an affine scheme.

Exercise 3. Using the fact that étale morphisms preserve codimension of points, prove the isomorphism $\oplus_{V^{(1)}} \mathbb{Z} \cong (\oplus_{X^{(1)}} i_{x*} \mathbb{Z})(V)$

Proposition 3 (Milne, Exam.II.3.9)**.** *For any connected normal scheme X, there is an exact sequence of sheaves on X_{et},*

$\eta \times_X V \leftarrow V$

$X^{(0)} = \{\eta\}$ $\quad Et(\eta) \leftarrow Et(X)$

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \operatorname{Div} \to 0,$$

$g_*: Sh(Et(\eta)) \to Sh(Et(X))$

$\mathbb{G}_m \mapsto g_*\mathbb{G}_m$

$\mathbb{G}_m = \mathbb{G}_{m,K}$

where $g : \eta \to X$ is the inclusion of the generic point.

Proof. It suffices to show that it is exact after evaluating on any connected affine $Y \in X_{\mathsf{et}}$. That is, that the sequence

$(g_*\mathbb{G}_m)(V)$ def. of g_*

e.g. $V = \mathbb{A}^1 = \operatorname{Spec}(k[t])$

$$0 \to \Gamma(Y, \mathcal{O}_Y)^* \to k(Y)^* \xrightarrow{v} \operatorname{Div}(Y) \to 0.$$

$= \mathbb{G}_m(\eta \times_X V)$

$= \mathbb{G}_m(\eta_V)$ generic point of V

$= k(V)^*$

(We take as given the fact that X normal implies Y normal, [Milne, I.3.17(b)]). The map v is defined as follows. Since Y is normal, for every $y \in Y^{(1)}$ the local ring $\mathcal{O}_{Y,y}$ is a discrete valuation ring. Let $v_y : k(Y)^* \cong \operatorname{Frac}(\mathcal{O}_{Y,y})^* \to \mathbb{Z}$ be its valuation. The map v is then

$\to v_y(f) = \#$ zeros of f at y

$$v : f \mapsto \sum_{y \in Y^{(1)}} v_y(f).y$$

Its a standard fact from commutative algebra that if A is a normal ring, then $A = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$. In particular, $A^* = \bigcap_{\operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}^*$. But $A_{\mathfrak{p}}^* = \ker(v_{\mathfrak{p}})$. $\square$

Theorem 4 (Milne, III.2.22(d), III.4.9)**.** *Let U be a curve. Then*

$$H^r_{\mathsf{et}}(U, \mathbb{G}_m) = \begin{cases} \Gamma(U, \mathcal{O}_U)^*, & r = 0 \\ \operatorname{Pic}(U), & r = 1 \\ 0, & r > 1. \end{cases}$$

Here, the Picard group $\operatorname{Pic}(U)$ can be defined by the exact sequence

$$k(U)^* \to \oplus_{x \in U} \mathbb{Z} \to \operatorname{Pic}(U) \to 0. \tag{1}$$

Proof. Consider the long exact sequence associated to the divisor sequence of Proposition 3

$\Gamma(U, \mathbb{G}_m)^* \qquad k(U)^* \quad \longrightarrow \quad \oplus_{U^{(1)}} \mathbb{Z}$

$$\begin{aligned} 0 \to H^0_{\mathsf{et}}(U, \mathbb{G}_m) \to H^0_{\mathsf{et}}(U, g_* \mathbb{G}_{m,K}) \to H^0_{\mathsf{et}}(U, \operatorname{Div}) \to \\ \to H^1_{\mathsf{et}}(U, \mathbb{G}_m) \to H^1_{\mathsf{et}}(U, g_* \mathbb{G}_{m,K}) \to H^1_{\mathsf{et}}(U, \operatorname{Div}) \to \\ \to H^2_{\mathsf{et}}(U, \mathbb{G}_m) \to H^2_{\mathsf{et}}(U, g_* \mathbb{G}_{m,K}) \to H^2_{\mathsf{et}}(U, \operatorname{Div}) \to \dots. \end{aligned}$$

? Pic = ?

0 = ?

0 = ?

Claim: these are zero

We always have $H^0_{\mathsf{et}}(U, \mathbb{G}_m) = \mathbb{G}_m(U)$, so it suffices to treat the case $r \geq 1$. For this, by the definition of Equation 1, it suffices to show

$$H^r_{\mathsf{et}}(U, g_* \mathbb{G}_{m,K}) = 0, \qquad \text{and} \qquad H^r_{\mathsf{et}}(U, \mathrm{Div}) = 0, \qquad \text{for all } r \geq 1.$$

The latter is easy. Since U is a curve, all codimension one points are closed. Moreover, since k is algebraically closed, the are all isomorphic to $\mathrm{Spec}(k)$. By $\mathrm{Div} = \oplus i_{u*}\mathbb{Z}$, for $r > 0$ we have

$$H^r_{\mathsf{et}}(U, \mathrm{Div}) = H^r_{\mathsf{et}}(U, \oplus i_{u*}\mathbb{Z}) \overset{Ex.4}{\cong} \oplus H^r_{\mathsf{et}}(U, i_{u*}\mathbb{Z}) \overset{Ex.5}{\cong} \oplus H^r_{\mathsf{et}}(\mathrm{Spec}(k), \mathbb{Z}) \overset{Ex.6}{\cong} 0.$$

Showing $H^r_{\mathsf{et}}(U, g_*\mathbb{G}_{m,K}) = 0$ is harder. We have $H^r_{\mathsf{et}}(U, g_*\mathbb{G}_{m,K}) = H^r_{\mathsf{et}}(\eta, \mathbb{G}_m)$, and then use Hilbert's Theorem 90 and Tsen's Theorem (see Milne III.4.9, III.2.22(d) for details). □

Exercise 4. Using the fact that (possibly infinite) sums of injective sheaves are injective, show $H^n_{\mathsf{et}}(X, \oplus_{i\in I} F_i) \cong \oplus_{i \in I} H^n_{\mathsf{et}}(X, F_i)$.

Exercise 5. Using the fact that if $i : Z \to X$ is a closed immersion, $i_* : \mathsf{Shv}_{\mathsf{et}}(Z) \to \mathsf{Shv}_{\mathsf{et}}(X)$ is exact and has an exact left adjoint (we saw this in the last lecture), show that $H^n_{\mathsf{et}}(X, i_*F) \cong H^n_{\mathsf{et}}(Z, F)$.

Exercise 6. Show that since k is algebraically closed, $H^n_{\mathsf{et}}(\mathrm{Spec}(k), F) = 0$ for any $F \in \mathsf{Shv}_{\mathsf{et}}(\mathrm{Spec}(k))$, and all $n > 0$.

4 μ_n-coefficients

Recall that μ_n is the sheaf

$$\mu_n : V \mapsto \{a \in \Gamma(V, \mathcal{O}_V)^* : a^n = 1\} \cong \hom(V, \mathrm{Spec}\, \mathbb{Z}[t]/(t^n - 1)).$$

Exercise 7. Prove the isomorphism above in the case that V is affine.

Exercise 8 (Milne, Pg.125)**.** Using the fact that $n : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[t, t^{-1}]; t \mapsto t^n$ is an étale morphism, prove that the sequence of étale sheaves

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$$

is exact (now n is the morphism $\Gamma(V, \mathcal{O}_V)^* \to \Gamma(V, \mathcal{O}_V)^*; a \mapsto a^n$, cf. Exercise 2).

Definition 5. *The exact sequence of Exercise 8 is called the* Kummer sequence.

Proposition 6. *Let X be a projective curve of genus g and $n \neq \mathrm{char.}(k)$. Then*

$$H^r_{\mathsf{et}}(X, \mu_n) = \begin{cases} \mu_n(k), & r = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g}, & r = 1 \\ \mathbb{Z}/n\mathbb{Z}, & r = 2 \\ 0, & r > 2 \end{cases}$$

Any automorphism of X acts trivially on H^2 (but not necessarily trivially on H^0 or H^1).

$\Gamma(X, \mathcal{O}_X) = k$

Proof. Consider the long exact sequence associated to the Kummer sequence

$$\begin{aligned}
0 &\to H^0_{\mathsf{et}}(X, \mu_n) \to H^0_{\mathsf{et}}(X, \mathbb{G}_m) \to H^0_{\mathsf{et}}(X, \mathbb{G}_m) \to \\
&\to H^1_{\mathsf{et}}(X, \mu_n) \to H^1_{\mathsf{et}}(X, \mathbb{G}_m) \to H^1_{\mathsf{et}}(X, \mathbb{G}_m) \to \\
&\to H^2_{\mathsf{et}}(X, \mu_n) \to H^2_{\mathsf{et}}(X, \mathbb{G}_m) \to H^2_{\mathsf{et}}(X, \mathbb{G}_m) \to \dots.
\end{aligned}$$

Since X is *projective* (and connected) we have $\mathbb{G}_m(X) = k^*$. So then by Theorem 4, this long exact sequence becomes

$$\begin{aligned}
0 &\to H^0_{\mathsf{et}}(X, \mu_n) \to k^* \overset{(-)^n}{\to} k^* \to \\
&\to H^1_{\mathsf{et}}(X, \mu_n) \to \mathrm{Pic}(X) \to \mathrm{Pic}(X) \to \\
&\to H^2_{\mathsf{et}}(X, \mu_n) \to 0 \to 0 \to \dots.
\end{aligned}$$

We automatically have $H^0_{\mathsf{et}}(X, \mu_n) = \mu_n(k)$. Since k is *algebraically closed*, the map $k^* \overset{(-)^n}{\to} k^*$ is surjective. So it remains only to show that

$$\ker(\mathrm{Pic}(X) \to \mathrm{Pic}(X)) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$$

$$\mathrm{coker}(\mathrm{Pic}(X) \to \mathrm{Pic}(X)) \cong (\mathbb{Z}/n\mathbb{Z})$$

These follow from the theory of abelian varieties. The group $\mathrm{Pic}(X)$ sits in a short exact sequence $0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \overset{\deg}{\to} \mathbb{Z} \to 0$, where deg is induced by the degree map $\mathrm{Div} \to \mathbb{Z}; \sum n_i x_i \mapsto \sum n_i$, and $\mathrm{Pic}^0(X)$ has the structure of an abelian variety of dimension $2g$. In general, for an abelian variety of dimension d over an algebraically closed field k and $\mathrm{char.}(k) \nmid n$, the multiplication by n map $A \to A$ is surjective with kernel isomorphic to $(\mathbb{Z}/n)^d$. $\square$

Remark 7. If $k = \mathbb{C}$, then $\mathrm{Pic}^0(X)$ can be identified with $\mathbb{C}^g/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^{2g}$ by integrating holomorphic differential forms around curves inside the Riemann surface $X(\mathbb{C})$.

5 Compact support (for curves)

Definition 8 (Milne, page 91, 93)**.** *Let U be a curve, and $j : U \to X$ its smooth compactification. That is, j is the unique dense open embedding into a smooth projective curve X.* Cohomology with compact support *of a sheaf $F \in \mathsf{Shv}_{\mathsf{et}}(U)$ are the cohomology groups of $j_! F \in \mathsf{Shv}_{\mathsf{et}}(X)$*

$$H^r_c(U, F) := H^r_{\mathsf{et}}(X, j_! F).$$

Remark 9. The functor $j_!$ does not preserves injectives, so these are not the right derived functors you might expect $H^r_{\mathsf{et}}(X, j_! F) \neq R^r\Gamma(X, j_! -)$.

Exercise 9. Let $i : Z \to X$ be the closed complement to $j : U \to X$ in the definition of cohomology with compact support. Using the short exact sequence $0 \to j_! j^* \to \mathrm{id} \to i_* i^* \to 0$ show that for any sheaf $F \in \mathsf{Shv}_{\mathsf{et}}(X)$, there is a long exact sequence

$$\dots \to H^r_c(U, j^* F) \to H^r_{\mathsf{et}}(X, F) \to H^r_{\mathsf{et}}(Z, i^* F) \to H^{r+1}_c(U, j^* F) \to \dots$$

Corollary 10. *Let U be a curve, $U \to X$ the smooth compactification, and $m = \#(X \setminus U)$. Choose an isomorphism $\mu_n \cong \mathbb{Z}/n$ (that is, choose a primitive nth root of unity in k^*). Then*

$$H^r_c(U, \mathbb{Z}/n) \cong \left\{ \begin{array}{c|ccc} r\backslash m & 0 & 1 & >1 \\ \hline 0 & \mathbb{Z}/n & 0 & 0 \\ 1 & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g+m-1} \\ 2 & \mathbb{Z}/n & \mathbb{Z}/n & \mathbb{Z}/n \\ >2 & 0 & 0 & 0 \end{array} \right.$$

Here g is the genus of the compactification, and these identifications depend on the isomorphism $\mathbb{Z}/n \cong \mu_n$.

Exercise 10. Prove Corollary 10 using Proposition 6, Exercise 9 and $Z \cong \amalg_{i=1}^N \operatorname{Spec}(k)$ (and that k is algebraically closed). Cf. Exercises 4, 5, 6.

Note: the groups $H^r_c(U, \mathbb{Z}/n)$ are all $\mathbb{Z}/n$-modules (since we can take the injective resolution inside the category of sheaves of $\mathbb{Z}/n$-modules). Moreover, every free module is projective. Hence, any short exact sequence of the form $0 \to (\mathbb{Z}/n)^a \to H^r_c(U, \mathbb{Z}/n) \to (\mathbb{Z}/n)^b \to 0$ is split.

6 Poincaré duality for curves

Definition 11. *An étale sheaf $F \in \mathsf{Shv}_{\mathsf{et}}(X)$ is* locally constant *if there is some étale covering $\{U_i \to X\}_{i \in I}$ such that each $F|_{U_i}$ is a constant sheaf.*

Remark 12. Any finite étale morphism $Y \to X$ induces a locally constant sheaf $\hom_X(-, Y)$ with finite fibres. In fact, there is an equivalence of categories between finite étale morphisms to X, and locally constant sheaves with finite fibres, cf. Milne Prop.V.1.1

Theorem 13 (Poincaré Duality. Milne Thm.V.2.1)**.** *Let F be a locally constant sheaf of $\mathbb{Z}/n$-modules with finite fibres on a curve U. There is a canonical perfect pairing of finite groups*

$$H^r_c(U, F) \times H^{2-r}_{\mathsf{et}}(U, \check{F}(1)) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Here $\check{F}(1)$ is the sheaf $V \mapsto \hom_{\mathsf{Shv}_{\mathsf{et}}(V)}(F|_V, \mu_n|_V)$.

Remark 14. This pairing is canonically isomorphic to the pairing induced by composition in $D(\mathsf{Shv}_{\mathsf{et}}(X, \mathbb{Z}/n))$

$$\hom(X, j_! F[r]) \times \hom(j_! F[r], \mu_n[2]) \to \hom(X, \mu_n[2])$$

Unfortunately, we do not have time for the proof.

Corollary 15. *Let U be a curve, $U \to X$ the smooth compactification, and $m = \#(X \setminus U)$. Choose an isomorphism $\mu_n \cong \mathbb{Z}/n$ (that is, choose a primitive* *root of unity in k^*). Then*

$$H^r_{\mathsf{et}}(U, \mathbb{Z}/n) \cong \left\{ \begin{array}{c|ccc} r\backslash m & 0 & 1 & > 1 \\ \hline 0 & \mathbb{Z}/n & \mathbb{Z}/n & \mathbb{Z}/n \\ 1 & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g} & (\mathbb{Z}/n)^{2g+m-1} \\ 2 & \mathbb{Z}/n & 0 & 0 \\ > 2 & 0 & 0 & 0 \end{array} \right.$$

Here g is the genus of the compactification g, and these identifications depend on the isomorphism $\mathbb{Z}/n \cong \mu_n$.

Exercise 11. Prove the corollary using Poincaré duality and Corollary 10.

7 Support in a closed subscheme

The proof of Poincaré Duality for curves uses cohomology with support in a closed subscheme. As we did not do the proof, we did not use this cohomology.

Definition 16 (Milne pg.91)**.** *Let $i : Z \to X$ be a closed immersion.* Cohomology with support in Z *is defined as the right derived functor of the functor left exact functor*[3] $\Gamma(X, i_* i^! -)$.

$$H^r_Z(X, F) = R^r\Gamma(X, i_* i^! -).$$

Exercise 12. Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$. Recall that j^* is exact and preserves injectives.[4] Using the short exact sequence

$$0 \to i_* i^! \to \mathrm{id} \to j_* j^* \to 0$$

show that there is a long exact sequence

$$\cdots \to H^r_Z(X, F) \to H^r_{\mathsf{et}}(X, F) \to H^r_{\mathsf{et}}(U, F) \to H^{r+1}_Z(X, F) \to \dots.$$

Exercise 13. Using Exercise 12, and the fact that $\Gamma(X, -)$ is left exact, show that the sections with support in Z functor admits the description

$$\Gamma(X, i_* i^! -) : F \mapsto \ker\Big(F(X) \to F(U)\Big) \tag{2}$$

Suppose that

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[i]{} & X \end{array}$$

[3]Note that this is also right Quillen. That is, it has a left adjoint $i_* i^* \gamma$ which preserves monomorphisms, and monomorphic weak equivalences (here $\gamma : \mathrm{Ab} \to \mathsf{Shv}_{\mathsf{et}}(X)$ is the constant sheaf functor; left adjoint to global sections $\Gamma(X, -) : \mathsf{Shv}_{\mathsf{et}}(X) \to \mathrm{Ab}$). So the right derived functor can be calculated on unbounded complexes via fibrant replacements.

[4]Since j^* has an exact left adjoint $j_!$, the functor j^* also preserves fibrant objects.

is a commutative square with i, i' closed immersions, and f, g étale morphisms. Show that there is a canonical morphism of functors

$$\Gamma_Z(X, -) \to \Gamma_{Z'}(X', f^*-)$$

(Note since f is étale, f^* is just the restriction from $\mathrm{Et}(X)$ to $\mathrm{Et}(X') \subset \mathrm{Et}(X)$).

Theorem 17 (Excision. Milne Prop.III.2.7)**.** *In the situation of Exercise 13, if the square is cartesian, and $Z' \to Z$ is an isomorphism, then*

$$R\Gamma_Z(X, -) \cong R\Gamma_{Z'}(X', -).$$

Proof. It suffices to show that $\alpha : \Gamma_Z(X, -) \to \Gamma_{Z'}(X', f^*-)$ is an isomorphism of functors (because f^* is exact and preserves injectives. Given a sheaf F, by Exercise 13 the morphism α fits into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_{Z'}(X', f^*F) \arrow[r] & F(X') \arrow[r] & F(U') \\ 0 \arrow[r] & \Gamma_Z(X, F) \arrow[r] \arrow[u, "\alpha"] & F(X) \arrow[r] \arrow[u] & F(U) \arrow[u] \end{tikzcd}$$

where $U = X \setminus Z$ and $U' = U \times_X X' = X' \setminus Z'$. Now the rows of this diagram are exact by Exercise 13, and if the right-most square is cartesian, then an easy diagram chase shows α is an isomorphism.

To show the square is cartesian, since $F(-) = \hom_{\mathsf{Shv}_{\mathsf{et}}(X)}(-, F)$, it suffices to show that

$$0 \to \mathbb{Z}(U') \to \mathbb{Z}(U) \oplus \mathbb{Z}(X') \to \mathbb{Z}(X) \to 0$$

is exact. The pair (j^*, i_*) detect exactness, so it suffices to show that this sequence is exact after applying these two functors. Since we have[5] $g^*\mathbb{Z}(W) = \mathbb{Z}(W \times_X Y)$ for any $W \in \mathrm{Et}(X)$ and morphism $g : Y \to X$, the two resulting sequences are

$$0 \to \mathbb{Z}(U') \to \mathbb{Z}(U) \oplus \mathbb{Z}(U') \to \mathbb{Z}(U) \to 0 \qquad \text{after } j^*$$

$$0 \to \mathbb{Z}(\varnothing) \to \mathbb{Z}(\varnothing) \oplus \mathbb{Z}(Z) \to \mathbb{Z}(Z) \to 0 \qquad \text{after } i^*$$

which are clearly exact. □

Exercise 14. Do the diagram chase in the proof of Theorem 17 which shows α is an isomorphism.

Exercise 15 (Milne Cor.III.1.28)**.** Let $x \in X$ be a closed point in a scheme and consider its henselisation $\mathcal{O}^h_{X,x}$. Show that

$$H^r_x(X, F) \cong H^r_x(\mathrm{Spec}(\mathcal{O}_{X,x}), F) \cong H^r_x(\mathrm{Spec}(\mathcal{O}^h_{X,x}), F).$$

[5] This was an exercise last week. It is proved using Yoneda and adjunction, by applying $\hom(-, F)$ for any sheaf $F \in \mathsf{Shv}_{\mathsf{et}}(Y)$.

Proposition 18. *Let U be a curve, $x \in U$ a point, and $n \neq \mathrm{char.}(k)$. Then*

$$H^r_x(U, \mu_n) = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & r = 2 \\ 0, & r \neq 2 \end{cases}$$

Remark 19. Heuristically, $\mathrm{Spec}(\mathcal{O}^h_{U,x})$ is a small neighbourhood of x in the curve U. Its generic point is this neighbourhood with the point x removed, i.e., a small annulus. This proposition is a cohomological consequence of this geometric heuristic.

Proof. By Exercise 15 we can replace U with $T = \mathrm{Spec}(\mathcal{O}^h_{U,x})$. Let $\eta \in T$ be the generic point (so $T = \{\eta, x\}$). Consider the long exact sequence of Exercise 12. Since $H^0_{\mathsf{et}}(T, \mu_n) = H^0_{\mathsf{et}}(\eta, \mu_n)$, the part

$$0 \to H^0_x(T, \mu_n) \to H^0_{\mathsf{et}}(T, \mu_n) \to H^0_{\mathsf{et}}(\eta, \mu_n) \to H^1_x(T, \mu_n) \to H^1_{\mathsf{et}}(T, \mu_n)$$

Show that $H^0_x(U, \mu_n) = 0$ and $H^1_x(T, \mu_n) \to H^1_{\mathsf{et}}(T, \mu_n)$ is injective. Now since $k \cong k(x)$ is algebraically closed, $\mathcal{O}^h_{U,u}$ is in fact strictly henselian, and so $H^r_{\mathsf{et}}(T, F) = 0$ for all F and all $r > 0$. So in fact, $H^1_x(T, \mu_n) = 0$, and by the exact sequence

$$H^r_{\mathsf{et}}(T, F) \to H^r_{\mathsf{et}}(\eta, \mu_n) \to H^{r+1}_x(T, \mu_n) \to H^{r+1}_{\mathsf{et}}(T, F)$$

we have $H^{r+1}_x(T, \mu_n) = H^r_{\mathsf{et}}(\eta, \mu_n)$ for all $r > 0$. Finally, the calculation for $H^r_{\mathsf{et}}(\eta, \mu_n)$, recall from the end of the proof of Theorem 4 that we had $H^r_{\mathsf{et}}(\eta, \mathbb{G}_m) = 0$ for all $r > 0$. It then follows from the Kummer long exact sequence that $H^r_{\mathsf{et}}(\eta, \mu_n) = 0$ for $r > 1$. Finally, we have the long exact sequence

$$H^0_{\mathsf{et}}(T, \mathbb{G}_m) \to H^0_{\mathsf{et}}(\eta, \mathbb{G}_m) \to H^1_x(T, \mathbb{G}_m) \to H^1_{\mathsf{et}}(T, \mathbb{G}_m)$$

shows that $H^1_x(T, \mathbb{G}_m) \cong \mathbb{Z}$, since $H^1_{\mathsf{et}}(T, \mathbb{G}_m) \cong \mathrm{Pic}(T) = 0$ and $\mathrm{Frac}(\mathcal{O}_{U,x})^*/\mathcal{O}^*_{U,x} \cong \mathbb{Z}$ because it is a discrete valuation ring. Then the Kummer exact sequence

$$\underbrace{H^1_x(T, \mathbb{G}_m)}_{\cong \mathbb{Z}} \xrightarrow{n} \underbrace{H^1_x(T, \mathbb{G}_m)}_{\cong \mathbb{Z}} \to H^2_x(T, \mu_n) \to \underbrace{H^2_x(T, \mathbb{G}_m)}_{\cong 0}$$

shows that $H^2_x(T, \mu_n) \cong \mathbb{Z}/n$. □